The Solution of the Generalized Boussinesq's Problem for Elastic Foundation. Part I

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Synopsis. This paper gives the solution of the generalized Boussinesq's problem for elastic foundation, in which any distribution of two kinds of shearing forces as well as of normal pressure is given on the bounding surface of the semi-infinite elastic soild. The procedure of solving the problem is due to my proposed one. The solution consists of double Fourier integral representation. Several examples at once follow from the present solution, and their evaluation was relied on the method of numerical integration proposed by me.

CONTENTS

§ 1. INTRODUCTORY

Since the time of Boussinesq, the theoretical basis for the problem of the safety of elastic foundation has been discussed and developed by various investigators, of which the works due to Prof. Terazawa[1)] and Love[2)] will especially be noteworthy. The former discussed it in detail when the loaded area is of circular form, and the latter gave the integration of Boussinesq's

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potentials when a uniform pressure extends over a rectangular area. But the Boussinesq's potential methed, as is well known, cannot at all be compatible with shearing forces, which would be of considerable importance especially in the case of soft foundation. In addition, the Boussinesq's potentials are difficult to perform their integrations, and only the simplest case cited has been treated by Love. The numerical process of their integration is also almost impossible by ordinary methods of numerical integration, since integrands involved have an infinite number of singular points.

The boundary-value problem here treated is that, within a rectangular form of loaded area, any distributions of two kinds of shearing forces as well as of normal pressure are given on the surface of the semi-infinite elastic solid, provided these three kinds of external forces are expressible in terms of Fourier's integral in two dimensions.

The procedure of the calculation is for convenience due to a new set of functions which has been proposed by me and might be called stress-functions in three dimensions.[3] The resulting solution is obtained in the forms of Fourier's integral, and the evaluation of the integrals was relied on the method of mechanical cubature[4),5)] because of the difficulty in its analytical performance. As regards stresses there is no singularity in the integrands, and to secure first two or three significant figures in the numerical result is not so laborious. Displacements can also be integrated, despite that each of integrands involved has one singularity at the origin of parametric coordinates.

Applications of the general solution will at once follow from the general solution, and the following cases are written out : (1) Uniform pressure (2) Uniformly varying pressure (3) Uniformly shearing force (4) Uniformly varying shear and (5) Quadratically varying pressure.

§ 2. BOUNDARY CONDITIONS

We take the semi-infinite elastic solid, and any distributions of three kinds of external forces, consisting of one kind of distributed pressure and two kinds of shearing forces, are given on the bounding plane of the solid within some prescribed area.

We take the origin of rectangular coordinates to be a point on the bounding surface of the semi-infinite solid, the plane $z=0$ to be the surface of the body, and the positive direction of the axis of z to be that which goes into the interior of the body. (Cf. Fig. 1.)

The boundary conditions with which we are now to deal are then:

1) $\left(\widehat{zz}\right)_{z=0}=F_1(x,y),$ (1)

2) $\left(\widehat{yz}\right)_{z=0}=F_2(x,y),$ and (2)

3) $\left(\widehat{zx}\right)_{z=0}=F_3(x,y)$ (3)

within some prescribed area, and these three functions must vanish outside the area.

4) All the stress-components, and also displacement-components, vanish when x, y, z become indefinitely great.

Here the functions $F_1(x,y)$, $F_2(x,y)$ and $F_3(x,y)$ are any functions which will be expansible in terms of Fourier's integral in two dimensions, and accordingly all of them vanish for every value of both x and y outside the given domain.

Now Fourier's integral theorem in two dimensions is in general written

$$f(x,y)=\frac{1}{\pi}\int_0^\infty d\alpha\int_{-\infty}^\infty f(\xi,y)\cos\alpha(x-\xi)\,d\xi$$
$$=\frac{1}{\pi^2}\int_0^\infty d\alpha\int_0^\infty d\beta\int_{-\infty}^\infty\int_{-\infty}^\infty f(\xi,\eta)\cos\alpha(x-\xi)\ \cos\beta(y-\eta)\,d\xi\,d\eta.$$

In virtue of this, the boundary conditions (1), (2) and (3) can be written down

$$\left(\widehat{zz}\right)_{z=0}=\frac{1}{\pi^2}\int_0^\infty d\alpha\int_0^\infty d\beta\int_{-\infty}^\infty\int_{-\infty}^\infty F_1(\xi,\eta)\cos\alpha(x-\xi)\cos\beta(y-\eta)\,d\xi\,d\eta$$
$$=\frac{1}{\pi^2}\int_0^\infty d\alpha\int_0^\infty d\beta\int_{-\infty}^\infty\int_{-\infty}^\infty F_1(\xi,\eta)$$
$$\times(\cos\alpha x\cos\beta y\cos\alpha\xi\cos\beta\eta+\cos\alpha x\sin\beta y\cos\alpha\xi\ \sin\beta\eta$$
$$+\sin\alpha x\cos\beta y\sin\alpha\xi\cos\beta\eta+\sin\alpha x\ \sin\beta y\ \sin\alpha\xi\ \sin\beta\eta)\,d\xi\,d\eta. \tag{4}$$

$$\left(\widehat{yz}\right)_{z=0}=\frac{1}{\pi^2}\int_0^\infty d\alpha\int_0^\infty d\beta\int_{-\infty}^\infty\int_{-\infty}^\infty F_2(\xi,\eta)\cos\alpha(x-\xi)\cos\beta(y-\eta)\,d\xi\,d\eta$$
$$=\frac{1}{\pi^2}\int_0^\infty d\alpha\int_0^\infty d\beta\int_{-\infty}^\infty\int_{-\infty}^\infty F_2(\xi,\eta)$$
$$\times(\cos\alpha x\cos\beta y\cos\alpha\xi\cos\beta\eta+\cos\alpha x\sin\beta y\cos\alpha\xi\sin\beta\eta$$
$$+\sin\alpha x\cos\beta y\sin\alpha\xi\cos\beta\eta+\sin\alpha x\sin\beta y\sin\alpha\xi\sin\beta\eta)\,d\xi\,d\eta. \tag{5}$$

$$\left(\widehat{zx}\right)_{z=0}=\frac{1}{\pi^2}\int_0^\infty d\alpha\int_0^\infty d\beta\int_{-\infty}^\infty\int_{-\infty}^\infty F_3(\xi,\eta)\cos\alpha(x-\xi)\cos\beta(y-\eta)\,d\xi\,d\eta$$
$$=\frac{1}{\pi^2}\int_0^\infty d\alpha\int_0^\infty d\beta\int_{-\infty}^\infty\int_{-\infty}^\infty F_3(\xi,\eta)$$

$$\times(\cos\alpha x\cos\beta y\cos\alpha\xi\cos\beta\eta+\cos\alpha x\sin\beta y\cos\alpha\xi\sin\beta\eta$$
$$+\sin\alpha x\cos\beta y\sin\alpha\xi\cos\beta\eta+\sin\alpha x\sin\beta y\sin\alpha\xi\sin\beta\eta)\,d\xi\,d\eta. \tag{6}$$

The solution to be obtained in perfect compatibility with the above boundary conditions (1)—(3) would not necessarily be restricted to a certain particular form of domain on which external forces are applied, for no restriction is imposed upon the domain of integration in the Fourier's integrals (4)—(6) above. But on applications of the solution to individual boundary-value problems our attention will for the present be confined to the rectangular domain, owing to the simplicity of performing definite integrals extending over the domain.

§ 3. PROPOSED STRESS-FUNCTIONS AND TYPICAL SOLUTIONS SUITABLE FOR THE PROBLEM

The proposed procedure for the three-dimensional elasticity is [3]

$$\left.\begin{aligned}
u&=\frac{1}{2\mu}\left\{-\frac{\partial}{\partial x}\nabla^2+(1-\sigma)\left(\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\nabla^2\right\}\chi+\frac{1}{2\mu}\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial z}\right)\phi, \quad \cdots\cdots,\\
\widehat{xx}&=\left\{\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\nabla^2-(1-\sigma)\frac{\partial^2}{\partial y\partial z}\nabla^2\right\}\chi+\left(\frac{\partial^2}{\partial x\partial y}-\frac{\partial^2}{\partial z\partial x}\right)\phi, \quad \cdots\cdots,\\
\widehat{yz}&=\left\{-\frac{\partial^2}{\partial y\partial z}\nabla^2+\frac{1-\sigma}{2}\frac{\partial}{\partial x}\left(-\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\nabla^2\right\}\chi\\
&\quad+\frac{1}{2}\left(\frac{\partial^2}{\partial x\partial y}-\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial z\partial x}\right)\phi, \quad \cdots\cdots,
\end{aligned}\right\}$$

where χ and ϕ satisfy respectively the equations

$$\nabla^4\chi=0 \qquad \text{and} \qquad \nabla^2\phi=0,$$

and $$\nabla^4=\nabla^2\nabla^2, \qquad \nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}, \qquad \boldsymbol{\nabla}^2=\frac{\partial^2}{\partial y\partial z}+\frac{\partial^2}{\partial z\partial x}+\frac{\partial^2}{\partial x\partial y},$$

μ being the modulus of rigidity and σ Poisson's ratio for the material. Other components of displacement and stress are given by cyclical interchange of x, y, z.

It can easily be verified by substitution that the above system of equations satisfy the stress-equations of the type

$$\frac{\partial\widehat{xx}}{\partial x}+\frac{\partial\widehat{xy}}{\partial y}+\frac{\partial\widehat{zx}}{\partial z}=0,$$

and the stress-strain relations of the types

$$\frac{\partial u}{\partial x}=\frac{1}{E}\left\{\widehat{xx}-\sigma(\widehat{yy}+\widehat{zz})\right\},\qquad \frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}=\frac{2(1+\sigma)}{E}\widehat{yz}.$$

A detailed description of deriving the proposed procedure above has appeared elsewhere.[1)]

The χ function is composed of harmonics and biharmonics proper. For the former functions, the above operations for displacement and stress-components become

$$\left.\begin{aligned}
u &= -\frac{1}{2\mu}\frac{\partial}{\partial x}\nabla^2\chi, \quad \cdots\cdots,\\
\widehat{xx} &= \left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\nabla^2\chi = -\frac{\partial^2}{\partial x^2}\nabla^2\chi, \quad \cdots\cdots,\\
\widehat{yz} &= -\frac{\partial^2}{\partial y\partial z}\nabla^2\chi, \cdots\cdots,
\end{aligned}\right\}$$

since in this case $\nabla^2\chi=0$. In these operations $\nabla^2\chi$ appears as a common factor, which is still harmonic. Then $-\nabla^2\chi$ may be replaced by a new harmonic function, ϕ say, so that the above operations may be written in the simple ones

$$u=\frac{1}{2\mu}\frac{\partial\phi}{\partial x}, \quad \cdots\cdots, \quad \widehat{xx}=\frac{\partial^2\phi}{\partial x^2}, \quad \cdots\cdots, \quad \widehat{yz}=\frac{\partial^2\phi}{\partial y\partial z}, \quad \cdots\cdots.$$

Thus the proposed procedures above may take the forms

$$\left.\begin{aligned}
u &= \frac{1}{2\mu}\frac{\partial\phi}{\partial x}+\frac{1}{2\mu}\left\{-\frac{\partial}{\partial x}\nabla^2+(1-\sigma)\left(\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\nabla^2\right\}\chi+\frac{1}{2\mu}\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial z}\right)\psi, \cdots,\\
\widehat{xx} &= \frac{\partial^2\phi}{\partial x^2}+\left\{\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\nabla^2-(1-\sigma)\frac{\partial^2}{\partial y\partial z}\nabla^2\right\}\chi+\left(\frac{\partial^2}{\partial x\partial y}-\frac{\partial^2}{\partial z\partial x}\right)\psi, \cdots\cdots,\\
\widehat{yz} &= \frac{\partial^2\phi}{\partial y\partial z}+\left\{-\frac{\partial^2}{\partial y\partial z}\nabla^2+\frac{1-\sigma}{2}\frac{\partial}{\partial x}\left(-\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\nabla^2\right\}\chi\\
&\quad +\frac{1}{2}\left(\frac{\partial^2}{\partial x\partial y}-\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial z\partial x}\right)\psi, \quad \cdots\cdots,
\end{aligned}\right\} \tag{7}$$

where

$$\nabla^2\phi=0, \qquad \nabla^2\psi=0, \text{ and } \nabla^4\chi=0,$$

χ being biharmonics proper. Thus it can be stated that the ϕ function is for convenience extracted from the χ function. Preference must be given to the procedures (7) for practical calculations.

Typical solutions suitable for the present boundary-value problem (cf.

boundary conditions 1)-3)) are

$$\left.\begin{aligned}\phi &= (A_1 \cos\alpha x \cos\beta y + A_2 \cos\alpha x \sin\beta y + A_3 \sin\alpha x \cos\beta y + A_4 \sin\alpha x \sin\beta y)\, e^{-\gamma z},\\ \psi &= (B_1 \cos\alpha x \cos\beta y + B_2 \cos\alpha x \sin\beta y + B_3 \sin\alpha x \cos\beta y + B_4 \sin\alpha x \sin\beta y)\, e^{-\gamma z},\\ \chi &= (C_1 \cos\alpha x \cos\beta y + C_2 \cos\alpha x \sin\beta y + C_3 \sin\alpha x \cos\beta y + C_4 \sin\alpha x \sin\beta y)\, z e^{-\gamma z},\end{aligned}\right\}\tag{8}$$

where $A_1, A_2, \cdots C_3, C_4$ are constants to be determined; α, β and γ being parameters provided

$$\alpha^2 + \beta^2 = \gamma^2. \tag{9}$$

The coodinate system here referred to is illustratced in Fig. 1.

The ϕ and ψ functions in (8) are entirely of the same form, but they can be independent solutions for boundary conditions, because of the different operations upon them, as is given in (7). Such instances have appeared elsewhere; for example, in the three-dimensional dynamical elasticity, in which, as is well known, two kinds of equivoluminal waves, together with one kind of irrotational wave, may develop in the elastic medium. This basic theory in the dynamical system has been confirmed in my work cited[6] as an extension of the statical system, the ϕ function might be called harmonics of the first kind, and the ψ function harmonics of the second kind, or from the view-point of their derivation the former the general harmonics, and the latter the singular harmonics.

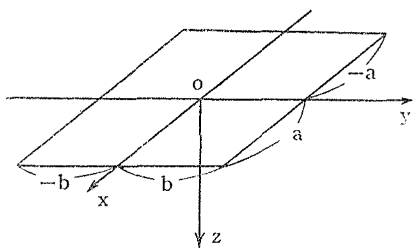

Fig. 1. Coordinate system

It is added also that in general the functions

$$\chi = x\omega,\quad y\omega,\quad z\omega,\quad (x^2 + y^2 + z^2)\omega$$

are all biharmonics, provided $\omega(x, y, z)$ is any harmonics; the third form having been adopted in (8).

§ 4. EXPRESSIONS FOR STRESS-COMPONENTS

The substitution of the typical solutions (8) into the proposed procedures (7) affords the following expressions:

$$\widehat{zz} = \{A_1\gamma^2 + B_2\beta\gamma - B_3\alpha\gamma - C_2\beta\gamma^2 - C_3\alpha\gamma^2 + C_4 2(1-\sigma)\alpha\beta\gamma\} \cos\alpha x \cos\beta y\, e^{-\gamma z}$$

$$+ \{A_2\gamma^2 - B_1\beta\gamma - B_4\alpha\gamma + C_1\beta\gamma^2 - C_3 2(1-\sigma)\alpha\beta\gamma - C_4\alpha\gamma^2\} \cos\alpha x \sin\beta y\, e^{-\gamma z}$$

$$+\{A_3\gamma^2+B_1\alpha\gamma+B_4\beta\gamma+C_1\alpha\gamma^2-C_2 2(1-\sigma)\alpha\beta\gamma-C_4\beta\gamma^2\}\sin\alpha x\cos\beta y\, e^{-\gamma z}$$
$$+\{A_4\gamma^2+B_2\alpha\gamma-B_3\beta\gamma+C_1 2(1-\sigma)\alpha\beta\gamma+C_2\alpha\gamma^2+C_3\beta\gamma^2\}\sin\alpha x\sin\beta y\, e^{-\gamma z}$$
$$+\gamma^2\{P_1\cos\alpha x\cos\beta y+P_2\cos\alpha x\sin\beta y+P_3\sin\alpha x\cos\beta y$$
$$+P_4\sin\alpha x\sin\beta y\}\, z\, e^{-\gamma z}, \qquad (10)$$

$$\widehat{yz}=\Big[-A_2\beta\gamma+\frac{1}{2}\{B_1(\beta^2+\gamma^2)+B_3\alpha\gamma+B_4\alpha\beta\}-C_1\{(1-\sigma)\alpha^2+2\beta^2\}\gamma$$
$$+C_3\{\beta^2+(1-\sigma)\gamma^2\}\alpha+C_4(1+\sigma)\alpha\beta\gamma\Big]\cos\alpha x\cos\beta y\, e^{-\gamma z}$$
$$+\Big[A_1\beta\gamma+\frac{1}{2}\Big\{B_2(\beta^2+\gamma^2)-B_3\alpha\beta+B_4\alpha\gamma\}-C_2\{(1-\sigma)\alpha^2+2\beta^2\}\gamma$$
$$-C_3(1+\sigma)\alpha\beta\gamma+C_4\{\beta^2+(1-\sigma)\gamma^2\}\alpha\Big]\cos\alpha x\sin\beta y\, e^{-\gamma z}$$
$$+\Big[-A_4\beta\gamma+\frac{1}{2}\{-B_1\alpha\gamma-B_2\alpha\beta+B_3(\beta^2+\gamma^2)\}-C_1\{\beta^2+(1-\sigma)\gamma^2\}\alpha$$
$$-C_2(1+\sigma)\alpha\beta\gamma-C_3\{(1-\sigma)\alpha^2+2\beta^2\}\gamma\Big]\sin\alpha x\cos\beta y\, e^{-\gamma z}$$
$$+\Big[A_3\beta\gamma+\frac{1}{2}\Big\{B_1\alpha\beta-B_2\alpha\gamma+B_4(\beta^2+\gamma^2)\}+C_1(1+\sigma)\alpha\beta\gamma$$
$$-C_2\{\beta^2+(1-\sigma)\gamma^2\}\alpha-C_4\{(1-\sigma)\alpha^2+2\beta^2\}\gamma\Big]\sin\alpha x\sin\beta y\, e^{-\gamma z}$$
$$+\beta\gamma[-P_2\cos\alpha x\cos\beta y+P_1\cos\alpha x\sin\beta y-P_4\sin\alpha x\cos\beta y$$
$$+P_3\sin\alpha x\sin\beta y]\, z\, e^{-\gamma z}, \qquad (11)$$

$$\widehat{zx}=\Big[-A_3\alpha\gamma+\frac{1}{2}\{-B_1(\alpha^2+\gamma^2)-B_2\beta\gamma-B_4\alpha\beta\}-C_1\{2\alpha^2+(1-\sigma)\beta^2\}\gamma$$
$$+C_2\{\alpha^2+(1-\sigma)\gamma^2\}\beta+C_4(1+\sigma)\alpha\beta\gamma\Big]\cos\alpha x\cos\beta y\, e^{-\gamma z}$$
$$+\Big[-A_4\alpha\gamma+\frac{1}{2}\{B_1\beta\gamma-B_2(\alpha^2+\gamma^2)+B_3\alpha\beta\}-C_1\{\alpha^2+(1-\sigma)\gamma^2\}\beta$$
$$-C_2\{2\alpha^2+(1-\sigma)\beta^2\}\gamma-C_3(1+\sigma)\alpha\beta\gamma\Big]\cos\alpha x\sin\beta y\, e^{-\gamma z}$$
$$+\Big[A_1\alpha\gamma+\frac{1}{2}\{B_2\alpha\beta-B_3(\alpha^2+\gamma^2)-B_4\beta\gamma\}-C_2(1+\sigma)\alpha\beta\gamma$$
$$-C_3\{2\alpha^2+(1-\sigma)\beta^2\}\gamma+C_4\{\alpha^2+(1-\sigma)\gamma^2\}\beta\Big]\sin\alpha x\cos\beta y\, e^{-\gamma z}$$
$$+\Big[A_2\alpha\gamma+\frac{1}{2}\{-B_1\alpha\beta+B_3\beta\gamma-B_4(\alpha^2+\gamma^2)\}+C_1(1+\sigma)\alpha\beta\gamma$$
$$-C_3\{\alpha^2+(1-\sigma)\gamma^2\}\beta-C_4\{2\alpha^2+(1-\sigma)\beta^2\}\gamma\Big]\sin\alpha x\sin\beta y\, e^{-\gamma z}$$
$$+\alpha\gamma[-P_3\cos\alpha x\cos\beta y-P_4\cos\alpha x\sin\beta y+P_1\sin\alpha x\cos\beta y$$
$$+P_2\sin\alpha x\sin\beta y]\, z\, e^{-\gamma z}, \qquad (12)$$

$$\widehat{xx}=[-A_1\alpha^2+B_3\alpha\gamma+B_4\alpha\beta+C_2(\alpha^2+2\sigma\gamma^2)\beta+C_3(\alpha^2+2\gamma^2)\alpha$$
$$-C_4 2\alpha\beta\gamma]\cos\alpha x\cos\beta y\, e^{-\gamma z}$$

$$+\left[-A_2\alpha^2-B_3\alpha\beta+B_4\alpha\gamma-C_1(\alpha^2+2\sigma\gamma^2)\beta+C_3 2\alpha\beta\gamma+C_4(\alpha^2+2\gamma^2)\alpha\right]$$
$$\times\cos\alpha x\sin\beta y\, e^{-\gamma z}$$
$$+\left[-A_3\alpha^2-B_1\alpha\gamma-B_2\alpha\beta-C_1(\alpha^2+2\gamma^2)\alpha+C_2 2\alpha\beta\gamma+C_4(\alpha^2+2\sigma\gamma^2)\beta\right]$$
$$\times\sin\alpha x\cos\beta y\, e^{-\gamma z}$$
$$+\left[-A_4\alpha^2+B_1\alpha\beta-B_2\alpha\gamma-C_1 2\alpha\beta\gamma-C_2(\alpha^2+2\gamma^2)\alpha-C_3(\alpha^2+2\sigma\gamma^2)\beta\right]$$
$$\times\sin\alpha x\sin\beta y\, e^{-\gamma z}$$
$$-\alpha^2\left[P_1\cos\alpha x\cos\beta y+P_2\cos\alpha x\sin\beta y+P_3\sin\alpha x\cos\beta y\right.$$
$$\left.+P_4\sin\alpha x\sin\beta y\right]z\, e^{-\gamma z}, \qquad (13)$$

$$\widehat{yy}=\left[-A_1\beta^2-B_2\beta\gamma-B_4\alpha\beta+C_2(\beta^2+2\gamma^2)\beta+C_3(\beta^2+2\sigma\gamma^2)\alpha\right.$$
$$\left.-C_4 2\alpha\beta\gamma\right]\cos\alpha x\cos\beta y\, e^{-\gamma z}$$
$$+\left[-A_2\beta^2+B_1\beta\gamma+B_3\alpha\beta-C_1(\beta^2+2\gamma^2)\beta+C_3 2\alpha\beta\gamma+C_4(\beta^2+2\sigma\gamma^2)\alpha\right]$$
$$\times\cos\alpha x\sin\beta y\, e^{-\gamma z}$$
$$+\left[-A_3\beta^2+B_2\alpha\beta-B_4\beta\gamma-C_1(\beta^2+2\sigma\gamma^2)\alpha+C_2 2\alpha\beta\gamma+C_4(\beta^2+2\gamma^2)\beta\right]$$
$$\times\sin\alpha x\cos\beta y\, e^{-\gamma z}$$
$$+\left[-A_4\beta^2-B_1\alpha\beta+B_3\beta\gamma-C_1 2\alpha\beta\gamma-C_2(\beta^2+2\sigma\gamma^2)\alpha-C_3(\beta^2+2\gamma^2)\beta\right]$$
$$\times\sin\alpha x\sin\beta y\, e^{-\gamma z}$$
$$-\beta^2\left[P_1\cos\alpha x\cos\beta y+P_2\cos\alpha x\sin\beta y+P_3\sin\alpha x\cos\beta y\right.$$
$$\left.+P_4\sin\alpha x\sin\beta y\right]z\, e^{-\gamma z}, \qquad (14)$$

$$\widehat{xy}=\left[A_4\alpha\beta+\frac{1}{2}\{B_1(\alpha^2-\beta^2)+B_2\beta\gamma-B_3\alpha\gamma\}+C_1(1-\sigma)\gamma^3\right.$$
$$\left.+C_2\{\alpha^2+(1-\sigma)\gamma^2\}\beta+C_3\{\beta^2+(1-\sigma)\gamma^2\}\alpha\right]\cos\alpha x\cos\beta y\, e^{-\gamma z}$$
$$+\left[-A_3\alpha\beta+\frac{1}{2}\{-B_1\beta\gamma+B_2(\alpha^2-\beta^2)-B_4\alpha\gamma\}-C_1\{\alpha^2+(1-\sigma)\gamma^2\}\beta\right.$$
$$\left.+C_2(1-\sigma)\gamma^3+C_4\{\beta^2+(1-\sigma)\gamma^2\}\alpha\right]\cos\alpha x\sin\beta y\, e^{-\gamma z}$$
$$+\left[-A_2\alpha\beta+\frac{1}{2}\{B_1\alpha\gamma+B_3(\alpha^2-\beta^2)+B_4\beta\gamma\}-C_1\{\beta^2+(1-\sigma)\gamma^2\}\alpha\right.$$
$$\left.+C_3(1-\sigma)\gamma^3+C_4\{\alpha^2+(1-\sigma)\gamma^2\}\beta\right]\sin\alpha x\cos\beta y\, e^{-\gamma z}$$
$$+\left[A_1\alpha\beta+\frac{1}{2}\{B_2\alpha\gamma-B_3\beta\gamma+B_4(\alpha^2-\beta^2)\}-C_2\{\beta^2+(1-\sigma)\gamma^2\}\alpha\right.$$
$$\left.-C_3\{\alpha^2+(1-\alpha)\gamma^2\}\beta+C_4(1-\sigma)\gamma^3\right]\sin\alpha x\sin\beta y\, e^{-\gamma z}$$
$$+\alpha\beta\left[P_4\cos\alpha x\cos\beta y-P_3\cos\alpha x\sin\beta y-P_2\sin\alpha x\cos\beta y\right.$$
$$\left.+P_1\sin\alpha x\sin\beta y\right]z\, e^{-\gamma z}, \qquad (15)$$

where and in what follows

$$\left.\begin{aligned}P_1&=C_2\beta\gamma+C_3\alpha\gamma-C_4\alpha\beta,\\P_2&=-C_1\beta\gamma+C_3\alpha\beta+C_4\alpha\gamma,\\P_3&=-C_1\alpha\gamma+C_2\alpha\beta+C_4\beta\gamma,\\P_4&=-C_1\alpha\beta-C_2\alpha\gamma-C_3\beta\gamma.\end{aligned}\right\} \qquad (16)$$

In the expressions (10)—(15) above, σ denotes, as before, Poisson's ratio of the elastic solid, and the constants A_1, $A_2, \cdots C_3$, C_4 will become determinate, after taking the boundary conditions (4), (5) and (6) into consideration.

It is to be noted that equations (16) can be written in the form

$$[P_1,\ P_2,\ P_3,\ P_4] = \begin{pmatrix} 0 & \beta\gamma & \gamma\alpha & -\alpha\beta \\ -\beta\gamma & 0 & \alpha\beta & \gamma\alpha \\ -\gamma\alpha & \alpha\beta & 0 & \beta\gamma \\ -\alpha\beta & -\gamma\alpha & -\beta\gamma & 0 \end{pmatrix} [C_1,\ C_2,\ C_3,\ C_4],$$

and that the square matrix constructed by the letters α, β and γ is a skew-symmetric one. This set of equations has to be solved simultaneously, as will be seen later on.

§ 5. CONSTRUCTION OF SIMULTANEOUS EQUATIONS

We shall obtain simultaneous equations for twelve unknowns A_1, A_2, $\cdots$ C_3, C_4. For instance we take the first term in equation (4), the first of the boundary conditions; viz.

(First term in equation (4))

$$= \frac{1}{\pi^2}\int_0^\infty d\alpha \int_0^\infty d\beta \int_{-\infty}^\infty \int_{-\infty}^\infty F_1(\xi, \eta) \cos\alpha x \cos\beta y \cos\alpha\xi \cos\beta\eta \, d\xi \, d\eta. \tag{17}$$

The corresponding term in the previous stress-components must be the first one in (10). This affords on the top surface $z = 0$ the following:

(First term in equation (10))

$$= \int_0^\infty \int_0^\infty \{A_1\gamma^2 + B_2\beta\gamma - B_3\alpha\gamma - C_2\beta\gamma^2 - C_3\alpha\gamma^2 + C_4 2(1-\sigma)\alpha\beta\gamma\} \cos\alpha x \cos\beta y \, d\alpha \, d\beta. \tag{18}$$

On comparing this with the preceding equation (17), we must have

$$\frac{1}{\pi^2}\int_{-\infty}^\infty \int_{-\infty}^\infty F_1(\xi, \eta) \cos\alpha\xi \cos\beta\eta \, d\xi \, d\eta = A_1\gamma^2 + B_2\beta\gamma - B_3\alpha\gamma - C_2\beta\gamma^2 - C_3\alpha\gamma^2 + C_4 2(1-\sigma)\alpha\beta\gamma.$$

If for shortness we write

$$K_1 = \frac{1}{\pi^2}\int_{-\infty}^\infty \int_{-\infty}^\infty F_1(\xi, \eta) \cos\alpha\xi \cos\beta\eta \, d\xi \, d\eta, \tag{19}$$

then the equation just written takes the form

$$A_1\gamma + B_2\beta - B_3\alpha - C_2\beta\gamma - C_3\alpha\gamma + C_4 2(1-\sigma)\alpha\beta = \frac{K_1}{\gamma}. \tag{20}$$

This is the first equation for determining the unknowns A_1, A_2, $\cdots$ C_3 C_4. Similar consideration on the remaining terms in $(\widehat{zz})_{z=0}$ and those in $(\widehat{yz})_{z=0}$ and $(\widehat{zx})_{z=0}$ will at once give the following equations (cf. (10), (11) and (12)):

$$A_2\gamma - B_1\beta - B_4\alpha + C_1\beta\gamma - C_3 2(1-\sigma)\alpha\beta - C_4\alpha\gamma = \frac{K_2}{\gamma}, \tag{21}$$

$$A_3\gamma + B_1\alpha + B_4\beta + C_1\alpha\gamma - C_2 2(1-\sigma)\alpha\beta - C_4\beta\gamma = \frac{K_3}{\gamma}, \tag{22}$$

$$A_4\gamma + B_2\alpha - B_3\beta + C_1 2(1-\sigma)\alpha\beta + C_2\alpha\gamma + C_3\beta\gamma = \frac{K_4}{\gamma}, \tag{23}$$

$$-A_2\beta\gamma + \frac{1}{2}\{B_1(\beta^2+\gamma^2) + B_3\alpha\gamma + B_4\alpha\beta\} - C_1\{(1-\sigma)\alpha^2 + 2\beta^2\}\gamma \\ + C_3\{\beta^2 + (1-\sigma)\gamma^2\}\alpha + C_4(1+\sigma)\alpha\beta\gamma = K_5, \tag{24}$$

$$A_1\beta\gamma + \frac{1}{2}\{B_2(\beta^2+\gamma^2) - B_3\alpha\beta + B_4\alpha\gamma\} - C_2\{(1-\sigma)\alpha^2 + 2\beta^2\}\gamma \\ - C_3(1+\sigma)\alpha\beta\gamma + C_4\{\beta^2 + (1-\sigma)\gamma^2\}\alpha = K_6, \tag{25}$$

$$-A_4\beta\gamma + \frac{1}{2}\{-B_1\alpha\gamma - B_2\alpha\beta + B_3(\beta^2+\gamma^2)\} - C_1\{\beta^2 + (1-\sigma)\gamma^2\}\alpha \\ - C_2(1+\sigma)\alpha\beta\gamma - C_3\{(1-\sigma)\alpha^2 + 2\beta^2\}\gamma = K_7, \tag{26}$$

$$A_3\beta\gamma + \frac{1}{2}\{B_1\alpha\beta - B_2\alpha\gamma + B_4(\beta^2+\gamma^2)\} + C_1(1+\sigma)\alpha\beta\gamma \\ - C_2\{\beta^2 + (1-\sigma)\gamma^2\}\alpha - C_4\{(1-\sigma)\alpha^2 + 2\beta^2\}\gamma = K_8, \tag{27}$$

$$-A_3\alpha\gamma + \frac{1}{2}\{-B_1(\alpha^2+\gamma^2) - B_2\beta\gamma - B_4\alpha\beta\} - C_1\{2\alpha^2 + (1-\sigma)\beta^2\}\gamma \\ + C_2\{\alpha^2 + (1-\sigma)\gamma^2\}\beta + C_4(1+\sigma)\alpha\beta\gamma = K_9, \tag{28}$$

$$-A_4\alpha\gamma + \frac{1}{2}\{B_1\beta\gamma - B_2(\alpha^2+\gamma^2) + B_3\alpha\beta\} - C_1\{\alpha^2 + (1-\sigma)\gamma^2\}\beta \\ - C_2\{2\alpha^2 + (1-\sigma)\beta^2\}\gamma - C_3(1+\sigma)\alpha\beta\gamma = K_{10}, \tag{29}$$

$$A_1\alpha\gamma + \frac{1}{2}\{B_2\alpha\beta - B_3(\alpha^2+\gamma^2) - B_4\beta\gamma\} - C_2(1+\sigma)\alpha\beta\gamma \\ - C_3\{2\alpha^2 + (1-\sigma)\beta^2\}\gamma + C_4\{\alpha^2 + (1-\sigma)\gamma^2\}\beta = K_{11}, \tag{30}$$

$$A_2\alpha\gamma + \frac{1}{2}\{-B_1\alpha\beta + B_3\beta\gamma - B_4(\alpha^2+\gamma^2)\} + C_1(1+\sigma)\alpha\beta\gamma \\ - C_3\{\alpha^2 + (1-\sigma)\gamma^2\}\beta - C_4\{2\alpha^2 + (1-\sigma)\beta^2\}\gamma = K_{12}, \tag{31}$$

where

$$K_2 = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(\xi,\eta)\cos\alpha\xi\sin\beta\eta\, d\xi\, d\eta,$$
$$K_3 = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(\xi,\eta)\sin\alpha\xi\cos\beta\eta\, d\xi\, d\eta,$$
$$K_4 = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(\xi,\eta)\sin\alpha\xi\sin\beta\eta\, d\xi\, d\eta,$$
$$K_5 = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(\xi,\eta)\cos\alpha\xi\cos\beta\eta\, d\xi\, d\eta,$$

$$\left.\begin{aligned}
K_6 &= \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(\xi,\eta)\cos\alpha\xi\sin\beta\eta\,d\xi\,d\eta,\\
K_7 &= \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(\xi,\eta)\sin\alpha\xi\cos\beta\eta\,d\xi\,d\eta,\\
K_8 &= \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(\xi,\eta)\sin\alpha\xi\sin\beta\eta\,d\xi\,d\eta,\\
K_9 &= \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_3(\xi,\eta)\cos\alpha\xi\cos\beta\eta\,d\xi\,d\eta,\\
K_{10} &= \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_3(\xi,\eta)\cos\alpha\xi\sin\beta\eta\,d\xi\,d\eta,\\
K_{11} &= \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_3(\xi,\eta)\sin\alpha\xi\cos\beta\eta\,d\xi\,d\eta,\\
K_{12} &= \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_3(\xi,\eta)\sin\alpha\xi\sin\beta\eta\,d\xi\,d\eta.
\end{aligned}\right\} \tag{32}$$

Equations (20)—(31) are twelve in number, and so they are for the determination of twelve unknowns A_1, A_2, $\cdots C_3$, C_4. (Cf. also equations (36).) K_1, K_2, $\cdots K_{12}$ may be taken as known quantities, since these will become determinate when $F_1(x,y)$, $F_2(x,y)$ and $F_3(x,y)$ are given, by processing double integrations indicated in (19) and (32).

When attention is restricted to a rectangular domain, whose sides are $2a$ and $2b$, the double integrations in (31) and (32) are to be written

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\xi\,d\eta = \int_{-b}^{b}\int_{-a}^{a} d\xi\,d\eta,$$

the rectangular coordinates being taken as illustrated in Fig. 1.

§ 6. SOLUTIONS OF SIMULTANEOUS EQUATIONS

We shall solve the simultaneous equations (20)—(31), in consequence of which the twelve unknowns A_1, A_2, $\cdots C_3$, C_4 will be determined. These equations may be written in schematic form as is written in (36). To solve these equations, it will be convenient to divide them into twelve sets of simultaneous equations by taking advantage of their linearity, and to solve these sets individually. The first set is equations consisting of

$$K_1 \neq 0, \qquad K_2 = K_3 = \cdots = K_{12} = 0, \tag{33}$$

and the second one of

$$K_1 = 0, \qquad K_2 \neq 0, \qquad K_3 = K_4 = \cdots = K_{12} = 0, \tag{34}$$

and in this way the last or twelfth set is equations consisting of

$$K_1 = K_2 = \cdots = K_{11} = 0, \qquad K_{12} \neq 0.$$

In accordance with this procedure, let us introduce notations with subscript such as

$$\widehat{zz}_1, \quad \widehat{zz}_2, \quad \cdots \widehat{zz}_{12}, \quad \text{etc.};$$

then the complete solution of the present boundary-value problem constitutes the aggregate of these individual solutions; that is to say, the stress and displacement-components in question are given by the sums

$$\left.\begin{array}{l} \widehat{zz} = \widehat{zz}_1 + \widehat{zz}_2 + \cdots + \widehat{zz}_{12}, \cdots\cdots, \ \widehat{xy} = \widehat{xy}_1 + \widehat{xy}_2 + \cdots + \widehat{xy}_{12}, \\ u = u_1 + u_2 + \cdots + u_{12}, \cdots\cdots. \end{array}\right\} \tag{35}$$

In the first place, we take the case (33) in which $K_1 \neq 0$ and $K_2 = K_3 = \cdots = K_{12} = 0$. To solve this set of equations our attention is directed to the eleven equations in (36) except for the first. This system of equations has twelve unknowns, so that one of them may be free from their determination. This freedom is denoted by letter Ω_1. After a simple transformation of these eleven equations we have the equations given in (37). That is, we arrive at the three equations

$$P_2 = 0, \qquad P_3 = 0, \qquad P_4 = 0,$$

P_i's having been defined in (16). This system of equations can be solved without difficulty, and thus we obtain the following solution (38) for the simultaneous equations (36), in which the subscript 1 is added to all letters in accordance with the first set of equations. In this connection, there will occur the three similar cases

$$P_1 = P_3 = P_4 = 0, \qquad P_1 = P_2 = P_4 = 0, \qquad P_1 = P_2 = P_3 = 0,$$

which, together with the preceding case, will require to solve the equations (16) simultaneously, as noted before. We now have

$$\left.\begin{array}{l} A_{11} = \{-2\alpha^9\beta + 3\alpha^7\beta^3 + 11\alpha^5\beta^5 + 3\alpha^3\beta^7 - 2\alpha\beta^9 \\ \qquad\qquad - \sigma(6\alpha^7\beta^3 + 14\alpha^5\beta^5 + 6\alpha^3\beta^7)\}\Omega_1, \\ A_{21} = \{5\alpha^7\beta^2\gamma + 3\alpha^5\beta^4\gamma - 7\alpha^3\beta^6\gamma - 2\alpha\beta^8\gamma \\ \qquad\qquad - \sigma(4\alpha^7\beta^2\gamma + 2\alpha^5\beta^4\gamma - 6\alpha^3\beta^6\gamma - 2\alpha\beta^8\gamma)\}\Omega_1, \\ A_{31} = \{-2\alpha^8\beta\gamma - 7\alpha^6\beta^3\gamma + 3\alpha^4\beta^5\gamma + 5\alpha^2\beta^7\gamma \\ \qquad\qquad - \sigma(-2\alpha^8\beta\gamma - 6\alpha^6\beta^3\gamma + 2\alpha^4\beta^5\gamma + 4\alpha^2\beta^7\gamma)\}\Omega_1, \\ A_{41} = \{2\alpha^8\beta^2 + 14\alpha^6\beta^4 + 14\alpha^4\beta^6 + 2\alpha^2\beta^8 \\ \qquad\qquad - \sigma(2\alpha^8\beta^2 + 12\alpha^6\beta^4 + 12\alpha^4\beta^6 + 2\alpha^2\beta^8)\}\Omega_1, \\ B_{11} = (1-\sigma)(2\alpha^9\beta + 6\alpha^7\beta^3 - 6\alpha^3\beta^7 - 2\alpha\beta^9)\Omega_1, \end{array}\right\} \tag{38}$$

Reference		A_1	A_2	A_3	A_4	B_1	B_2	B_3	B_4	C_1	C_2	C_3	C_4	= Const.
$\widehat{zz}$	cos cos	γ	0	0	0	0	β	$-\alpha$	0	0	$-\beta\gamma$	$-\alpha\gamma$	$2(1-\sigma)\alpha\beta$	$\frac{K_1}{\gamma}$
	cos sin	0	γ	0	0	$-\beta$	0	0	$-\alpha$	$\beta\gamma$	0	$-2(1-\sigma)\alpha\beta$	$-\alpha\gamma$	$\frac{K_2}{\gamma}$
	sin cos	0	0	γ	0	α	0	0	β	$\alpha\gamma$	$-2(1-\sigma)\alpha\beta$	0	$-\beta\gamma$	$\frac{K_3}{\gamma}$
	sin sin	0	0	0	γ	0	α	$-\beta$	0	$2(1-\sigma)\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	0	$\frac{K_4}{\gamma}$
$\widehat{yz}$	cos cos	0	$-\beta\gamma$	0	0	$\frac{1}{2}(\beta^2+\gamma^2)$	0	$\frac{1}{2}\alpha\gamma$	$\frac{1}{2}\alpha\beta$	$-\{(1-\sigma)\alpha^2+2\beta^2\}\gamma$	0	$\{\beta^2+(1-\sigma)\gamma^2\}\alpha$	$(1+\sigma)\alpha\beta\gamma$	K_5
	cos sin	$\beta\gamma$	0	0	0	0	$\frac{1}{2}(\beta^2+\gamma^2)$	$-\frac{1}{2}\alpha\beta$	$\frac{1}{2}\alpha\gamma$	0	$-\{(1-\alpha)\alpha^2+2\beta^2\}\gamma$	$-(1+\sigma)\alpha\beta\gamma$	$\{\beta^2+(1-\sigma)\gamma^2\}\alpha$	K_6
	sin cos	0	0	0	$-\beta\gamma$	$-\frac{1}{2}\alpha\gamma$	$-\frac{1}{2}\alpha\beta$	$\frac{1}{2}(\beta^2+\gamma^2)$	0	$-\{\beta^2+(1-\sigma)\gamma^2\}\alpha$	$-(1+\sigma)\alpha\beta\gamma$	$-\{(1-\sigma))\alpha^2+2\beta^2\}\gamma$	0	K_7
	sin sin	0	0	$\beta\gamma$	0	$\frac{1}{2}\alpha\beta$	$-\frac{1}{2}\alpha\gamma$	0	$\frac{1}{2}(\beta^2+\gamma^2)$	$(1+\sigma)\alpha\beta\gamma$	$-\{\beta^2+(1-\sigma)\gamma^2\}\alpha$	0	$-\{(1-\sigma)\alpha^2+2\beta^2\}\gamma$	K_8
$\widehat{zx}$	cos cos	0	0	$-\alpha\gamma$	0	$-\frac{1}{2}(\alpha^2+\gamma^2)$	$-\frac{1}{2}\beta\gamma$	0	$-\frac{1}{2}\alpha\beta$	$-\{2\alpha^2+(1-\sigma)\beta^2\}\gamma$	$\{\alpha^2+(1-\sigma)\gamma^2\}\beta$	0	$(1+\sigma)\alpha\beta\gamma$	K_9
	cos sin	0	0	0	$-\alpha\gamma$	$\frac{1}{2}\beta\gamma$	$-\frac{1}{2}(\alpha^2+\gamma^2)$	$\frac{1}{2}\alpha\beta$	0	$-\{\alpha^2+(1-\sigma)\gamma^2\}\beta$	$-\{2\alpha^2+(1-\sigma)\beta^2\}\gamma$	$-(1+\sigma)\alpha\beta\gamma$	0	K_{10}
	sin cos	$\alpha\gamma$	0	0	0	0	$\frac{1}{2}\alpha\beta$	$-\frac{1}{2}(\alpha^2+\gamma^2)$	$-\frac{1}{2}\beta\gamma$	0	$-(1+\sigma)\alpha\beta\gamma$	$-\{2\alpha^2+(1-\sigma)\beta^2\}\gamma$	$\{\alpha^2+(1-\sigma)\gamma^2\}\beta$	K_{11}
	sin sin	0	$\alpha\gamma$	0	0	$-\frac{1}{2}\alpha\beta$	0	$\frac{1}{2}\beta\gamma$	$-\frac{1}{2}(\alpha^2+\gamma^2)$	$(1+\sigma)\alpha\beta\gamma$	0	$-\{\alpha^2+(1-\sigma)\gamma^2\}\beta$	$-\{2\alpha^2+(1-\sigma)\beta^2\}\gamma$	K_{12}

(36)

$A_{11}\gamma$	$A_{21}\gamma$	$A_{31}\gamma$	$A_{41}\gamma$	$\frac{1}{2}B_{11}$	$\frac{1}{2}B_{21}$	$\frac{1}{2}B_{31}$	$\frac{1}{2}B_{41}$	C_{11}	C_{21}	C_{31}	C_{41}	
0	1	0	0	-2β	0	0	-2α	$\beta\gamma$	0	$-2(1-\sigma)\alpha\beta$	$-\alpha\gamma$	
0	0	1	0	2α	0	0	2β	$\alpha\gamma$	$-2(1-\sigma)\alpha\beta$	0	$-\beta\gamma$	
0	0	0	1	0	2α	-2β	0	$2(1-\sigma)\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	0	
0	0	0	0	α^2	0	$\alpha\gamma$	$-\alpha\beta$	$(\sigma\alpha^2-\gamma^2)\gamma$	0	$\{\alpha^2-\sigma(\alpha^2-\beta^2)\}\alpha$	$\sigma\alpha\beta\gamma$	
β	0	0	0	0	$\beta^2+\gamma^2$	$-\alpha\beta$	$\alpha\gamma$	0	$-\{(1-\sigma)\alpha^2+2\beta^2\}\gamma$	$-(1+\sigma)\alpha\beta\gamma$	$\{\beta^2+(1-\sigma)\gamma^2\}\alpha$	
0	0	0	0	$-\alpha\gamma$	$\alpha\beta$	α^2	0	$-\{\alpha^2-\sigma(\alpha^2-\beta^2)\}\alpha$	$-\sigma\alpha\beta\gamma$	$(\sigma\alpha^2-\gamma^2)\gamma$	0	
0	0	0	0	$-\alpha\beta$	$-\alpha\gamma$	0	α^2	$\sigma\alpha\beta\gamma$	$-\{\alpha^2-\sigma(\alpha^2-\beta^2)\}\alpha$	0	$(\sigma\alpha^2-\gamma^2)\gamma$	
0	0	0	0	0	0	0	0	$-\alpha\gamma$	$\alpha\beta$	0	$\beta\gamma$	$=P_3$
0	0	0	0	0	0	0	0	$-\alpha\beta$	$-\alpha\gamma$	$-\beta\gamma$	0	$=P_4$
0	0	0	0	0	$-\alpha$	$-\beta$	$-\gamma$	0	$(1-\sigma)\alpha\gamma$	$-(1-\sigma)\beta\gamma$	$-(1-\sigma)(\alpha^2-\beta^2)$	
0	0	0	0	0	0	0	0	$-\beta\gamma$	0	$\alpha\beta$	$\alpha\gamma$	$=P_2$

(37)

$$\left.\begin{aligned}
B_{21} &= (1-\sigma)(-2\,\alpha^7\beta^2\gamma - 10\,\alpha^5\beta^4\gamma - 6\,\alpha^3\beta^6\gamma)\,\Omega_1,\\
B_{31} &= (1-\sigma)(6\,\alpha^6\beta^3\gamma + 10\,\alpha^4\beta^5\gamma + 2\,\alpha^2\beta^7\gamma)\,\Omega_1,\\
B_{41} &= (1-\sigma)(4\,\alpha^8\beta^2 + 4\,\alpha^6\beta^4 - 4\,\alpha^4\beta^6 - 4\,\alpha^2\beta^8)\,\Omega_1,\\
C_{11} &= (2\,\alpha^5\beta^3\gamma + 2\,\alpha^3\beta^5\gamma)\,\Omega_1,\\
C_{21} &= (\alpha^7\beta^2 - 2\,\alpha^3\beta^6 - \alpha\beta^8)\,\Omega_1,\\
C_{31} &= (-\alpha^8\beta - 2\,\alpha^6\beta^3 + \alpha^2\beta^7)\,\Omega_1,\\
C_{41} &= (\alpha^6\beta^2\gamma + 3\,\alpha^4\beta^4\gamma + \alpha^2\beta^6\gamma)\,\Omega_1,\\
P_{11} &= -(\alpha^9\beta\gamma + 2\,\alpha^7\beta^3\gamma + 3\,\alpha^5\beta^5\gamma + 2\,\alpha^3\beta^7\gamma + \alpha\beta^9\gamma)\,\Omega_1,
\end{aligned}\right\}$$

where Ω_1 is a constant and $\alpha^2+\beta^2=\gamma^2$ as before.

The second set of equations (34) has also its solution, which is obtained in the forms

$$\left.\begin{aligned}
A_{12} &= -A_{21}\,\Omega_2, & A_{22} &= A_{11}\,\Omega_2, & A_{32} &= -A_{41}\,\Omega_2, & A_{42} &= A_{31}\,\Omega_2,\\
B_{12} &= -B_{21}\,\Omega_2, & B_{22} &= B_{11}\,\Omega_2, & B_{32} &= -B_{41}\,\Omega_2, & B_{42} &= B_{31}\,\Omega_2,\\
C_{12} &= -C_{21}\,\Omega_2, & C_{22} &= C_{11}\,\Omega_2, & C_{32} &= -C_{41}\,\Omega_2, & C_{42} &= C_{31}\,\Omega_2.
\end{aligned}\right\} \quad (39)$$

In this way Table 1 is obtained, in which the additional subscript 1 is for simplicity suppressed; i.e.,

$$A_1 = A_{11}, \quad A_2 = A_{21}, \quad \cdots\cdots \quad C_3 = C_{31}, \quad C_4 = C_{41}.$$

This table gives the solution of the simultaneous equations (20)–(30) or (36).

Table 1. Solution of simultaneous equations (36).

Reference		i	A_{1i}	A_{2i}	A_{3i}	A_{4i}	B_{1i}	B_{2i}	B_{3i}	B_{4i}	C_{1i}	C_{2i}	C_{3i}	C_{4i}
$\widehat{zz}$	cos cos	1	A_1	A_2	A_3	A_4	B_1	B_2	B_3	B_4	C_1	C_2	C_3	C_4
	cos sin	2	$-A_2$	A_1	$-A_4$	A_3	$-B_2$	B_1	$-B_4$	B_3	$-C_2$	C_1	$-C_4$	C_3
	sin cos	3	$-A_3$	$-A_4$	A_1	A_2	$-B_3$	$-B_4$	B_1	B_2	$-C_3$	$-C_4$	C_1	C_2
	sin sin	4	A_4	$-A_3$	$-A_2$	A_1	B_4	$-B_3$	$-B_2$	B_1	C_4	$-C_3$	$-C_2$	C_1
$\widehat{yz}$	cos cos	5	A_1	A_2	A_3	A_4	B_1	B_2	B_3	B_4	C_1	C_2	C_3	C_4
	cos sin	6	A_2	$-A_1$	A_4	$-A_3$	B_2	$-B_1$	B_4	$-B_3$	C_2	$-C_1$	C_4	$-C_3$
	sin cos	7	$-A_3$	$-A_4$	A_1	A_2	$-B_3$	$-B_4$	B_1	B_2	$-C_3$	$-C_4$	C_1	C_2
	sin sin	8	$-A_4$	A_3	A_2	$-A_1$	$-B_4$	B_3	B_2	$-B_1$	$-C_4$	C_3	C_2	$-C_1$
$\widehat{zx}$	cos cos	9	A_1	A_2	A_3	A_4	B_1	B_2	B_3	B_4	C_1	C_2	C_3	C_4
	cos sin	10	$-A_2$	A_1	$-A_4$	A_3	$-B_2$	B_1	$-B_4$	B_3	$-C_2$	C_1	$-C_4$	C_3
	sin cos	11	A_3	A_4	$-A_1$	$-A_2$	B_3	B_4	$-B_1$	$-B_2$	C_3	C_4	$-C_1$	$-C_2$
	sin sin	12	$-A_4$	A_3	A_2	$-A_1$	$-B_4$	B_3	B_2	$-B_1$	$-C_4$	C_3	C_2	$-C_1$

The above table implies that the first line corresponding to the omission of $\cos\alpha x\cos\beta y$ in $\widehat{zz}$ and to the case $i=1$ gives the solution (38), the second one the solution (39), and so on.

§ 7. GENERAL SOLUTION OF THE PROBLEM

From the foregoing calculations we can get the general solution of the present boundary-value problem. First we take the case (33), in which $K_1 \neq 0$. The solution (38) is substituted into (20), from which Ω_1 becomes determinate, and then from the last epuation of (38), P_{11} also becomes determinate. We thus obtain

$$\gamma P_{11} = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(\xi,\eta)\cos\alpha\xi\cos\beta\eta\, d\xi\, d\eta, \;= K_1, \tag{41}$$

and the corresponding part of the complete stress-components is given by the equations

$$\left.\begin{aligned}
\widehat{zz}_1 &= \int_0^\infty\int_0^\infty K_1(1+\gamma z)\cos\alpha x\cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{yz}_1 &= \int_0^\infty\int_0^\infty K_1\beta z\cos\alpha x\sin\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{zx}_1 &= \int_0^\infty\int_0^\infty K_1\alpha z\sin\alpha x\cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{xx}_1 &= \int_0^\infty\int_0^\infty K_1\frac{\alpha^2}{\gamma^2}\left(1+2\sigma\frac{\beta^2}{\alpha^2}-\gamma z\right)\cos\alpha x\cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{yy}_1 &= \int_0^\infty\int_0^\infty K_1\frac{\beta^2}{\gamma^2}\left(1+2\sigma\frac{\alpha^2}{\beta^2}-\gamma z\right)\cos\alpha x\cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{xy}_1 &= -\int_0^\infty\int_0^\infty K_1\frac{\alpha\beta}{\gamma^2}(1-2\sigma-\gamma z)\sin\alpha x\sin\beta y\, e^{-\gamma z}\, d\alpha\, d\beta.
\end{aligned}\right\} \tag{42}$$

The displacement can be found to be given by the equations

$$\left.\begin{aligned}
u_1 &= \frac{1}{2\mu}\int_0^\infty\int_0^\infty K_1\frac{\alpha}{\gamma^2}(1-2\sigma-\gamma z)\sin\alpha x\cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
v_1 &= \frac{1}{2\mu}\int_0^\infty\int_0^\infty K_1\frac{\beta}{\gamma^2}(1-2\sigma-\gamma z)\cos\alpha x\sin\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
w_1 &= -\frac{1}{2\mu}\int_0^\infty\int_0^\infty K_1\frac{1}{\gamma}\{2(1-\sigma)+\gamma z\}\cos\alpha x\cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,
\end{aligned}\right\} \tag{43}$$

where, as before, μ is the modulus of rigidity, σ Poisson's ratio, and $\gamma^2=\alpha^2+\beta^2$. The above results (41)—(43) constitute the first part of the solution for our problem. (Cf. (35).)

The second part of the complete solution is given by the equations

$$\left.\begin{aligned}
\widehat{zz}_2 &= \int_0^\infty \int_0^\infty K_2 (1 + \gamma z) \cos \alpha x \sin \beta y \, e^{-\gamma z} \, d\alpha \, d\beta, \\
\widehat{yz}_2 &= -\int_0^\infty \int_0^\infty K_2 \, \beta z \cos \alpha x \cos \beta y \, e^{-\gamma z} \, d\alpha \, d\beta, \\
\widehat{zx}_2 &= \int_0^\infty \int_0^\infty K_2 \, \alpha z \sin \alpha x \sin \beta y \, e^{-\gamma z} \, d\alpha \, d\beta, \\
\widehat{xx}_2 &= \int_0^\infty \int_0^\infty K_2 \frac{\alpha^2}{\gamma^2} \left(1 + 2\sigma \frac{\beta^2}{\alpha^2} - \gamma z\right) \cos \alpha x \sin \beta y \, e^{-\gamma z} \, d\alpha \, d\beta, \\
\widehat{yy}_2 &= \int_0^\infty \int_0^\infty K_2 \frac{\beta^2}{\gamma^2} \left(1 + 2\sigma \frac{\alpha^2}{\beta^2} - \gamma z\right) \cos \alpha x \sin \beta y \, e^{-\gamma z} \, d\alpha \, d\beta, \\
\widehat{xy}_2 &= \int_0^\infty \int_0^\infty K_2 \frac{\alpha\beta}{\gamma^2} (1 - 2\sigma - \gamma z) \sin \alpha x \cos \beta y \, e^{-\gamma z} \, d\alpha \, d\beta,
\end{aligned}\right\} \tag{44}$$

where

$$K_2 = \frac{1}{\pi^2} \int_{-\infty}^\infty \int_{-\infty}^\infty F_1 (\xi, \eta) \cos \alpha\xi \sin \beta\eta \, d\xi \, d\eta. \tag{45}$$

The corresponding displacement may be shown to be given be the equations

$$\left.\begin{aligned}
u_2 &= \frac{1}{2\mu} \int_0^\infty \int_0^\infty K_2 \frac{\alpha}{\gamma^2} (1 - 2\sigma - \gamma z) \sin \alpha x \sin \beta y \, e^{-\gamma z} \, d\alpha \, d\beta, \\
u_2 &= -\frac{1}{2\mu} \int_0^\infty \int_0^\infty K_2 \frac{\beta}{\gamma^2} (1 - 2\sigma - \gamma z) \cos \alpha x \cos \beta y \, e^{-\gamma z} \, d\alpha \, d\beta, \\
w_2 &= -\frac{1}{2\mu} \int_0^\infty \int_0^\infty K_2 \frac{1}{\gamma} \{2(1 - \sigma) + \gamma z\} \cos \alpha x \sin \beta y \, e^{-\gamma z} \, d\alpha \, d\beta.
\end{aligned}\right\} \tag{46}$$

The above results (44)—(46) constitute the second part of the complete solution, which is due to the case where $K_2 \neq 0$, and all the remaining K_i's vanish.

The third part of the complete solution is given by the equations

$$\left.\begin{aligned}
\widehat{zz}_3 &= \int_0^\infty \int_0^\infty K_3 (1 + \gamma z) \sin \alpha x \cos \beta y \, e^{-\gamma z} \, d\alpha \, d\beta, \\
\widehat{yz}_3 &= \int_0^\infty \int_0^\infty K_3 \, \beta z \sin \alpha x \sin \beta y \, e^{-\gamma z} \, d\alpha \, d\beta, \\
\widehat{zx}_3 &= -\int_0^\infty \int_0^\infty K_3 \, \alpha z \cos \alpha x \cos \beta y \, e^{-\gamma z} \, d\alpha \, d\beta, \\
\widehat{xx}_3 &= \int_0^\infty \int_0^\infty K_3 \frac{\alpha^2}{\gamma^2} \left(1 + 2\sigma \frac{\beta^2}{\alpha^2} - \gamma z\right) \sin \alpha x \cos \beta y \, e^{-\gamma z} \, d\alpha \, d\beta, \\
\widehat{yy}_3 &= \int_0^\infty \int_0^\infty K_3 \frac{\beta^2}{\gamma^2} \left(1 + 2\sigma \frac{\alpha^2}{\beta^2} - \gamma z\right) \sin \alpha x \cos \beta y \, e^{-\gamma z} \, d\alpha \, d\beta, \\
\widehat{xy}_3 &= \int_0^\infty \int_0^\infty K_3 \frac{\alpha\beta}{\gamma^2} (1 - 2\sigma - \gamma z) \cos \alpha x \sin \beta y \, e^{-\gamma z} \, d\alpha \, d\beta,
\end{aligned}\right\} \tag{47}$$

where

$$K_3 = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(\xi, \eta) \sin \alpha\xi \cos \beta\eta \, d\xi \, d\eta. \tag{48}$$

The corresponding displacement may be shown to be given by the equations

$$\left.\begin{aligned}
u_3 &= -\frac{1}{2\mu}\int_0^\infty\int_0^\infty K_3 \frac{\alpha}{\gamma^2}(1-2\sigma-\gamma z)\cos \alpha x \cos \beta y \, e^{-\gamma z} \, d\alpha \, d\beta, \\
v_3 &= \frac{1}{2\mu}\int_0^\infty\int_0^\infty K_3 \frac{\beta}{\gamma^2}(1-2\sigma-\gamma z)\sin \alpha x \sin \beta y \, e^{-\gamma z} \, d\alpha \, d\beta, \\
w_3 &= -\frac{1}{2\mu}\int_0^\infty\int_0^\infty K_3 \frac{1}{\gamma}\{2(1-\sigma)+\gamma z\}\sin \alpha x \cos \beta y \, e^{-\gamma z} \, d\alpha \, d\beta.
\end{aligned}\right\} \tag{49}$$

The above results (47)—(49) constitute the third part of the complete solution, which is due to the case where $K_3 \neq 0$ and all the remaining K_i's vanish.

The fourth part of the complete solution is given by the equations

$$\left.\begin{aligned}
\widehat{zz}_4 &= \int_0^\infty\int_0^\infty K_4 (1+\gamma z)\sin \alpha x \sin \beta y \, e^{-\gamma z} \, d\alpha \, d\beta, \\
\widehat{yz}_4 &= -\int_0^\infty\int_0^\infty K_4 \beta z \sin \alpha x \cos \beta y \, e^{-\gamma z} \, d\alpha \, d\beta, \\
\widehat{zx}_4 &= -\int_0^\infty\int_0^\infty K_4 \alpha z \cos \alpha x \sin \beta y \, e^{-\gamma z} \, d\alpha \, d\beta, \\
\widehat{xx}_4 &= \int_0^\infty\int_0^\infty K_4 \frac{\alpha^2}{\gamma^2}\left(1+2\sigma\frac{\beta^2}{\alpha^2}-\gamma z\right)\sin \alpha x \sin \beta y \, e^{-\gamma z} \, d\alpha \, d\beta, \\
\widehat{yy}_4 &= \int_0^\infty\int_0^\infty K_4 \frac{\beta^2}{\gamma^2}\left(1+2\sigma\frac{\alpha^2}{\beta^2}-\gamma z\right)\sin \alpha x \sin \beta y \, e^{-\gamma z} \, d\alpha \, d\beta, \\
\widehat{xy}_4 &= -\int_0^\infty\int_0^\infty K_4 \frac{\alpha\beta}{\gamma^2}(1-2\sigma-\gamma z)\cos \alpha x \cos \beta y \, e^{-\gamma z} \, d\alpha \, d\beta,
\end{aligned}\right\} \tag{50}$$

where

$$K_4 = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(\xi, \eta) \sin \alpha\xi \sin \beta\eta \, d\xi \, d\eta. \tag{51}$$

The corresponding displacement may be shown to be given by the equations

$$\left.\begin{aligned}
u_4 &= -\frac{1}{2\mu}\int_0^\infty\int_0^\infty K_4 \frac{\alpha}{\gamma^2}(1-2\sigma-\gamma z)\cos \alpha x \sin \beta y \, e^{-\gamma z} \, d\alpha \, d\beta, \\
v_4 &= -\frac{1}{2\mu}\int_0^\infty\int_0^\infty K_4 \frac{\beta}{\gamma^2}(1-2\sigma-\gamma z)\sin \alpha x \cos \beta y \, e^{-\gamma z} \, d\alpha \, d\beta, \\
w_4 &= -\frac{1}{2\mu}\int_0^\infty\int_0^\infty K_4 \frac{1}{\gamma}\{2(1-\sigma)+\gamma z\}\sin \alpha x \sin \beta y \, e^{-\gamma z} \, d\alpha \, d\beta.
\end{aligned}\right\} \tag{52}$$

The above results (50)—(52) constitute the fourth part of the complete solution, which is due to the case where $K_4 \neq 0$, and all the remaining K_i's vanish.

The fifth part of the complete solution is given by the equations

$$\left.\begin{aligned}
\widehat{zz}_5 &= \int_0^\infty\int_0^\infty K_5\,\beta z \cos\alpha x \sin\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{yz}_5 &= \int_0^\infty\int_0^\infty K_5\left(1-\frac{\beta^2}{\gamma}z\right)\cos\alpha x\cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{zx}_5 &= \int_0^\infty\int_0^\infty K_5\,\frac{\alpha\beta}{\gamma}z \sin\alpha x\sin\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{xx}_5 &= \int_0^\infty\int_0^\infty K_5\,\frac{\beta}{\gamma^2}\left(2\sigma\frac{\beta^2}{\gamma}-\alpha^2 z\right)\cos\alpha x\sin\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{yy}_5 &= \int_0^\infty\int_0^\infty K_5\,\frac{2\beta}{\gamma}\left(1+\sigma\frac{\alpha^2}{\gamma^2}-\frac{\beta^2}{2\gamma}z\right)\cos\alpha x\sin\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{xy}_5 &= \int_0^\infty\int_0^\infty K_5\,\frac{\alpha}{\gamma}\left(1-2\sigma\frac{\beta^2}{\gamma^2}-\frac{\beta^2}{\gamma}z\right)\sin\alpha x\cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,
\end{aligned}\right\}\tag{53}$$

where

$$K_5 = \frac{1}{\pi^2}\int_{-\infty}^\infty\int_{-\infty}^\infty F_2(\xi,\eta)\cos\alpha\xi\cos\beta\eta\, d\xi\, d\eta. \tag{54}$$

The corresponding displacement may be shown to be given by the equations

$$\left.\begin{aligned}
u_5 &= -\frac{1}{2\mu}\int_0^\infty\int_0^\infty K_5\,\frac{\alpha\beta}{\gamma^3}(2\sigma+\gamma z)\sin\alpha x\sin\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
v_5 &= -\frac{1}{2\mu}\int_0^\infty\int_0^\infty K_5\,\frac{1}{\gamma}\left\{2-\frac{\beta^2}{\gamma^2}(2\sigma+\gamma z)\right\}\cos\alpha x\cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
w_5 &= -\frac{1}{2\mu}\int_0^\infty\int_0^\infty K_5\,\frac{\beta}{\gamma^2}(1-2\sigma+\gamma z)\cos\alpha x\sin\beta y\, e^{-\gamma z}\, d\alpha\, d\beta.
\end{aligned}\right\}\tag{55}$$

The above results (53)—(55) constitute the fifth part of the complete solution, which is due to the case where $K_5 \neq 0$, and all the remaining K_i's vanish.

The sixth part of the complete solution is given by the equations

$$\left.\begin{aligned}
\widehat{zz}_6 &= -\int_0^\infty\int_0^\infty K_6\,\beta z \cos\alpha x \cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{yz}_6 &= \int_0^\infty\int_0^\infty K_6\left(1-\frac{\beta^2}{\gamma}z\right)\cos\alpha x\sin\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{zx}_6 &= -\int_0^\infty\int_0^\infty K_6\,\frac{\alpha\beta}{\gamma}z \sin\alpha x\cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{xx}_6 &= -\int_0^\infty\int_0^\infty K_6\,\frac{\beta}{\gamma^2}\left(2\sigma\frac{\beta^2}{\gamma}-\alpha^2 z\right)\cos\alpha x\cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{yy}_6 &= -\int_0^\infty\int_0^\infty K_6\,\frac{2\beta}{\gamma}\left(1+\sigma\frac{\alpha^2}{\gamma^2}-\frac{\beta^2}{2\gamma}z\right)\cos\alpha x\cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{xy}_6 &= \int_0^\infty\int_0^\infty K_6\,\frac{\alpha}{\gamma}\left(1-2\sigma\frac{\beta^2}{\gamma^2}-\frac{\beta^2}{\gamma}z\right)\sin\alpha x\sin\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,
\end{aligned}\right\}\tag{56}$$

where

$$K_6 = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(\xi,\eta)\cos\alpha\xi\sin\beta\eta\, d\xi\, d\eta. \tag{57}$$

The corresponding displacement may be shown to be given by the equations

$$\left.\begin{aligned}
u_6 &= \frac{1}{2\mu}\int_0^\infty\int_0^\infty K_6\frac{\alpha\beta}{\gamma^3}(2\sigma+\gamma z)\sin\alpha x\cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
v_6 &= -\frac{1}{2\mu}\int_0^\infty\int_0^\infty K_6\frac{1}{\gamma}\Big\{2-\frac{\beta^2}{\gamma^2}(2\sigma+\gamma z)\Big\}\cos\alpha x\sin\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
w_6 &= \frac{1}{2\mu}\int_0^\infty\int_0^\infty K_6\frac{\beta}{\gamma^2}(1-2\sigma+\gamma z)\cos\alpha x\cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta.
\end{aligned}\right\} \tag{58}$$

The above results (56)—(58) constitute the sixth part of the complete solution, which is due to the case where $K_6 \neq 0$, and all the remaining K_i's vanish.

The seventh part of the complete solution is given by the equations

$$\left.\begin{aligned}
\widehat{zz}_7 &= \int_0^\infty\int_0^\infty K_7\,\beta z\sin\alpha x\sin\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{yz}_7 &= \int_0^\infty\int_0^\infty K_7\Big(1-\frac{\beta^2}{\gamma}z\Big)\sin\alpha x\cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{zx}_7 &= -\int_0^\infty\int_0^\infty K_7\frac{\alpha\beta}{\gamma}z\cos\alpha x\sin\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{xx}_7 &= \int_0^\infty\int_0^\infty K_7\frac{\beta}{\gamma^2}\Big(2\sigma\frac{\beta^2}{\gamma}-\alpha^2 z\Big)\sin\alpha x\sin\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{yy}_7 &= \int_0^\infty\int_0^\infty K_7\frac{2\beta}{\gamma}\Big(1+\sigma\frac{\alpha^2}{\gamma^2}-\frac{\beta^2}{2\gamma}z\Big)\sin\alpha x\sin\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{xy}_7 &= -\int_0^\infty\int_0^\infty K_7\frac{\alpha}{\gamma}\Big(1-2\sigma\frac{\beta^2}{\gamma^2}-\frac{\beta^2}{\gamma}z\Big)\cos\alpha x\cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,
\end{aligned}\right\} \tag{59}$$

where

$$K_7 = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(\xi,\eta)\sin\alpha\xi\cos\beta\eta\, d\xi\, d\eta. \tag{60}$$

The corresponding displacement may be shown to be given by the equations

$$\left.\begin{aligned}
u_7 &= \frac{1}{2\mu}\int_0^\infty\int_0^\infty K_7\frac{\alpha\beta}{\gamma^3}(2\sigma+\gamma z)\cos\alpha x\sin\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
v_7 &= -\frac{1}{2\mu}\int_0^\infty\int_0^\infty K_7\frac{1}{\gamma}\Big\{2-\frac{\beta^2}{\gamma^2}(2\sigma+\gamma z)\Big\}\sin\alpha x\cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
w_7 &= -\frac{1}{2\mu}\int_0^\infty\int_0^\infty K_7\frac{\beta}{\gamma^2}(1-2\sigma+\gamma z)\sin\alpha x\sin\beta y\, e^{-\gamma z}\, d\alpha\, d\beta.
\end{aligned}\right\} \tag{61}$$

The above results (59)—(61) constitute the seventh part of the complete solution, which is due to the case where $K_7 \neq 0$, and all the remaining K_i's vanish.

The eighth part of the complete solution is given by the equations

$$\left.\begin{aligned}
\widehat{zz}_8 &= -\int_0^\infty\int_0^\infty K_8\,\beta z \sin\alpha x \cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{yz}_8 &= \int_0^\infty\int_0^\infty K_8\left(1-\frac{\beta^2}{\gamma}z\right)\sin\alpha x \sin\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{zx}_8 &= \int_0^\infty\int_0^\infty K_8\,\frac{\alpha\beta}{\gamma}z \cos\alpha x \cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{xx}_8 &= -\int_0^\infty\int_0^\infty K_8\,\frac{\beta}{\gamma^2}\left(2\sigma\frac{\beta^2}{\gamma}-\alpha^2 z\right)\sin\alpha x \cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{yy}_8 &= -\int_0^\infty\int_0^\infty K_8\,\frac{2\beta}{\gamma}\left(1+\sigma\frac{\alpha^2}{\gamma^2}-\frac{\beta^2}{2\gamma}z\right)\sin\alpha x \cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{xy}_8 &= -\int_0^\infty\int_0^\infty K_8\,\frac{\alpha}{\gamma}\left(1-2\sigma\frac{\beta^2}{\gamma^2}-\frac{\beta^2}{\gamma}z\right)\cos\alpha x \sin\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,
\end{aligned}\right\} \tag{62}$$

where

$$K_8 = \frac{1}{\pi^2}\int_{-\infty}^\infty\int_{-\infty}^\infty F_2(\xi,\eta)\sin\alpha\xi \sin\beta\eta\, d\xi\, d\eta. \tag{63}$$

The corresponding displacement may be shown to be given by the equations

$$\left.\begin{aligned}
u_8 &= -\frac{1}{2\mu}\int_0^\infty\int_0^\infty K_8\,\frac{\alpha\beta}{\gamma^3}(2\sigma+\gamma z)\cos\alpha x \cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
v_8 &= -\frac{1}{2\mu}\int_0^\infty\int_0^\infty K_8\,\frac{1}{\gamma}\left\{2-\frac{\beta^2}{\gamma^2}(2\sigma+\gamma z)\right\}\sin\alpha x \sin\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
w_8 &= \frac{1}{2\mu}\int_0^\infty\int_0^\infty K_8\,\frac{\beta}{\gamma^2}(1-2\sigma+\gamma z)\sin\alpha x \cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta.
\end{aligned}\right\} \tag{64}$$

The above results (62)—(64) constitute the eighth part of the complete solution, which is due to the case where $K_8 \neq 0$, and all the remaining K_i's vanish.

The ninth part of the complete solution is given by the equations

$$\left.\begin{aligned}
\widehat{zz}_9 &= \int_0^\infty\int_0^\infty K_9\,\alpha z \sin\alpha x \cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{yz}_9 &= \int_0^\infty\int_0^\infty K_9\,\frac{\alpha\beta}{\gamma}z \sin\alpha x \sin\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{zx}_9 &= \int_0^\infty\int_0^\infty K_9\left(1-\frac{\alpha^2}{\gamma}z\right)\cos\alpha x \cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{xx}_9 &= \int_0^\infty\int_0^\infty K_9\,\frac{2\alpha}{\gamma}\left(1+\sigma\frac{\beta^2}{\gamma^2}-\frac{\alpha^2}{2\gamma}z\right)\sin\alpha x \cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{yy}_9 &= \int_0^\infty\int_0^\infty K_9\,\frac{\alpha}{\gamma^2}\left(2\sigma\frac{\alpha^2}{\gamma}-\beta^2 z\right)\sin\alpha x \cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{xy}_9 &= \int_0^\infty\int_0^\infty K_9\,\frac{\beta}{\gamma}\left(1-2\sigma\frac{\alpha^2}{\gamma^2}-\frac{\alpha^2}{\gamma}z\right)\cos\alpha x \sin\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,
\end{aligned}\right\} \tag{65}$$

where

$$K_9 = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_3(\xi,\eta)\cos\alpha\xi\cos\beta\eta\,d\xi\,d\eta. \tag{66}$$

The corresponding displacement may be shown to be given by the equations

$$\left.\begin{aligned}
u_9 &= -\frac{1}{2\mu}\int_0^\infty\int_0^\infty K_9\frac{1}{\gamma}\left\{2-\frac{\alpha^2}{\gamma^2}(2\sigma+\gamma z)\right\}\cos\alpha x\cos\beta y\,e^{-\gamma z}\,d\alpha\,d\beta,\\
v_9 &= -\frac{1}{2\mu}\int_0^\infty\int_0^\infty K_9\frac{\alpha\beta}{\gamma^3}(2\sigma+\gamma z)\sin\alpha x\sin\beta y\,e^{-\gamma z}\,d\alpha\,d\beta,\\
w_9 &= -\frac{1}{2\mu}\int_0^\infty\int_0^\infty K_9\frac{\alpha}{\gamma^2}(1-2\sigma+\gamma z)\sin\alpha x\cos\beta y\,e^{-\gamma z}\,d\alpha\,d\beta.
\end{aligned}\right\} \tag{67}$$

The above results (65)—(67) constitute the ninth part of the complete solution, which is due to the case where $K_9 \neq 0$, and all the remaining K_i's vanish.

The tenth part of the complete solution is given by the equations

$$\left.\begin{aligned}
\widehat{zz}_{10} &= \int_0^\infty\int_0^\infty K_{10}\,\alpha z\sin\alpha x\sin\beta y\,e^{-\gamma z}\,d\alpha\,d\beta,\\
\widehat{yz}_{10} &= -\int_0^\infty\int_0^\infty K_{10}\frac{\alpha\beta}{\gamma}z\sin\alpha x\cos\beta y\,e^{-\gamma z}\,d\alpha\,d\beta,\\
\widehat{zx}_{10} &= \int_0^\infty\int_0^\infty K_{10}\left(1-\frac{\alpha^2}{\gamma}z\right)\cos\alpha x\sin\beta y\,e^{-\gamma z}\,d\alpha\,d\beta,\\
\widehat{xx}_{10} &= \int_0^\infty\int_0^\infty K_{10}\frac{2\alpha}{\gamma}\left(1+\sigma\frac{\beta^2}{\gamma^2}-\frac{\alpha^2}{2\gamma}z\right)\sin\alpha x\sin\beta y\,e^{-\gamma z}\,d\alpha\,d\beta,\\
\widehat{yy}_{10} &= \int_0^\infty\int_0^\infty K_{10}\frac{\alpha}{\gamma^2}\left(2\sigma\frac{\alpha^2}{\gamma}-\beta^2 z\right)\sin\alpha x\sin\beta y\,e^{-\gamma z}\,d\alpha\,d\beta,\\
\widehat{xy}_{10} &= -\int_0^\infty\int_0^\infty K_{10}\frac{\beta}{\gamma}\left(1-2\sigma\frac{\alpha^2}{\gamma^2}-\frac{\alpha^2}{\gamma}z\right)\cos\alpha x\cos\beta y\,e^{-\gamma z}\,d\alpha\,d\beta,
\end{aligned}\right\} \tag{68}$$

where

$$K_{10} = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_3(\xi,\eta)\cos\alpha\xi\sin\beta\eta\,d\xi\,d\eta. \tag{69}$$

The corresponding displacement may be shown to be given by the equations

$$\left.\begin{aligned}
u_{10} &= -\frac{1}{2\mu}\int_0^\infty\int_0^\infty K_{10}\frac{1}{\gamma}\left\{2-\frac{\alpha^2}{\gamma^2}(2\sigma+\gamma z)\right\}\cos\alpha x\sin\beta y\,e^{-\gamma z}\,d\alpha\,d\beta,\\
v_{10} &= \frac{1}{2\mu}\int_0^\infty\int_0^\infty K_{10}\frac{\alpha\beta}{\gamma^3}(2\sigma+\gamma z)\sin\alpha x\cos\beta y\,e^{-\gamma z}\,d\alpha\,d\beta,\\
w_{10} &= -\frac{1}{2\mu}\int_0^\infty\int_0^\infty K_{10}\frac{\alpha}{\gamma^2}(1-2\sigma+\gamma z)\sin\alpha x\sin\beta y\,e^{-\gamma z}\,d\alpha\,d\beta.
\end{aligned}\right\} \tag{70}$$

The above results (68)—(70) constitute the tenth part of the complete solution, which is due to the case where $K_{10} \neq 0$, and all the remaining K_i's vanish.

The eleventh part of the complete solution is given by the equations

$$\left.\begin{aligned}
\widehat{zz}_{11} &= -\int_0^\infty\int_0^\infty K_{11}\,\alpha z \cos\alpha x \cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{yz}_{11} &= -\int_0^\infty\int_0^\infty K_{11}\frac{\alpha\beta}{\gamma} z \cos\alpha x \sin\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{zx}_{11} &= \int_0^\infty\int_0^\infty K_{11}\left(1-\frac{\alpha^2}{\gamma}z\right) \sin\alpha x \cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{xx}_{11} &= -\int_0^\infty\int_0^\infty K_{11}\frac{2\alpha}{\gamma}\left(1+\sigma\frac{\beta^2}{\gamma^2}-\frac{\alpha^2}{2\gamma}z\right) \cos\alpha x \cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{yy}_{11} &= -\int_0^\infty\int_0^\infty K_{11}\frac{\alpha}{\gamma^2}\left(2\sigma\frac{\alpha^2}{\gamma}-\beta^2 z\right) \cos\alpha x \cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{xy}_{11} &= \int_0^\infty\int_0^\infty K_{11}\frac{\beta}{\gamma}\left(1-2\sigma\frac{\alpha^2}{\gamma^2}-\frac{\alpha^2}{\gamma}z\right) \sin\alpha x \sin\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,
\end{aligned}\right\} \tag{71}$$

where

$$K_{11} = \frac{1}{\pi^2}\int_{-\infty}^\infty\int_{-\infty}^\infty F_3(\xi,\eta)\sin\alpha\xi\cos\beta\eta\, d\xi\, d\eta. \tag{72}$$

The corresponding displacement may be shown to be given by the equations

$$\left.\begin{aligned}
u_{11} &= -\frac{1}{2\mu}\int_0^\infty\int_0^\infty K_{11}\frac{1}{\gamma}\left\{2-\frac{\alpha^2}{\gamma^2}(2\sigma+\gamma z)\right\}\sin\alpha x\cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
v_{11} &= \frac{1}{2\mu}\int_0^\infty\int_0^\infty K_{11}\frac{\alpha\beta}{\gamma^3}(2\sigma+\gamma z)\cos\alpha x\sin\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
w_{11} &= \frac{1}{2\mu}\int_0^\infty\int_0^\infty K_{11}\frac{\alpha}{\gamma^2}(1-2\sigma+\gamma z)\cos\alpha x\cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta.
\end{aligned}\right\} \tag{73}$$

The above results (71)—(73) constitute the eleventh part of the complete solution, which is due to the case where $K_{11}\neq 0$, and all the remaining K_i's vanish.

The twelfth part of the complete solution is given by the equations

$$\left.\begin{aligned}
\widehat{zz}_{12} &= -\int_0^\infty\int_0^\infty K_{12}\,\alpha z \cos\alpha x \sin\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{yz}_{12} &= \int_0^\infty\int_0^\infty K_{12}\frac{\alpha\beta}{\gamma} z \cos\alpha x \cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{zx}_{12} &= \int_0^\infty\int_0^\infty K_{12}\left(1-\frac{\alpha^2}{\gamma}z\right) \sin\alpha x \sin\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{xx}_{12} &= -\int_0^\infty\int_0^\infty K_{12}\frac{2\alpha}{\gamma}\left(1+\sigma\frac{\beta^2}{\gamma^2}-\frac{\alpha^2}{2\gamma}z\right) \cos\alpha x \sin\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{yy}_{12} &= -\int_0^\infty\int_0^\infty K_{12}\frac{\alpha}{\gamma^2}\left(2\sigma\frac{\alpha^2}{\gamma}-\beta^2 z\right) \cos\alpha x \sin\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,\\
\widehat{xy}_{12} &= -\int_0^\infty\int_0^\infty K_{12}\frac{\beta}{\gamma}\left(1-2\sigma\frac{\alpha^2}{\gamma^2}-\frac{\alpha^2}{\gamma}z\right) \sin\alpha x \cos\beta y\, e^{-\gamma z}\, d\alpha\, d\beta,
\end{aligned}\right\} \tag{74}$$

where

$$K_{12} = \frac{1}{\pi^2}\int_{-\infty}^\infty\int_{-\infty}^\infty F_3(\xi,\eta)\sin\alpha\xi\sin\beta\eta\, d\xi\, d\eta. \tag{75}$$

The corresponding displacement may be shown to be given by the equations

$$\left.\begin{aligned} u_{12} &= -\frac{1}{2\mu}\int_0^\infty\int_0^\infty K_{12}\frac{1}{\gamma}\left\{2-\frac{\alpha^2}{\gamma^2}(2\sigma+\gamma z)\right\}\sin\alpha x\sin\beta y\, e^{-\gamma z}\,d\alpha\, d\beta,\\ v_{12} &= -\frac{1}{2\mu}\int_0^\infty\int_0^\infty K_{12}\frac{\alpha\beta}{\gamma^3}(2\sigma+\gamma z)\cos\alpha x\cos\beta y\, e^{-\gamma z}\,d\alpha\, d\beta,\\ w_{12} &= \frac{1}{2\mu}\int_0^\infty\int_0^\infty K_{12}\frac{\alpha}{\gamma^2}(1-2\sigma+\gamma z)\cos\alpha x\sin\beta y\, e^{-\gamma z}\,d\alpha\, d\beta. \end{aligned}\right\} \tag{76}$$

The above results(74)—(76)constitute the twelfth part of the complete solution, which is due to the case where $K_{12}\neq 0$, and all the remaining K_i's vanish. As was stated in (35), the complete solution of the present boundary-value problem is given by the aggregate of the twelve constituent solutions obtained in equations (41)—(76).

§ 8. SUMMARY OF THE GENRAL SOLUTION

The foregoing solutions (41)—(76) can be rearranged in compact forms, though rather inconvenient for individual applications which will appear in a subsequent article. For instance, the component $\widehat{zz}$ can be written

$$\begin{aligned} \widehat{zz} &= \left(\widehat{zz}_1+\widehat{zz}_2+\widehat{zz}_3+\widehat{zz}_4\right)+\left(\widehat{zz}_5+\widehat{zz}_6+\widehat{zz}_7+\widehat{zz}_8\right)+\left(\widehat{zz}_9+\widehat{zz}_{10}+\widehat{zz}_{11}+\widehat{zz}_{12}\right)\\ &= \int_0^\infty\int_0^\infty(1+\gamma z)\,(K_1\cos\alpha x\cos\beta y+K_2\cos\alpha x\sin\beta y+K_3\sin\alpha x\cos\beta y\\ &\qquad\qquad +K_4\sin\alpha x\sin\beta y)\,e^{-\gamma z}\,d\alpha\, d\beta\\ &+\int_0^\infty\int_0^\infty\beta z\,(K_5\cos\alpha x\sin\beta y-K_6\cos\alpha x\cos\beta y+K_7\sin\alpha x\sin\beta y\\ &\qquad\qquad -K_8\sin\alpha x\cos\beta y)\,e^{-\gamma z}\,d\alpha\, d\beta\\ &+\int_0^\infty\int_0^\infty\alpha z\,(K_9\sin\alpha x\cos\beta y+K_{10}\sin\alpha x\sin\beta y-K_{11}\cos\alpha x\cos\beta y\\ &\qquad\qquad -K_{12}\cos\alpha x\sin\beta y)\,e^{-\gamma z}\,d\alpha\, d\beta, \end{aligned}$$

which, with the values of K_i 's, is rearranged in the form

$$\begin{aligned} \widehat{zz} = \frac{1}{\pi^2}\int_0^\infty d\alpha\int_0^\infty d\beta\int_{-\infty}^\infty\int_{-\infty}^\infty\Big[&(1+\gamma z)\,F_1(\xi,\eta)\cos\alpha(x-\xi)\cos\beta(y-\eta)\\ &+\beta z\,F_2(\xi,\eta)\cos\alpha(x-\xi)\sin\beta(y-\eta)\\ &+\alpha z\,F_3(\xi,\eta)\sin\alpha(x-\xi)\cos\beta(y-\eta)\Big]e^{-\gamma z}\,d\xi\, d\eta. \end{aligned} \tag{77}$$

The remaining stress-components can similarly be rearranged in the forms

$$\begin{aligned} \widehat{yz} = \frac{1}{\pi^2}\int_0^\infty d\alpha\int_0^\infty d\beta\int_{-\infty}^\infty\int_{-\infty}^\infty\Big[&\beta z\,F_1(\xi,\eta)\cos\alpha(x-\xi)\sin\beta(y-\eta)\\ &+\left(1-\frac{\beta^2}{\gamma}z\right)F_2(\xi,\eta)\cos\alpha(x-\xi)\cos\beta(y-\eta)\\ &+\frac{\alpha\beta}{\gamma}z\,F_3(\xi,\eta)\sin\alpha(x-\xi)\sin\beta(y-\eta)\Big]e^{-\gamma z}\,d\xi\, d\eta, \end{aligned} \tag{78}$$

$$\widehat{zx} = \frac{1}{\pi^2}\int_0^\infty d\alpha \int_0^\infty d\beta \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[\alpha z\, F_1(\xi,\eta)\sin\alpha(x-\xi)\cos\beta(y-\eta)$$
$$+\frac{\alpha\beta}{\gamma}z\, F_2(\xi,\eta)\sin\alpha(x-\xi)\sin\beta(y-\eta)$$
$$+\Big(1-\frac{\alpha^2}{\gamma}z\Big)F_3(\xi,\eta)\cos\alpha(x-\xi)\cos\beta(y-\eta)\Big]e^{-\gamma z}\,d\xi\,d\eta, \quad (79)$$

$$\widehat{xx} = \frac{1}{\pi^2}\int_0^\infty d\alpha \int_0^\infty d\beta \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[\frac{\alpha^2}{\gamma^2}\Big(1+2\sigma\frac{\beta^2}{\alpha^2}-\gamma z\Big)F_1(\xi,\eta)\cos\alpha(x-\xi)\cos\beta(y-\eta)$$
$$+\frac{\beta}{\gamma^2}\Big(2\sigma\frac{\beta^2}{\gamma}-\alpha^2 z\Big)F_2(\xi,\eta)\cos\alpha(x-\xi)\sin\beta(y-\eta)$$
$$+\frac{2\alpha}{\gamma}\Big(1+\sigma\frac{\beta^2}{\gamma^2}-\frac{\alpha^2}{2\gamma}z\Big)F_3(\xi,\eta)\sin\alpha(x-\xi)\cos\beta(y-\eta)\Big]e^{-\gamma z}\,d\xi\,d\eta, \quad (80)$$

$$\widehat{yy} = \frac{1}{\pi^2}\int_0^\infty d\alpha \int_0^\infty d\beta \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[\frac{\beta^2}{\gamma^2}\Big(1+2\sigma\frac{\alpha^2}{\beta^2}-\gamma z\Big)F_1(\xi,\eta)\cos\alpha(x-\xi)\cos\beta(y-\eta)$$
$$+\frac{2\beta}{\gamma}\Big(1+\sigma\frac{\alpha^2}{\gamma^2}-\frac{\beta^2}{2\gamma}z\Big)F_2(\xi,\eta)\cos\alpha(x-\xi)\sin\beta(y-\eta)$$
$$+\frac{\alpha}{\gamma^2}\Big(2\sigma\frac{\alpha^2}{\gamma}-\beta^2 z\Big)F_3(\xi,\eta)\sin\alpha(x-\xi)\cos\beta(y-\eta)\Big]e^{-\gamma z}\,d\xi\,d\eta, \quad (81)$$

$$\widehat{xy} = \frac{1}{\pi^2}\int_0^\infty d\alpha \int_0^\infty d\beta \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[-\frac{\alpha\beta}{\gamma^2}(1-2\sigma-\gamma z)F_1(\xi,\eta)\sin\alpha(x-\xi)\sin\beta(y-\eta)$$
$$+\frac{\alpha}{\gamma}\Big(1-2\sigma\frac{\beta^2}{\gamma^2}-\frac{\beta^2}{\gamma}z\Big)F_2(\xi,\eta)\sin\alpha(x-\xi)\cos\beta(y-\eta)$$
$$+\frac{\beta}{\gamma}\Big(1-2\sigma\frac{\alpha^2}{\gamma^2}-\frac{\alpha^2}{\gamma}z\Big)F_3(\xi,\eta)\cos\alpha(x-\xi)\sin\beta(y-\eta)\Big]e^{-\gamma z}\,d\xi\,d\eta. \quad (82)$$

Also the displacement-components can be rearranged in the forms

$$u = \frac{1}{2\pi^2\mu}\int_0^\infty d\alpha \int_0^\infty d\beta \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[\frac{\alpha}{\gamma^2}(1-2\sigma-\gamma z)F_1(\xi,\eta)\sin\alpha(x-\xi)\cos\beta(y-\eta)$$
$$-\frac{\alpha\beta}{\gamma^3}(2\sigma+\gamma z)F_2(\xi,\eta)\sin\alpha(x-\xi)\sin\beta(y-\eta)$$
$$-\frac{1}{\gamma}\Big\{2-\frac{\alpha^2}{\gamma^2}(2\sigma+\gamma z)\Big\}F_3(\xi,\eta)\cos\alpha(x-\xi)\cos\beta(y-\eta)\Big]e^{-\gamma z}\,d\xi\,d\eta, \quad (83)$$

$$v = \frac{1}{2\pi^2\mu}\int_0^\infty d\alpha \int_0^\infty d\beta \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[\frac{\beta}{\gamma^2}(1-2\sigma-\gamma z)F_1(\xi,\eta)\cos\alpha(x-\xi)\sin\beta(y-\eta)$$
$$-\frac{1}{\gamma}\Big\{2-\frac{\beta^2}{\gamma^2}(2\sigma+\gamma z)\Big\}F_2(\xi,\eta)\cos\alpha(x-\xi)\cos\beta(y-\eta)$$
$$-\frac{\alpha\beta}{\gamma^3}(2\sigma+\gamma z)F_3(\xi,\eta)\sin\alpha(x-\xi)\sin\beta(y-\eta)\Big]e^{-\gamma z}\,d\xi\,d\eta, \quad (84)$$

$$w = \frac{1}{2\pi^2\mu}\int_0^\infty d\alpha \int_0^\infty d\beta \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[-\frac{1}{\gamma}\{2(1-\sigma)+\gamma z\}F_1(\xi,\eta)\cos\alpha(x-\xi)\cos\beta(y-\eta)$$
$$-\frac{\beta}{\gamma^2}(1-2\sigma+\gamma z)F_2(\xi,\eta)\cos\alpha(x-\xi)\sin\beta(y-\eta)$$
$$-\frac{\alpha}{\gamma^2}(1-2\sigma+\gamma z)F_3(\xi,\eta)\sin\alpha(x-\xi)\cos\beta(y-\eta)\Big]e^{-\gamma z}\,d\xi\,d\eta. \quad (85)$$

In these equations σ denotes Poisson's ratio, and μ modulus of rigidity, which are connected with Young's modulus, E say, by the relation $\mu = E/2(1+\sigma)$, and $\gamma^2 = \alpha^2+\beta^2$. $F_1(x,y)$, $F_2(x,y)$ and $F_3(x,y)$ are given functions, representing the three kinds of external forces applied on the bounding plane of the semi-infinite solid. (Cf. equations (4)—(6).)

It can be seen at once that, on the bounding plane $z=0$ of the semi-infinite solid, the three components $\widehat{zz}$, $\widehat{yz}$ and $\widehat{zx}$, which are given in equations (77), (78) and (79) respectively, reduce to equations (4), (5) and (6), or to the assumed boundary conditions (1), (2) and (3). In addition all the stress- and displacement-components will vanish for large values of x, y and z; as for z, this is at once seen in virtue of the factor $e^{-\gamma z}$, both γ and z being positive, while, as for x and y, this is also valid because of the proper characteristics of Fourier's integral.

It has been verified also that the solutions (77)—(85) satisfy the three stress-equations

$$\frac{\partial \widehat{xx}}{\partial x}+\frac{\partial \widehat{xy}}{\partial y}+\frac{\partial \widehat{zx}}{\partial z}=0, \quad \cdots\cdots,$$

and the six stress-strain relations

$$\frac{\partial u}{\partial x}=\frac{1}{E}\left\{\widehat{xx}-\sigma\,(\widehat{yy}+\widehat{zz})\right\}, \quad \cdots\cdots \quad \frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}=\frac{1}{\mu}\widehat{yz}, \quad \cdots\cdots.$$

Thus the equations (77)—(85) above are the required solution of the generalized Boussinesq's problem, in which any distributions of the three kinds of external forces are given on the bounding plane of the semi-infinite elastic solid.

Now since the complete solution has been obtained in compact forms as in (77)—(85), the following remarks should be noted. That is to say, it would be rather preferable that the original assumption in the typical solutions (8) may be replaced by the alternative forms

$$\begin{aligned}\phi=&\{A_1\cos\alpha(x-\xi)\cos\beta(y-\eta)+A_2\cos\alpha(x-\xi)\sin\beta(y-\eta)\\&+A_3\sin\alpha(x-\xi)\cos\beta(y-\eta)+A_4\sin\alpha(x-\xi)\sin\beta(y-\eta)\}\,e^{-\gamma z},\\ \psi=&\{B_1\cos\alpha(x-\xi)\cos\beta(y-\eta)+B_2\cos\alpha(x-\xi)\sin\beta(y-\eta)\\&+B_3\sin\alpha(x-\xi)\cos\beta(y-\eta)+B_4\sin\alpha(x-\xi)\sin\beta(y-\eta)\}\,e^{-\gamma z},\\ \chi=&\{C_1\cos\alpha(x-\xi)\cos\beta(y-\eta)+C_2\cos\alpha(x-\xi)\sin\beta(y-\eta)\\&+C_3\sin\alpha(x-\xi)\cos\beta(y-\eta)+C_4\sin\alpha(x-\xi)\sin\beta(y-\eta)\}\,ze^{-\gamma z}.\end{aligned}$$

Several examples of the general solution (77)—(85) will appear in the succeeding issue of this Journal, together with their numerical evaluation. It is added that an abridged version of the present work has appeared elsewhere[8].

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