A Note on the Stress Distribution in a Plate Flanged with a Circular Plug

By Shotaro NATSUME and Bennosuke TANIMOTO

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Synopsis. This article gives a note on the old problem that a large plate flanged with a circular plug is subjected to a pair of edge forces. It is based on the principle of the minimum strain-energy of the whole system, by allowing the discrepancy between tangential displacements at the junction of the plate and the plug. Numerical calculation reveals a considerable increase in the maximum circumferential tensile stress.

The analytical solution for a plate, with a circular hole and no insertion in it, was given by G. Kirsch as old as in 1898[1], and the same problem was independently treated by S. Timoshenko[2]. It is well known that the maximum circumferential normal stress is three times the stress at infinity for tension, and four times the stress at infinity for shear, and occurs at the edge of the hole. This is the first and oldest among solutions of the "perforated problems."

The present article gives a note on the stress distribution around a circular hole in a plate subjected to a pair of parallel and uniform edge forces, the hole being inserted with a heterogenous solid plug. The problem is said to be related with the practical construction that a plate is riveted by a solid bolt.

The similar problem that a heterogeneous circular ring is inserted will be related to the so-called reinforcement of a plate. This often occurs in practice especially in aeroplane construction, where welding of material is of no use, and the plate is subjected to a considerable magnitude of tension and in addition to a strong vibration.

These two problems have been treated by the late K. Sezawa and Professor G. Nishimura of Tokyo University by means of their method of dilatation and rotation since more than twenty years ago[2]. Later C. Gurney treated independently the latter problem, viz., a plate with a concentric circular ring, by means of Airy's stress-function[3].

These two kinds of solutions are of course in accordance with each other provided that several misprints are corrected, since the boundary conditions assumed are the same. These are based on the perfect continuity of stresses and displacements along the circumferential boundary, which would seem to involve a careless oversight.

In this regard the late K. Suyehiro treated the problem on the assumption that there is no frictional resistance along the circumferential boundary and that the discrepancy between tangential displacements along the boundary may occur under the restriction that the plate and the circular plug are of the same material[4]. It will be seen, however, that this solution is based on a rather artificial assumption, and that from the theoretical point of view such a boundary condition can by no means be realized, the reason being in the violence of the principle of minimum strain-energy.

Gurney worked out many numerical examples with elaborate but plausible—according to the authors' viewpoint—curves, and he insisted on the reinforcement around a hole by the insertion of a concentric ring which is thicker than the surrounding plate. But this will call for a scrutiny as will be seen later on.

The present device for attacking this problem lies in our intuition that would admit of the existence of the case in which no separation at the junction takes place, which is meant by the perfect continuity of $\widehat{rr}$, $\widehat{r\theta}$ and u (radial displacement) referred to plane polar coordinates. But as for the fourth condition for v (tangential displacement), the perfect continuity would be open to doubt. In fact, if a loose contact at the junction is realized, the tangential displacements there may easily be discrepant, whilst if a firm and perfect contact is secured, no sliding takes place.

It should then be assumed that the fourth condition cited is replaced by that of minimum strain-energy of the whole system considered. It follows that the possible state of stress distribution is not unique, but is between the two cited conditions; that is, these two give the limiting states of stress distribution.

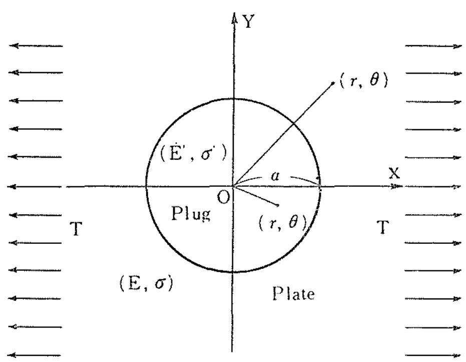

Fig. 1.

Boundary conditions for the present problem are here taken to be (Fig. 1):

1. At the junction of the plug and the plate, where $r = a$,

$$\left.\begin{aligned}&(\widehat{rr})_{r=a}=(\widehat{rr}')_{r=a},\quad (\widehat{r\theta})_{r=a}=(\widehat{r\theta}')_{r=a},\quad (u)_{r=a}=(u')_{r=a},\\&(v)_{r=a}=(v')_{r=a}+V\sin 2\theta,\end{aligned}\right\}\quad(1)$$

in which tangential displacements there may be discrepant, and the quantity V in the last equation will be determined from the condition of minimum strain-energy of the whole elastic system.

2. At an indefinitely great distance from the origin of coordinates, the plate is subjected to a pair of parallel and uniform tensile forces, say T, in the direction of the axis of x, which is expressed by

$$(\widehat{xx})_{r=\infty}=T,\qquad (\widehat{yy})_{r=\infty}=0,\qquad (\widehat{xy})_{r=\infty}=0.\qquad (2)$$

In the above and in what follows, quantities with no prime refer to the plate and those with prime to the plug.

Referring to plane polar coordinates, Airy's stress-functions suitable for the above boundary-value problem are as follows:

1) Stress-function for the plate:

$$\chi(r,\theta)=A_0r^2+B_0\log r+(Br^2+C+Dr^{-2})\cos 2\theta.\qquad (3)$$

2) Stress-function for the plug:

$$\chi'(r,\theta)=A_0'r^2+(A'r^4+B'r^2)\cos 2\theta.\qquad (4)$$

Here $A_0, B_0, \cdots D$ and A_0', A', B' are constants to be determined from the boundary conditions (1); the coordinates (r, θ) being taken as indicated in Fig. 1.

Stresses are derived by the operations

$$\widehat{rr}=\frac{1}{r}\frac{\partial\chi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\chi}{\partial\theta^2},\qquad \widehat{\theta\theta}=\frac{\partial^2\chi}{\partial r^2},\qquad \widehat{r\theta}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi}{\partial\theta}\right);\qquad (5)$$

$$\widehat{rr}'=\frac{1}{r}\frac{\partial\chi'}{\partial r}+\frac{1}{r^2}\frac{\partial^2\chi'}{\partial\theta^2},\qquad \widehat{\theta\theta}'=\frac{\partial^2\chi'}{\partial r^2},\qquad \widehat{r\theta}'=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi'}{\partial\theta}\right).\qquad (6)$$

Then the substitution of (3) into (5) affords

$$\left.\begin{aligned}\widehat{rr}&=\left(2A_0+\frac{B_0}{r^2}\right)+\left(-2B-4\frac{C}{r^2}-6\frac{D}{r^4}\right)\cos 2\theta,\\ \widehat{\theta\theta}&=\left(2A_0-\frac{B_0}{r^2}\right)+\left(2B+6\frac{D}{r^4}\right)\cos 2\theta,\\ \widehat{r\theta}&=\left(2B-2\frac{C}{r^2}-6\frac{D}{r^4}\right)\sin 2\theta,\end{aligned}\right\}\qquad (7)$$

which are the expressions for the plate. The substitution of (4) into (6) affords

$$
\left.\begin{aligned}
\widehat{rr}' &= 2A_0' + (\qquad\qquad -2B')\cos 2\theta,\\
\widehat{\theta\theta}' &= 2A_0' + (12A'r^2 + 2B')\cos 2\theta,\\
\widehat{r\theta}' &= \qquad\quad (6A'r^2 + 2B')\sin 2\theta,
\end{aligned}\right\} \tag{8}
$$

which are the expressions for the plug.

Expressions for displacements can in general be obtained by the operations

$$
\left.\begin{aligned}
u &= \int e_{rr}dr + f_1(\theta),\\
v &= r\int e_{\theta\theta}d\theta - \iint e_{rr}drd\theta - \int f_1(\theta)d\theta + f_2(r),
\end{aligned}\right\} \tag{9}
$$

where strains are defined by

$$
e_{rr} = \frac{\partial u}{\partial r}, \qquad e_{\theta\theta} = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad e_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}.
$$

In an elastic solid slightly strained from the unstressed state the components of strain are linear functions of the components of stress, and then, assuming the plane-stress state, the stress-strain relations are

$$
e_{rr} = \frac{1}{E}\left(\widehat{rr} - \sigma\widehat{\theta\theta}\right), \qquad e_{\theta\theta} = \frac{1}{E}\left(\widehat{\theta\theta} - \sigma\widehat{rr}\right), \qquad e_{r\theta} = \frac{2(1+\sigma)}{E}\widehat{r\theta}.
$$

Then after some amount of calculations, the expressions for displacements become as follows:

1) Displacements in the plate:

$$
\left.\begin{aligned}
u &= \frac{1}{E}\Big[2(1-\sigma)A_0 r - (1+\sigma)\frac{B_0}{r}\\
&\qquad + \left\{-2(1+\sigma)Br + 4\frac{C}{r} + 2(1+\sigma)\frac{D}{r^3}\right\}\cos 2\theta\Big],\\
v &= \frac{1}{E}\left\{2(1+\sigma)Br - 2(1-\sigma)\frac{C}{r} + 2(1+\sigma)\frac{D}{r^3}\right\}\sin 2\theta.
\end{aligned}\right\} \tag{10}
$$

2) Displacements in the plug:

$$
\left.\begin{aligned}
u' &= \frac{1}{E'}\left[2(1-\sigma')A_0'r + \left\{-4\sigma' A'r^3 - 2(1+\sigma')B'r\right\}\cos 2\theta\right],\\
v' &= \frac{1}{E'}\left\{2(3+\sigma')A'r^3 + 2(1+\sigma')B'r\right\}\sin 2\theta.
\end{aligned}\right\} \tag{11}
$$

Here E and σ denote respectively Young's modulus and Poisson's ratio for the plate, and E' and σ' those for the plug. It can be shown that the arbitrary functions $f_1(\theta)$ and $f_2(r)$ in (9) vanish, when there is no rigid displacement as a whole of the solid considered. This is the reason why these terms are suppressed in equations (10) and (11).

The total strain-energy, say W, of the system is the sum of that in the plate and that in the plug, so that we have

$$W=\frac{1}{2E}\oint\int_a^\infty\left[\left(\widehat{rr}^2+\widehat{\theta\theta}^2\right)-2\sigma\widehat{rr}\,\widehat{\theta\theta}+2(1+\sigma)\widehat{r\theta}^2\right]rdrd\theta$$
$$+\frac{1}{2E'}\oint\int_0^a\left[\left(\widehat{rr}'^2+\widehat{\theta\theta}'^2\right)-2\sigma'\widehat{rr}'\,\widehat{\theta\theta}'+2(1+\sigma')\widehat{r\theta}'^2\right]rdrd\theta. \quad (12)$$

Then we must have

$$\frac{\partial W}{\partial V}=0, \quad (13)$$

which implies that the total strain-energy is a minimum.

The transformation of stress systems is in general effected by the equations

$$\left.\begin{aligned}\widehat{rr}&=\frac{1}{2}\left(\widehat{xx}+\widehat{yy}\right)+\frac{1}{2}\left(\widehat{xx}-\widehat{yy}\right)\cos 2\theta+\widehat{xy}\sin 2\theta,\\ \widehat{\theta\theta}&=\frac{1}{2}\left(\widehat{xx}+\widehat{yy}\right)-\frac{1}{2}\left(\widehat{xx}-\widehat{yy}\right)\cos 2\theta-\widehat{xy}\sin 2\theta,\\ \widehat{r\theta}&=\qquad\qquad\qquad\;-\frac{1}{2}\left(\widehat{xx}-\widehat{yy}\right)\sin 2\theta+\widehat{xy}\cos 2\theta.\end{aligned}\right\}$$

Then (2) is written in the forms

$$\left.\begin{aligned}&\left(\widehat{rr}\right)_{r=\infty}=\frac{T}{2}+\frac{T}{2}\cos 2\theta, \qquad \left(\widehat{\theta\theta}\right)_{r=\infty}=\frac{T}{2}-\frac{T}{2}\cos 2\theta,\\ &\left(\widehat{r\theta}\right)_{r=\infty}=-\frac{T}{2}\sin 2\theta.\end{aligned}\right\} \quad (14)$$

On the other hand, at a great distance from the origin, equations (7) reduce to

$$\left.\begin{aligned}\left(\widehat{rr}\right)_{r=\infty}&=2A_0-2B\cos 2\theta,\\ \left(\widehat{\theta\theta}\right)_{r=\infty}&=2A_0+2B\cos 2\theta,\\ \left(\widehat{r\theta}\right)_{r=\infty}&=\qquad\;\; 2B\sin 2\theta.\end{aligned}\right\}$$

On comparing these with (14), we must have

$$2A_0 = \frac{T}{2}, \qquad 2B = -\frac{T}{2}. \tag{15}$$

Equations (7) and (10) take for the present the forms

$$\left.\begin{aligned}
\widehat{rr} &= \left(\frac{T}{2} + \frac{B_0}{r^2}\right) + \left(\frac{T}{2} - 4\frac{C}{r^2} - 6\frac{D}{r^4}\right)\cos 2\theta, \\
\widehat{\theta\theta} &= \left(\frac{T}{2} - \frac{B_0}{r^2}\right) + \left(-\frac{T}{2} \qquad + 6\frac{D}{r^4}\right)\cos 2\theta, \\
\widehat{r\theta} &= \qquad\qquad\quad \left(-\frac{T}{2} - 2\frac{C}{r^2} - 6\frac{D}{r^4}\right)\sin 2\theta;
\end{aligned}\right\} \tag{16}$$

$$\left.\begin{aligned}
u &= \frac{r}{E}\left[(1-\sigma)\frac{T}{2} - (1+\sigma)\frac{B_0}{r^2} + \left\{(1+\sigma)\frac{T}{2} + 4\frac{C}{r^2} + 2(1+\sigma)\frac{D}{r^4}\right\}\cos 2\theta\right], \\
v &= \frac{r}{E}\left\{-(1+\sigma)\frac{T}{2} - 2(1-\sigma)\frac{C}{r^2} + 2(1+\sigma)\frac{D}{r^4}\right\}\sin 2\theta.
\end{aligned}\right\} \tag{17}$$

The next step to the problem is to determine the remaining constants

$$B_0, \ A_0'; \ C, \ D, \ A', \ B',$$

all of which will involve the unknown constant V in linear form. These constants can be determined from the boundary conditions (1), which gives rise with (16), (8), (17) and (11) to the conditional equations

$$\frac{T}{2} + \frac{B_0}{a^2} = 2A_0', \qquad \frac{T}{2} - 4\frac{C}{a^2} - 6\frac{D}{a^4} = -2B',$$

$$-\frac{T}{2} - 2\frac{C}{a^2} - 6\frac{D}{a^4} = 6A'a^2 + 2B',$$

$$\frac{1}{E}\left[(1-\sigma)\frac{T}{2} - (1+\sigma)\frac{B_0}{a^2}\right] = \frac{1}{E'}2(1-\sigma')A_0',$$

$$\frac{1}{E}\left[(1+\sigma)\frac{T}{2} + 4\frac{C}{a^2} + 2(1+\sigma)\frac{D}{a^4}\right] = \frac{1}{E'}\left[-4\sigma'A'a^2 - 2(1+\sigma')B'\right],$$

$$\frac{1}{E}\left[-(1+\sigma)\frac{T}{2} - 2(1-\sigma)\frac{C}{a^2} + 2(1+\sigma)\frac{D}{a^4}\right]$$
$$= \frac{1}{E'}\left[2(3+\sigma')A'a^2 + 2(1+\sigma')B'\right] + \frac{V}{a}.$$

These equations can be solved simultaneously, and be rearranged in the following forms:

$$\begin{Vmatrix}
\dfrac{B_0}{a^2} & 2A_0' & \dfrac{T}{2} \\
1 & -1 & 1 \\
1 & \dfrac{1-\sigma'}{1+\sigma'}n & -\dfrac{1-\sigma}{1+\sigma}
\end{Vmatrix} = 0, \tag{18}$$

$$\begin{Vmatrix} \dfrac{C}{a^2} & \dfrac{D}{a^4} & A' a^2 & B' & \dfrac{T}{4} \\ 1 & 1 & -1 & -1 & -1 \\ 1 & 2 & 1 & 0 & 0 \\ \dfrac{2}{1+\sigma} & 1 & \dfrac{2n\sigma'}{1+\sigma'} & n & 1 \\ 0 & 0 & -1-\dfrac{n(3-\sigma')}{1+\sigma'} & 0 & -\dfrac{4\mu}{Ta}V \end{Vmatrix} = 0, \quad (19)$$

where

$$n = \frac{\mu}{\mu'}, \quad \mu = \frac{E}{2(1+\sigma)}, \quad \mu' = \frac{E'}{2(1+\sigma')}, \tag{20}$$

so that, if $n < 1$, the plug is more rigid than the surrounding plate; and vice versa. On solving (18) and (19), we obtain

$$\left.\begin{aligned} \frac{B_0}{a^2} &= \frac{1}{\varDelta_0}\left(\frac{1-\sigma}{1+\sigma} - \frac{1-\sigma'}{1+\sigma'}n\right)\frac{T}{2}, \qquad A_0' = \frac{1}{\varDelta_0}\frac{1}{1+\sigma}\frac{T}{2}, \\ \varDelta_0 &= \frac{1+\sigma'+(1-\sigma')n}{1+\sigma'}; \end{aligned}\right\} \tag{21}$$

$$\left.\begin{aligned} \frac{C}{a^2} &= \frac{1}{\varDelta}\left[-2(1-n)\left(1-n+\frac{4n}{1+\sigma'}\right)\frac{T}{4} + \left(1-n+\frac{4n}{1+\sigma'}\right)\left(\frac{-\mu}{a}V\right)\right], \\ \frac{D}{a^4} &= \frac{1}{\varDelta}\left[(1-n)\left(1-n+\frac{4n}{1+\sigma'}\right)\frac{T}{4} + 2\left(-\frac{1}{1+\sigma} - \frac{n}{1+\sigma'}\right)\left(\frac{-\mu}{a}V\right)\right], \\ A' a^2 &= \frac{1}{\varDelta}\left(-1+n+\frac{4}{1+\sigma}\right)\left(\frac{-\mu}{a}V\right), \\ B' &= \frac{1}{\varDelta}\left[\frac{-4}{1+\sigma}\left(1-n+\frac{4n}{1+\sigma'}\right)\frac{T}{4} + 2\left(1-n-\frac{3}{1+\sigma}+\frac{n}{1+\sigma'}\right)\left(\frac{-\mu}{a}V\right)\right], \\ \varDelta &= \left(-1+n+\frac{4}{1+\sigma}\right)\left(1-n+\frac{4n}{1+\sigma'}\right); \end{aligned}\right\} \tag{22}$$

$\varDelta_0$ and $\varDelta$ being determinants of the simultaneous equations (18) and (19) respectively.

Our last step to the problem is to determine V in (22), which will be effected by (13). Equation (12) which is for W is a quadratic form with respect to stresses, and equations (22) are of linear form with respect to V, and hence (13) will give rise to a linear equation for V. After some amount of calculation, we thus obtain

$$V=\frac{\frac{3-\sigma}{1+\sigma}\varphi\,\phi_1+n\varphi'\,(\phi_1'+2\,\phi_2')}{\frac{3+\sigma}{1+\sigma}\phi_1^2+2\,\phi_1\phi_2+\frac{2}{3}\phi_2^2+n\left\{\frac{3+\sigma'}{3(1+\sigma')}\phi_1'^2+2\phi_1'\phi_2'+2\phi_2'^2\right\}}\frac{T}{2}. \quad (23)$$

Hence the resulting stresses become as follows:

1) Stresses in the plate $(r\geqq a)$:

$$\left.\begin{aligned}\widehat{rr}&=\frac{T}{2}\left[1+\frac{f}{\rho^2}+\left\{1+\left(-\frac{4}{\rho^2}+\frac{3}{\rho^4}\right)\varphi+\left(\frac{2\,\phi_1}{\rho^2}+\frac{\phi_2}{\rho^4}\right)K\right\}\cos 2\,\theta\right],\\ \widehat{\theta\theta}&=\frac{T}{2}\left[1-\frac{f}{\rho^2}+\left(-1-\frac{3\varphi}{\rho^4}-\frac{\phi_2}{\rho^4}K\right)\cos 2\,\theta\right],\\ \widehat{r\theta}&=\frac{T}{2}\left\{-1+\left(-\frac{2}{\rho^2}+\frac{3}{\rho^4}\right)\varphi+\left(\frac{\phi_1}{\rho^2}+\frac{\phi_2}{\rho^4}\right)K\right\}\sin 2\,\theta.\end{aligned}\right\} \quad (24)$$

2) Stresses in the plug $(r\leqq a)$:

$$\left.\begin{aligned}\widehat{rr}'&=\frac{T}{2}\left\{f'+\left(-\varphi'+\phi_2'K\right)\cos 2\,\theta\right\},\\ \widehat{\theta\theta}'&=\frac{T}{2}\left[f'+\left\{\varphi'+(-2\,\phi_1'\rho^2-\phi_2')K\right\}\cos 2\,\theta\right],\\ \widehat{r\theta}'&=\frac{T}{2}\left\{\varphi'+(-\phi_1'\rho^2-\phi_2')\,K\right\}\sin 2\,\theta,\end{aligned}\right\} \quad (25)$$

where, for shortness,

$$\rho=\frac{r}{a},$$

and f, f', φ, φ', ϕ_1, ϕ_2, ϕ_1', ϕ_2' are represented by certain functions of σ, σ' and n, which follow.

$$f=\frac{1}{\Delta_0}\left(\frac{1-\sigma}{1+\sigma}-\frac{1-\sigma'}{1+\sigma'}n\right),\qquad f'=\frac{1}{\Delta_0}\frac{2}{1+\sigma},$$

$$\varphi=-\frac{1}{\Delta}(1-n)\left(1-n+\frac{4n}{1+\sigma'}\right),\qquad \varphi'=-\frac{4}{\Delta(1+\sigma)}\left(1-n+\frac{4n}{1+\sigma'}\right),$$

$$\phi_1=\frac{2}{\Delta}\left(1-n+\frac{4n}{1+\sigma'}\right)\frac{\mu}{a},\qquad \phi_2=\frac{12}{\Delta}\left(-\frac{1}{1+\sigma}-\frac{n}{1+\sigma'}\right)\frac{\mu}{a},$$

$$\phi_1'=\frac{6}{\Delta}\left(-1+n+\frac{4}{1+\sigma}\right)\frac{\mu}{a},\qquad \phi_2'=\frac{4}{\Delta}\left(1-n-\frac{3}{1+\sigma}+\frac{n}{1+\sigma'}\right)\frac{\mu}{a},$$

$$K=\frac{\frac{3-\sigma}{1+\sigma}\varphi\phi_1+n\varphi'\,(\phi_1'+2\,\phi_2')}{\frac{3+\sigma}{1+\sigma}\phi_1^2+2\,\phi_1\phi_2+\frac{2}{3}\phi_2^2+n\left\{\frac{3+\sigma'}{3(1+\sigma')}\phi_1'^2+2\,\phi_1'\phi_2'+2\,\phi_2'^2\right\}}.$$

It can easily be seen that, if in (24) and (25) above we put $V=0$, the above solution reduces to the ordinary one which is due to K. Sezawa and Professor G. Nishimura[3]. The effect of V on the stress distribution in the plate is, so far as our numerical calculations have revealed, not desirable, that is, it makes the maximum circumferential stress greater to a considerable degree. The same thing is true in the case of the ring insertion. This will be reserved for the subsequent paper.

Numerical examples will be given, in which we take $\sigma'=\sigma=0.25$ throughout. Three of numerical examples will be for $n=\mu/\mu'=1.0$, 1/1.5, 1/2.0.

Example 1. We take

$$\mu'=\mu, \qquad \sigma'=\sigma=0.25, \qquad \text{so that } E'=E.$$

In this case equations (24) and (25) reduce to:

1) Stresses in the plate ($r \geqq a$, or $\rho \geqq 1$):

$$\left.\begin{aligned} \widehat{rr} &= T\left[0.5+\left\{0.5+\left(-\frac{0.156}{\rho^2}+\frac{0.234}{\rho^4}\right)\right\}\cos 2\theta\right], \\ \widehat{\theta\theta} &= T\left[0.5+\left(-0.5-\frac{0.234}{\rho^4}\right)\cos 2\theta\right], \\ \widehat{r\theta} &= T\left\{-0.5+\left(-\frac{0.078}{\rho^2}+\frac{0.234}{\rho^4}\right)\right\}\sin 2\theta. \end{aligned}\right\}$$

2) Stresses in the plug ($r \geqq a$, or $\rho \leqq 1$):

$$\left.\begin{aligned} \widehat{rr}' &= T\left[0.5+\left\{0.5+(0.078)\right\}\cos 2\theta\right], \\ \widehat{\theta\theta}' &= T\left[0.5+\left\{-0.5+(0.469\,\rho^2-0.078)\right\}\cos 2\theta\right], \\ \widehat{r\theta}' &= T\left\{-0.5+(0.234\rho^2-0.078)\right\}\sin 2\theta. \end{aligned}\right\}$$

In the above solution, the terms in parentheses ( ) express the effect of the discrepancy between tangential displacements. If these terms are suppressed, then the solution reduces to the one for an even plate with no hole and no plug insertion, and the cited effect at the junction entirely disappears.

Fig. 5 shows the circumferential stress $\widehat{\theta\theta}$ at the junction, in which the maximum stress is increased by 23% compared with that of the ordinary solution.

Fig. 6 shows the change in $\widehat{\theta\theta}$ along radius vectors for several angles from the direction of edge forces, in which we see rapid convergences to the values of the ordinary solution.

Example 2. We take

$$\mu' = 1.5\,\mu, \quad \sigma' = \sigma = 0.25, \quad \text{so that} \quad E' = 1.5E.$$

In this case equations (24) and (25) reduce to:

1) Stresses in the plate ($r \geqq a$, or $\rho \geqq 1$):

$$\left.\begin{aligned}
\widehat{rr} &= T\left[0.5 + \frac{0.071}{\rho^2} + \left\{0.5 + \left(\frac{0.233}{\rho^2} - \frac{0.174}{\rho^4}\right) + \left(-\frac{0.049}{\rho^2} + \frac{0.079}{\rho^4}\right)\right\}\cos 2\theta\right],\\
\widehat{\theta\theta} &= T\left[0.5 - \frac{0.071}{\rho^2} + \left\{-0.5 + \frac{0.174}{\rho^4} + \left(\frac{0.079}{\rho^4}\right)\right\}\cos 2\theta\right],\\
\widehat{r\theta} &= T\left\{-0.5 + \left(\frac{0.115}{\rho^2} - \frac{0.174}{\rho^4}\right) + \left(-\frac{0.025}{\rho^2} + \frac{0.079}{\rho^4}\right)\right\}\sin 2\theta.
\end{aligned}\right\}$$

2) Stresses in the plug ($r \leqq a$, or $\rho \leqq 1$):

$$\left.\begin{aligned}
\widehat{rr}' &= T\left[0.571 + \left\{0.559 + (0.030)\right\}\cos 2\theta\right],\\
\widehat{\theta\theta}' &= T\left[0.571 + \left\{-0.559 + (0.170\rho^2 - 0.030)\right\}\cos 2\theta\right],\\
\widehat{r\theta}' &= T\left\{-0.559 + (0.085\rho^2 - 0.030)\right\}\sin 2\theta.
\end{aligned}\right\}$$

Example 3. We take

$$\mu' = 2.0\,\mu, \quad \sigma' = \sigma = 0.25, \quad \text{so that} \quad E' = 2.0E.$$

In this case equations (24) and (25) reduce to:

1) Stresses in the plate ($r \geqq a$, or $\rho \geqq 1$):

$$\left.\begin{aligned}
\widehat{rr} &= T\left[0.5 + \frac{0.115}{\rho^2} + \left\{0.5 + \left(\frac{0.370}{\rho^2} - \frac{0.277}{\rho^4}\right) + \left(-\frac{0.062}{\rho^2} + \frac{0.107}{\rho^4}\right)\right\}\cos 2\theta\right],\\
\widehat{\theta\theta} &= T\left[0.5 - \frac{0.115}{\rho^2} + \left\{-0.5 + \frac{0.277}{\rho^4} + \left(-\frac{0.107}{\rho^4}\right)\right\}\cos 2\theta\right],\\
\widehat{r\theta} &= T\left\{-0.5 + \left(\frac{0.185}{\rho^2} - \frac{0.277}{\rho^4}\right) + \left(-\frac{0.031}{\rho^2} + \frac{0.107}{\rho^4}\right)\right\}\sin 2\theta.
\end{aligned}\right\}$$

2) Stresses in the plug ($r \leqq a$, or $\rho \leqq 1$):

$$\widehat{rr}' = T\left[0.615 + \left\{0.593 + (0.044)\right\}\cos 2\theta\right],$$

$$\left.\begin{aligned}\widehat{\theta\theta}' &= T\Big[0.615+\Big\{-0.593+(0.240\,\rho^2-0.044)\Big\}\cos 2\,\theta\Big],\\ \widehat{r\theta}' &= T\Big\{-0.593+(0.120\,\rho^2-0.044)\Big\}\sin 2\,\theta.\end{aligned}\right\}$$

The last terms in parentheses ( ) on the right-hand side in these equations express also the effect of the discrepancy between tangential displacements at the junction of the plate and the plug. If these terms be suppressed, all the above solutions reduce to the ordinary ones due to K. Sezawa and G. Nishimura. Figs. 5, 6 and 7 were drawn for illustration by the above solutions, Examples 1—3.

From the standpoint of the total strain-energy of the elastic system, the situation of the three kinds of solution is figuratively illustrated in Fig. 2.

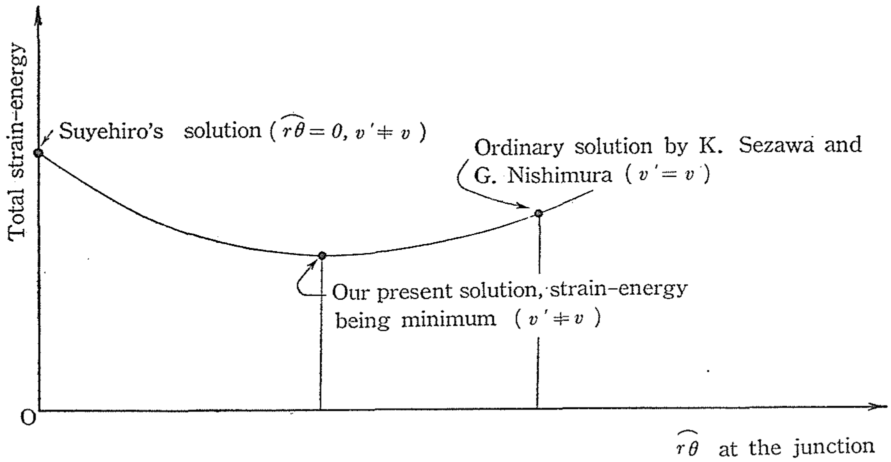

Fig. 2. Figurative illustration for the situation of the three solutions

That the maximum tensile stress does increase compared with the ordinary solution will be due to the fact that the tangential displacement

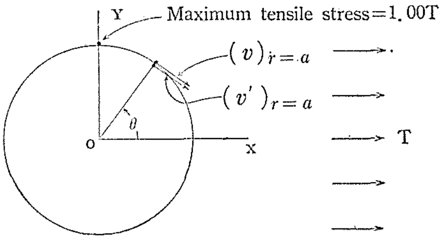

Fig. 3. Ordinary solution
($E'=E,\ \sigma'=\sigma=0.25$)

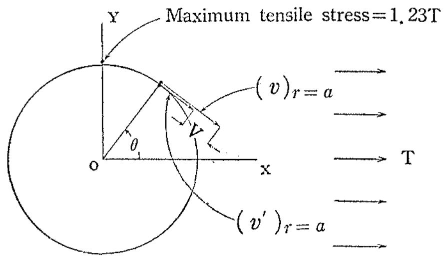

Fig. 4. Our present solution
($E'=E,\ \sigma'=\sigma=0.25$)

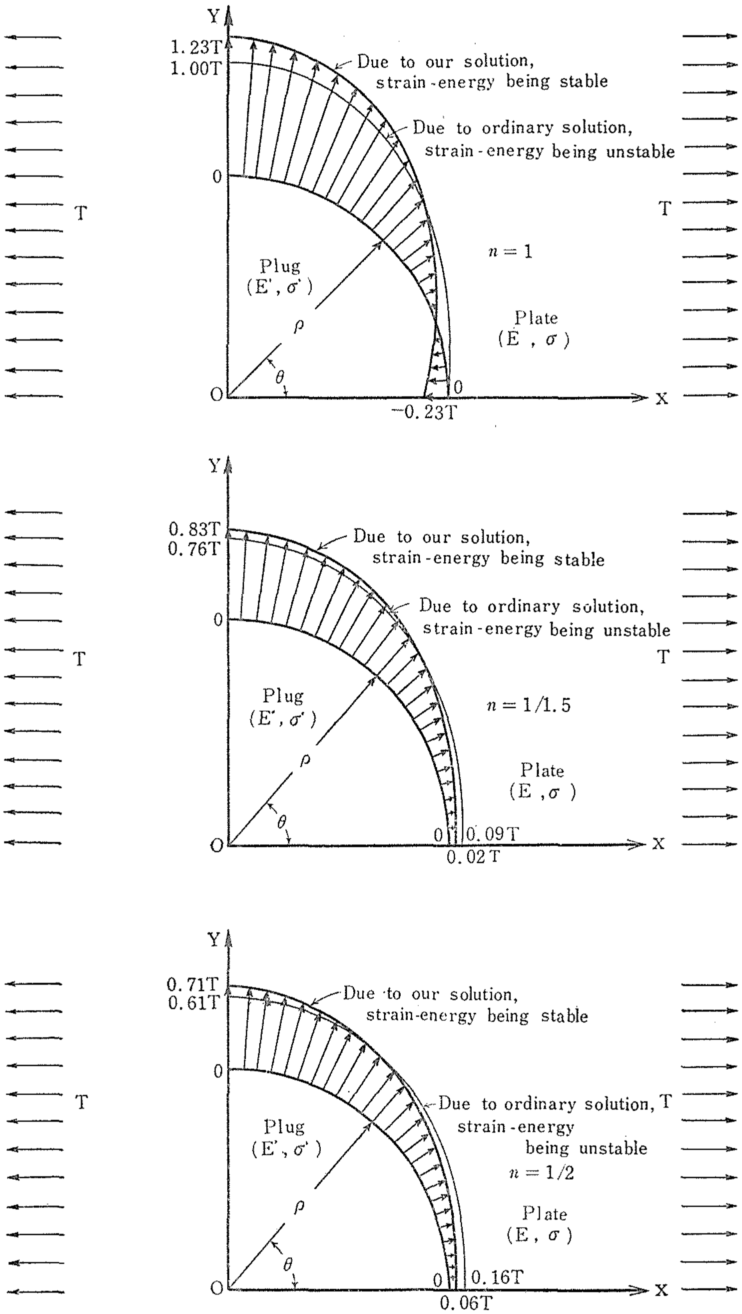

Fig. 5. Circumferential normal stress $\widehat{\theta\theta}$ at the junction of the plate and the plug ($\sigma' = \sigma = 0.25$)

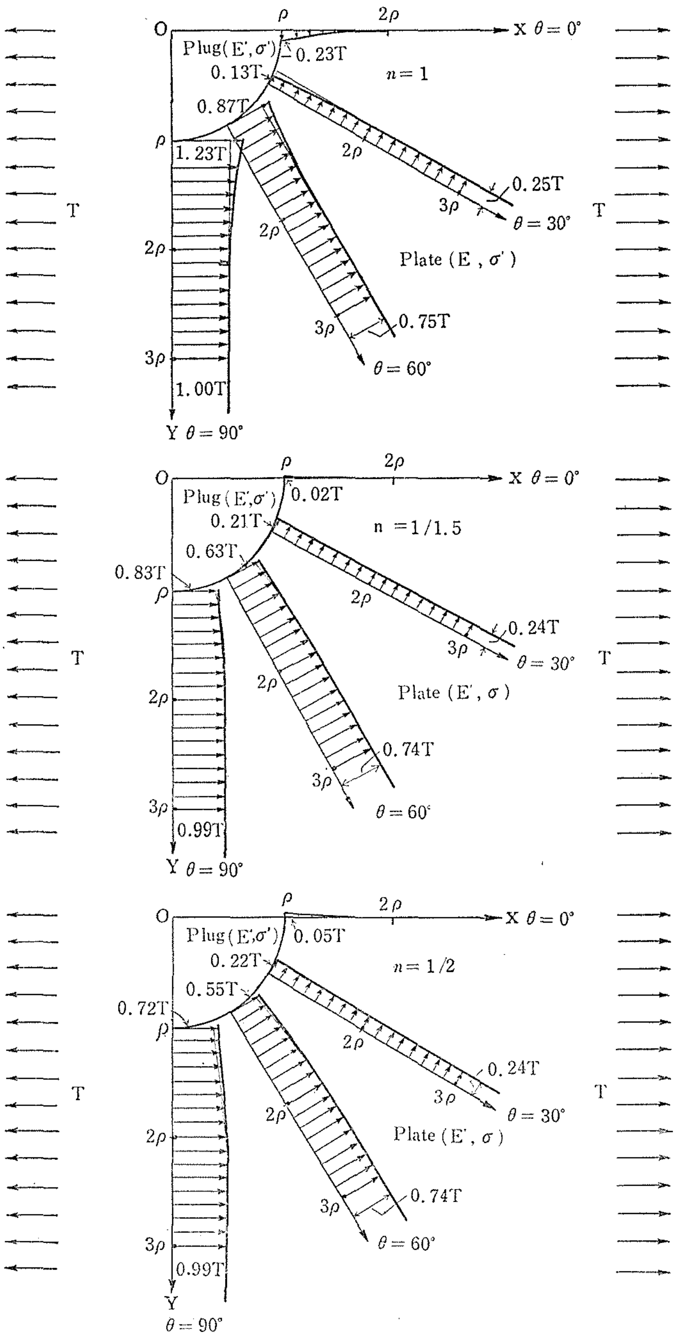

Fig. 6. Change in $\widehat{\theta\theta}$ along radius vectors for several angles from the direction of edge forces ($\sigma' = \sigma = 0.25$)

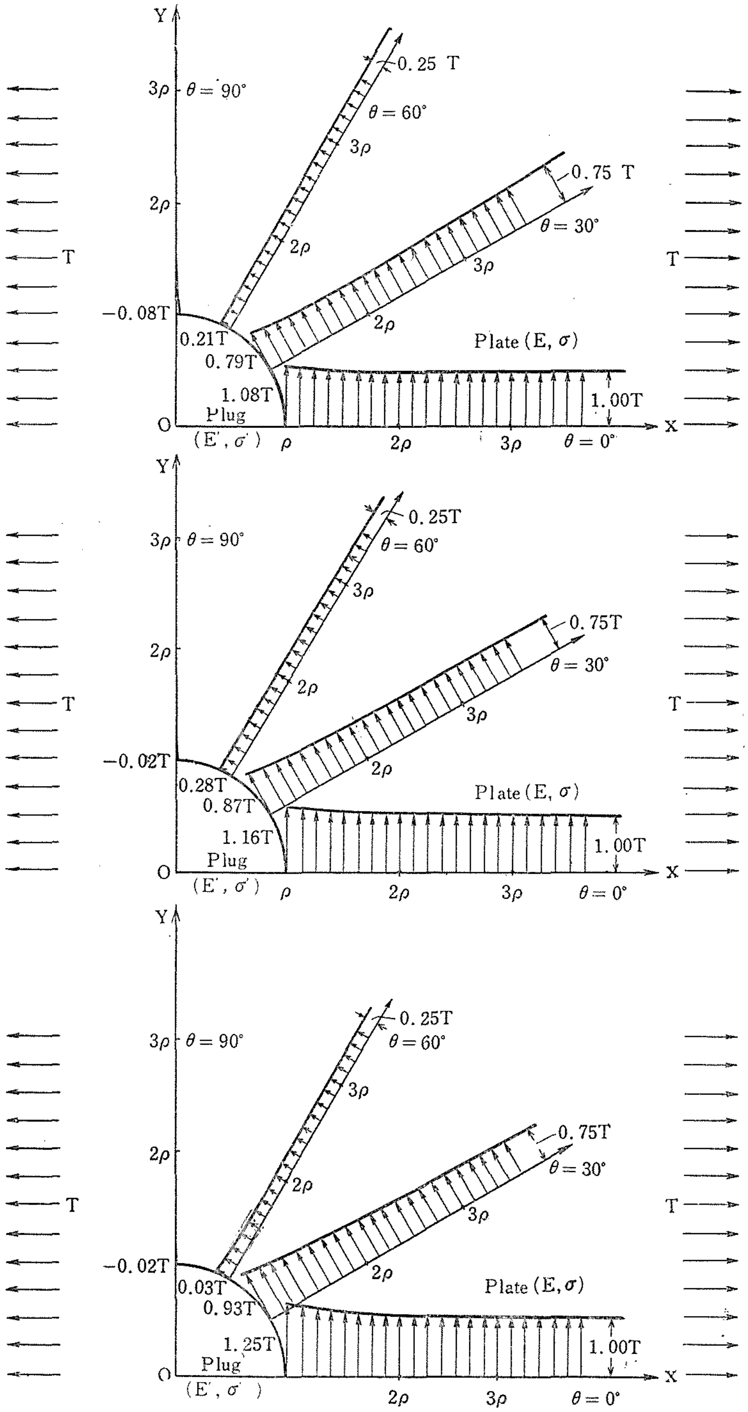

Fig. 7. Change in $\widehat{rr}$ along radius vectors for several angles from the direction of edge forces ($\sigma' = \sigma = 0.25$)

at the junction of the plate is greater than that of the ordinary solution, as shown in Figs. 3 and 4. In fact, it may naturally be inferred that the occurrence of the difference, V, between tangential displacements at the junction in Fig. 4 results in greater maximum tensile stress than that in Fig. 3, where no difference between tangential displacements occurs.

Emphasis should be laid on the meaning of the term "boundary conditions." In such a boundary-value problem as the present one, we say boundary conditions at infinity $r=\infty$, and at the junction $r=a$. The former may, to be sure, be called boundary conditions, while the latter should not be termed boundary conditions in the true meaning of word. That is to say, in the former case our intuition decisively admits of the only one state of given external forces. Contrarily, in the latter case we may assume many possible, but at times plausible, conditions. Our present solution may be justified, when a certain amount of initial radial compression exists at the junction of the plate and the plug, so that no radial tension may occur there after strain. If there is no such initial compression, then a separation at the junction may easily occur after strain in the case of monocoque construction. But, if a firm contact at the junction is secured by means of welding or the like, then no separation will occur in spite of the tensile stress.

The possible behaviour of the elastic system at the neighbourhood of the junction is in reality that the discrepancy between radial displacements also occurs; that is to say, along some part of the circumference at the junction, the radial normal stress ($\widehat{rr}$ or $\widehat{rr'}$) will be a tension, while along other part of it, this will be a compression. In the case of monocoque construction, which is one of prevailing practices, the occurrence of such behaviour is to be expected.

In this respect, the present work would not represent the true and possible state of the elastic system considered. But, it can point out at least the unreasonableness of the ordinary solution in a certain case, and the true maximum tensile stress will be greater than 1.23 T (Fig. 4) owing to the occurrence of the separation at the junction of the plate and the plug. We have reason to assume, however, that the effect of the separation between radial displacements would be less than that between tangential displacements. For Fig. 7 above indicates an insignificant change in the radial normal stresses compared with those of the ordinary solution, and moreover a Coker and Filon's treatment also suggests the same thing[7].

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