17-*Figure* e^n-*Table and its Applications*

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Synopsis. An engineer does not necessarily keep with him a logarithm table of more than seven places, but he is at times faced with the urgent necessity for getting more accurate value of logarithm of a given number. The present e^n-Table will suffice effectively for such a need with a hand-driven calculating machine. Accurate values of e^ν and N^ν can also be found by the table, ν and N being given numbers. Examples are added for illustration.

1. A 7-figure logarithm table is constantly kept with an engineer, but he is seldom provided with more accurate table, though the need for this at times occurs. Even if the more accurate table is kept, time and labour needed for computing logarithm of a given number is generally not so small, since interpolation of considerably high order must be worked out. This si especially the case, when large places of decimals are desired for logarithm of a number.

Such a case can be treated effectively by the aid of the present e^n-Table, when a 20-figure hand-driven calculating machine is available. The e^n-Table affords 18 places of decimals, though the last digits are not necessarily correct, differing at times by one unit from their correct values, because of the unfavourable accumulation of errors due to rounding-off of the nineteenth place of decimals in the course of computation.

2. The principle of the procedure to find accurate logarithm of a given number is as follows. Let us suppose to find natural logarithm of a number, N say; and write

$$\log_e N = n + \varepsilon, \qquad (1)$$

or

$$N = e^n \times e^\varepsilon.$$

By Taylor's theorem e^ε may be expanded in power series of ε, in virtue of which the equation last written takes the form

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$$\varepsilon = \left(\frac{N}{e^n} - 1\right) - \frac{\varepsilon^2}{2} - \frac{\varepsilon^3}{6} - \cdots. \tag{2}$$

N is divided by an e^n which is nearest to N, and the quotient is divided by the second e^n which is again nearest to the first quotient; such divisions, or often multiplications, are performed several times, ordinarily four times. The last quotient thus obtained is almost equal to unity. ε in equation (2) will then be found easily, in which the procedure of iteration is effectively employed. The sum of the fractional values of the above n's and ε gives the wanted logarithm.

All numbers, large or small, whose logarithms are desired, can be reduced to those of first place, since for example

$$\log_e 326.8 = \frac{2}{M} + \log_e 3.268.$$

It is added also that multiplication and division for large numbers by a 20–figure calculating machine can be performed through one–hand system by the procedures

$$(A + B \times 10^{-9})(C + D \times 10^{-9}) = (AD + BC) \times 10^{-9} + AC,$$

$$\frac{C + D \times 10^{-9}}{A + B \times 10^{-9}} = \left\{\left(-\frac{B}{A}C + D\right) \times 10^{-9} + C\right\}\frac{1}{A}.$$

Several examples which follow will serve as illustration.

3. Example 1. To find log 2. Referring to the e^n–Table,

$$\log_e 2 = \left(\frac{1}{2} + \frac{1}{6} + \frac{1}{40} + \frac{1}{704} + \frac{1}{16\,384}\right) + \varepsilon. \tag{3}$$

We now have

$$e^n = \exp\left(\frac{1}{2} + \frac{1}{6} + \frac{1}{40} + \frac{1}{704} + \frac{1}{16\,384}\right) = 2.000\,001\,951\,617\,804\,044.$$

Then by division

$$\frac{2}{e^n} = 0.9^6\,024\,192\,050\,180,$$

so that ε can be found to be

$$\varepsilon = -0.0^6\,975\,807\,949\,820 - 0.0^{12}\,476\,101 = -0.0^6\,975\,808\,425\,921.$$

Thus we obtain

$$\log_e 2 = 0.500 + 0.1\dot{6}\dot{6} + 0.025 + 0.001\,420\,4\dot{5}\dot{4}$$

$$+ 0.0^3\,061\,035\,156\,250 - 0.0^6\,975\,808\,425\,921$$

$$= 0.693\,147\,180\,559\,945\,291, \qquad (4)$$

the last digit of decimals being of course not reliable.

In addition we have

$$M = \log_{10} e = 0.434\,294\,481\,903\,251\,841,$$

so that we obtain

$$\log_{10} 2 = M \log_e 2 = 0.301\,029\,995\,663\,981\,196.$$

In place of the selection of e^n made in (3), Example 1, a more rapid convergence for ε will result from another selection, that is

$$e^n = \exp\left(\frac{3}{4} - \frac{1}{20} - \frac{1}{160} - \frac{1}{1\,536} + \frac{1}{20\,480}\right),$$

and, in fact, we have

$$\frac{2}{e^n} = \frac{2.117\,102\,105\,528\,670\,962}{2.117\,103\,388\,277\,812\,634} = 0.9^6\,394\,101\,795\,512.$$

We then have

$$\varepsilon = -0.0^6\,605\,898\,204\,488 - \frac{\varepsilon^2}{2} - \cdots$$

$$= -0.0^6\,605\,898\,204\,488 - 183\,556 = -0.0^6\,605\,898\,388\,044.$$

Hence we obtain

$$\log_e 2 = 0.693\,147\,786\,458\,333\,333 - 0.0^6\,605\,898\,388\,044$$

$$= 0.693\,147\,180\,559\,945\,289,$$

which is practically in accordance with the preceding value (4), differing by 2 units in the last place of decimals.

Example 2. To find $\log_e \pi$,

$$\log_e \pi = \log_e 3.141\,592\,653\,589\,793\,238$$

$$= \left(1 + \frac{1}{7} + \frac{1}{640} + \frac{1}{3\,584} + \frac{1}{32\,768}\right) + \varepsilon.$$

We now have

$$\exp\left(1+\frac{1}{7}+\frac{1}{640}+\frac{1}{3\,584}+\frac{1}{32\,768}\right)=3.141\,590\,430\,734\,739\,655,$$

so that the equation for ε becomes

$$\varepsilon=0.0^6\,707\,557\,239\,747-\frac{\varepsilon^2}{2}-\cdots.$$

On solving this equation by the simple process of iteration, we at once have

$$\varepsilon=0.0^6\,707\,557\,239\,747-250\,319=0.0^6\,707\,556\,989\,428.$$

We thus obtain

$$\log_e \pi=1+0.\dot{1}42\,85\dot{7}+0.001\,562\,5+0.0^3\,279\,017\,857\,142\,857$$
$$+0.0^3\,030\,517\,578\,125+0.0^6\,707\,556\,989\,428$$
$$=1.144\,729\,885\,849\,400\,142.$$

4. The present e^n–Table serves also to find e^N, N being any given number.

Example 3. Let us for example compute $e^{\sqrt{2}}$, $\sqrt{2}$ being here taken to be approximately 1.414 213 562 373 095 049. On referring to the e^n–Table, we write

$$\sqrt{2}=n+\varepsilon=\left(\frac{3}{2}-\frac{1}{12}-\frac{1}{384}+\frac{1}{7\,168}+\frac{1}{57\,344}\right)+\varepsilon,$$

which gives

$$\varepsilon=-0.0^3\,005\,885\,171\,547\,809.$$

We now have

$$e^n=\exp\left(\frac{3}{2}-\frac{1}{12}-\frac{1}{384}+\frac{1}{7\,168}+\frac{1}{57\,344}\right)$$

$$=\frac{4.482\,392\,515\,634\,069\,336}{1.089\,738\,217\,538\,093\,233}=4.113\,274\,586\,038\,257\,770,$$

and

$$e^\varepsilon=1+\varepsilon+\frac{\varepsilon^2}{2}+\frac{\varepsilon^3}{6}+\cdots$$

$$= 1.000 - 0.0^3\,005\,885\,171\,547\,809 + 17\,317\,622 - 34$$

$$= 0.9^3\,994\,114\,845\,769\,779.$$

Multiplying e^n by e^ε above, we obtain the required value, which follows.

$$e^{\sqrt{2}} = 4.113\,250\,378\,782\,927\,686.$$

5. Accurate value of N^ν can also be computed by using the present e^n-Table, both of N and ν representing given numbers. In fact, if we put

$$P = N^\nu,$$

and take logarithm of both members, i.e.

$$\log_e P = \nu \log_e N, \quad = \alpha \quad \text{say},$$

we then have

$$P = e^\alpha,$$

which is the required result, $\log_e N$ and e^α being computed by the preceding procedures.

Example 4. As an example, we shall get the accurate value of

$$P = \pi^{\sqrt{2}}.$$

Taking logarithm of both members, and referring to the preceding values, we have

$$\log_e P = \sqrt{2}\,\log_e \pi = 1.414\,213\,562\,373\,095\,049$$

$$\times\, 1.144\,729\,885\,849\,400\,142$$

$$= 1.618\,892\,529\,822\,026\,623, \quad = \alpha \quad \text{say}.$$

With the e^n-Table, we can write

$$\alpha = n + \varepsilon = \frac{3}{2} + \frac{1}{9} + \frac{1}{160} + \frac{1}{640} - \frac{1}{32\,768} + \varepsilon$$
$$= 1.618\,893\,093\,532\,986\,111 + \varepsilon,$$

so that ε amounts to

$$\varepsilon = 1.618\,892\,529\,822\,026\,623 - 1.618\,893\,093\,532\,986\,111$$
$$= -\,0.0^6\,563\,710\,959\,488.$$

We then have

$$e^{\varepsilon}=1+\varepsilon+\frac{\varepsilon^2}{2}+\cdots$$

$$=1-0.0^6\,563\,710\,959\,488+158\,885$$

$$=0.999\,999\,436\,289\,199\,397.$$

On the other hand we have by successive multiplication and division

$$e^n=\exp\left(\frac{3}{2}+\frac{1}{9}+\frac{1}{160}+\frac{1}{640}-\frac{1}{32\,768}\right)$$

$$=5.047\,500\,112\,701\,240\,709.$$

By multiplication we obtain

$$P=e^{\alpha}=e^n\times e^{\varepsilon}=\pi^{\sqrt{2}}=5.047\,497\,267\,370\,911\,134,$$

which is the required result.

6. When only nine or ten places of effective figures are desired, which will be of frequent occurrence, time and labour required for the computation is greatly reduced, since multiplication and division with a 20-figure calculating machine can then be effected by ordinary and simple operation.

Exampe 5. For example, let us compute $\log_e \pi$ to ten places of effective figures, in which $\pi=3.141\,592\,654$.

Referring to the e^n-Table, we have

$$3.141\,592\,654=\exp\left(1+\frac{1}{7}+\frac{1}{512}-\frac{1}{12\,288}\right)\times 1.000000998,$$

so that we at once have

$$\varepsilon=0.000\,000\,998.$$

Thus we obtain

$$\log_e \pi=\left(1+\frac{1}{7}+\frac{1}{512}-\frac{1}{12\,288}\right)+\varepsilon=1.144\,729\,886,$$

which is correct to the last place of decimals, the true value being 1.144 729 885 849 ⋯.

It will also be a very easy work to compute e^{ν} or N^{ν}, ν and N being given, by the present e^n-Table, when only nine or ten places of effective figures are desired.

e^n-Table

n	n	e^n
2	2.000	7.389 056 098 930 650 650
$^3/_2$	1.500	4.481 689 070 338 065 015
1	1.000	2.718 281 828 459 045 313
$^3/_4$	0.750	2.117 000 016 612 674 714
$^1/_2$	0.500	1.648 721 270 700 128 170
$^3/_8$	0.375	1.454 991 414 618 201 352
$^1/_3$	$0.33\dot{3}$	1.395 612 425 086 089 542
$^1/_4$	0.250	1.284 025 416 687 741 493
$^1/_5$	0.200	1.221 402 758 160 169 841
$^3/_{16}$	0.187 500	1.206 230 249 420 980 717
$^1/_6$	$0.16\dot{6}$	1.181 360 412 865 645 986
$^1/_7$	$0.\dot{1}42\,85\dot{7}$	1.153 564 994 895 107 758
$^1/_8$	0.125	1.133 148 453 066 826 321
$^1/_9$	$0.11\dot{1}$	1.117 519 068 741 863 653
$^1/_{10}$	0.100	1.105 170 918 075 647 628
$^3/_{32}$	0.093 750	1.098 285 140 307 825 852
$^1/_{11}$	$0.0\dot{9}\dot{0}$	1.095 169 439 874 664 288
$^1/_{12}$	$0.083\,33\dot{3}$	1.086 904 049 521 228 891
$^1/_{14}$	$0.071\,\dot{4}28\,57\dot{1}$	1.074 041 430 716 295 859
$^1/_{16}$	0.062 500	1.064 494 458 917 859 432
$^1/_{20}$	0.050	1.051 271 096 376 024 041
$^3/_{64}$	0.046 875	1.047 991 002 016 632 704
$^1/_{22}$	$0.0\dot{4}\dot{5}$	1.046 503 435 194 870 389
$^1/_{24}$	$0.041\,66\dot{6}$	1.042 546 905 189 991 387
$^1/_{28}$	$0.035\,\dot{7}14\,28\dot{5}$	1.036 359 701 414 666 094
$^1/_{32}$	0.031 250	1.031 743 407 499 102 672
$^1/_{40}$	0.025	1.025 315 120 524 428 841
$^3/_{128}$	0.023 437 500	1.023 714 316 602 357 918
$^1/_{44}$	$0.022\,7\dot{2}\dot{7}$	1.022 987 504 906 521 519
$^1/_{48}$	$0.020\,83\dot{3}$	1.021 051 862 145 107 391
$^1/_{56}$	$0.017\,\dot{8}57\,14\dot{2}$	1.018 017 534 924 947 201
$^1/_{64}$	0.015 625	1.015 747 708 586 685 748
$^1/_{80}$	0.012 500	1.012 578 451 540 634 377
$^3/_{256}$	0.011 718 750	1.011 787 683 559 331 492
$^1/_{88}$	$0.011\,3\dot{6}\dot{3}$	1.011 428 447 744 338 225
$^1/_{96}$	$0.010\,41\dot{6}$	1.010 471 109 010 597 784

n	n	e^n
$^1/_{112}$	0.008 928 $\dot{5}$71 42$\dot{8}$	1.008 968 550 017 763 042
$^1/_{128}$	0.007 812 500	1.007 843 097 206 447 978
$^1/_{160}$	0.006 250	1.006 269 572 003 762 010
$^3/_{512}$	0.005 859 375	1.005 876 574 714 478 323
$^1/_{176}$	0.005 6$\dot{8}\dot{1}$	1.005 697 990 325 295 532
$^1/_{192}$	0.005 208 33$\dot{3}$	1.005 221 920 279 595 666
$^1/_{224}$	0.004 464 $\dot{2}$85 71$\dot{4}$	1.004 474 265 483 075 003
$^1/_{256}$	0.003 906 250	1.003 913 889 338 347 574
$^1/_{320}$	0.003 125	1.003 129 887 902 739 149
$^3/_{1\,024}$	0.002 929 687 500	1.002 933 983 228 446 758
$^1/_{352}$	0.002 840 9$\dot{0}\dot{9}$	1.002 844 948 297 240 779
$^1/_{384}$	0.002 604 16$\dot{6}$	1.002 607 560 454 037 104
$^1/_{448}$	0.002 232 $\dot{1}$42 85$\dot{7}$	1.002 234 635 942 639 434
$^1/_{512}$	0.001 953 125	1.001 955 033 591 002 812
$^1/_{640}$	0.001 562 500	1.001 563 721 339 156 308
$^3/_{2\,048}$	0.001 464 843 750	1.001 465 917 157 666 808
$^1/_{704}$	0.001 420 4$\dot{5}\dot{4}$	1.001 421 463 868 855 159
$^1/_{768}$	0.001 302 08$\dot{3}$	1.001 302 931 411 886 512
$^1/_{896}$	0.001 116 071 $\dot{4}$28 57$\dot{1}$	1.001 116 694 468 052 228
$^1/_{1\,024}$	0.000 976 562 500	1.000 977 039 492 416 535
$^1/_{1\,280}$	0.000 781 250	1.000 781 555 255 269 634
$^3/_{4\,096}$	0.000 732 421 875	1.000 732 690 161 397 100
$^1/_{1\,408}$	0.000 710 2$\dot{2}\dot{7}$	1.000 710 479 543 836 475
$^1/_{1\,536}$	0.000 651 041 66$\dot{6}$	1.000 651 253 640 291 260
$^1/_{1\,792}$	0.000 558 035 $\dot{7}$14 28$\dot{5}$	1.000 558 191 445 181 377
$^1/_{2\,048}$	0.000 488 281 250	1.000 488 400 478 694 473
$^1/_{2\,560}$	0.000 390 625	1.000 390 701 303 880 390
$^3/_{8\,192}$	0.000 366 210 937 500	1.000 366 278 000 911 574
$^1/_{2\,816}$	0.000 355 113 6$\dot{3}\dot{6}$	1.000 355 176 696 675 307
$^1/_{3\,072}$	0.000 325 520 83$\dot{3}$	1.000 325 573 820 989 173
$^1/_{3\,584}$	0.000 279 017 $\dot{8}$57 14$\dot{2}$	1.000 279 056 786 245 714
$^1/_{4\,096}$	0.000 244 140 625	1.000 244 170 429 747 855
$^1/_{5\,120}$	0.000 195 312 500	1.000 195 331 574 728 152
$^3/_{16\,384}$	0.000 183 105 468 750	1.000 183 122 233 579 571
$^1/_{5\,632}$	0.000 177 556 8$\dot{1}\dot{8}$	1.000 177 572 582 326 657
$^1/_{6\,144}$	0.000 162 760 41$\dot{6}$	1.000 162 773 662 861 926
$^1/_{7\,168}$	0.000 139 508 928 $\dot{5}$71 42$\dot{8}$	1.000 139 518 660 394 558

n	n	e^n
$^1/_{8\,192}$	0. 000 122 070 312 500	1. 000 122 077 763 383 771
$^1/_{10\,240}$	0. 000 097 656 250	1. 000 097 661 018 526 806
$^3/_{32\,768}$	0. 000 091 552 734 375	1. 000 091 556 925 454 486
$^1/_{11\,264}$	$0.\,000\,088\,778\,4\dot{0}\dot{9}$	1. 000 088 782 350 010 492
$^1/_{12\,288}$	$0.\,000\,081\,380\,208\,33\dot{3}$	1. 000 081 383 519 792 316
$^1/_{14\,336}$	$0.\,000\,069\,754\,464\,\dot{2}85\,71\dot{4}$	1. 000 069 756 897 184 927
$^1/_{16\,384}$	0. 000 061 035 156 250	1. 000 061 037 018 933 045
$^1/_{20\,480}$	0. 000 048 828 125	1. 000 048 829 317 112 298
$^3/_{65\,536}$	0. 000 045 776 367 187 500	1. 000 045 777 414 941 384
$^1/_{22\,528}$	$0.\,000\,044\,389\,204\,5\dot{4}\dot{5}$	1. 000 044 390 189 760 773
$^1/_{24\,576}$	$0.\,000\,040\,690\,104\,16\dot{6}$	1. 000 040 690 932 020 184
$^1/_{28\,672}$	0. 000 034 877 232 142 857	1. 000 034 877 840 360 590
$^1/_{32\,768}$	0. 000 030 517 578 125	1. 000 030 518 043 791 024
$^1/_{40\,960}$	0. 000 024 414 062 500	1. 000 024 414 360 525 649
$^3/_{131\,072}$	0. 000 022 888 183 593 750	1. 000 022 888 445 530 223
$^1/_{45\,056}$	$0.\,000\,022\,194\,602\,2\dot{7}\dot{2}$	1. 000 022 194 848 574 735
$^1/_{49\,152}$	$0.\,000\,020\,345\,052\,08\dot{3}$	1. 000 020 345 259 045 309
$^1/_{57\,344}$	0. 000 017 438 616 071 429	1. 000 017 438 768 124 978

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