The Derivation of the Proposed Stress-Functions in Three Dimensions

By Bennosuke TANIMOTO*, Dr. Eng.

Synopsis. This is a detailed description of the derivation of my proposed stress-functions in three-dimensional elasticity, which have been briefly reported on the Bulletin of the Earthquake Research Institute, Tokyo University, 1948. The resulting equations are that certain operations are performed on one biharmonic and one harmonic. It is noted that the fundamental equations have successfully been applied to solving the generalized Boussinesq's problem of elastic foundation.

1. In the development of the theory of elasticity we have been much occupied with bodies in equilibrium under forces applied over their surfaces only. In the case of two dimensions Airy's stress-function has been found, which is of some simple construction, and its many interesting attainments have been accumulated.

This paper is devoted to the derivation of the proposed stress-functions in three dimensions when elastic solids are in a state of equilibrium. In this regard there have been proposed the so-called Maxwell's stress-functions and the Morera's,[1] either of which seems to be widely accepted as the most general expressions for stress-components. They are, to be sure, completely general, so long as the stress-equations only are concerned, but would not necessarily be general for the whole system of our fundamental equations, which consists of the stress-equations, the compatibility conditions of strain, and the stress-strain relations.

In fact it is obvious that our true aim is to attack the whole system in question, and not the stress-equations alone. The unknown quantities in question are fifteen in number, that is the six of stress-components, the six of strain-components, and the three of displacement-components. On the other hand the fundamental equations are also fifteen in number, that is the three of stress-equations, the six of compatibility conditions, and the six

* Professor of Civil Engineering, Faculty of Engineering, Shinshu University, Nagano, Japan.

1) Love, Elasticity, 4th ed., pp. 87–88.

of stress-strain relations. With regard to the last if we take the Hooke's law between stress and strain as is usually done, the system of equations reduces to three equations involving three unknowns, that is the displacement-equations.

2. In a body in equilibrium under no body forces the six components of stress satisfy three of the following equations at every point of the body:

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial Z_x}{\partial z} = 0, \cdots\cdots, \tag{1}$$

which are known as the stress-equations.

As for the stress-strain relations our attention will be confined to Hooke's law, viz.

$$e_{xx} = \frac{1}{E}\{X_x - \sigma\,(Y_y + Z_z)\}, \cdots\cdots, \qquad e_{yz} = \frac{2\,(1+\sigma)}{E}\,Y_z\,, \;\cdots\cdots, \tag{2}$$

E being Young's modulus and σ Poisson's ratio of the material.

The six components of strain have to satisfy the following equations:

$$\frac{\partial^2 e_{yy}}{\partial z^2} + \frac{\partial^2 e_{zz}}{\partial y^2} = \frac{\partial^2 e_{yz}}{\partial y \partial z}, \;\cdots\cdots, \qquad 2\frac{\partial^2 e_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial e_{yz}}{\partial x} + \frac{\partial e_{zx}}{\partial y} + \frac{\partial e_{xy}}{\partial z}\right), \;\cdots\cdots, \tag{3}$$

the so-called Saint-Venant's identical relations between strains. These, however, would only be necessary conditions for securing the compatibility conditions of strain, viz.

$$e_{xx} = \frac{\partial u}{\partial x}\,, \;\cdots\cdots, \qquad e_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\,, \;\cdots\cdots,$$

and will be not sufficient.

3. We shall begin by putting, with the extension of both of Maxwell's stress-functions and of Morera's,

$$\begin{aligned} Y_z = \Big[a\,\nabla^2 + b\,\frac{\partial^2}{\partial x^2} + c\,\frac{\partial^2}{\partial y \partial z} + d\,\frac{\partial}{\partial x}\Big(\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\Big)\Big]\,\chi_1 \\ + e\left(\frac{\partial^2 \chi_2}{\partial y^2} + \frac{\partial^2 \chi_3}{\partial z^2}\right) + f\left(\frac{\partial^2 \chi_2}{\partial z^2} + \frac{\partial^2 \chi_3}{\partial y^2}\right) + g\,\frac{\partial^2}{\partial x^2}\,(\chi_2 + \chi_3) + h\,\frac{\partial^2}{\partial y \partial z}\,(\chi_2 + \chi_3) \\ + i\,\frac{\partial}{\partial x}\left(\frac{\partial \chi_2}{\partial y} + \frac{\partial \chi_3}{\partial z}\right) + j\,\frac{\partial}{\partial x}\left(\frac{\partial \chi_2}{\partial z} + \frac{\partial \chi_3}{\partial y}\right), \end{aligned} \tag{4}$$

Z_x, X_y being given by cyclical interchange of x, y, z, and χ_1, χ_2, χ_3; a, b, c, d, e, f, g, h, i, j being all constants, and ∇^2 standing for

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

χ_1, χ_2, χ_3 are some functions of x, y, z. The form of the substitution made in (4) may be the most general one, so long as symmetrical quadratic forms of differential coefficients are concerned. In particular if we a priori take $c = -1$, and all other constants to be zero, we then have the so-called Maxwell's stress-functions; and if we take $b = \frac{1}{2}$, $i = -\frac{1}{2}$, and all other constants to be zero, we then have the so-called Morera's stress-functions. But these restricted substitutions are in general not sufficient for expressing stress-components, even if they are necessary. This is a reason why I made the substitution (4).

Next we might assume a single function χ such that

$$\chi_1 = \left[\alpha \nabla^2 + \beta \frac{\partial^2}{\partial x^2} + \gamma \frac{\partial^2}{\partial y \partial z} + \delta \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\right]\chi, \quad \cdots\cdots, \tag{5}$$

where α, β, γ, δ are constants; χ_2, χ_3 being given by cyclical interchange of x, y, z.

On substitution for the above expressions (5) into (4) we obtain, after a little rearrangement,[2)]

$$\begin{aligned} Y_z = & [c_1 \nabla^4 + c_2 \mathbf{x}^2 \nabla^2 + c_3 \mathbf{yz} \nabla^2 + c_4 \mathbf{x}(\mathbf{y}+\mathbf{z})\nabla^2 \\ & + c_5 \mathbf{x}^4 + c_6 \mathbf{x}^2 \mathbf{yz} + c_7 \mathbf{x}^3(\mathbf{y}+\mathbf{z}) + c_8 \mathbf{y}^2\mathbf{z}^2 + c_9 \mathbf{xyz}(\mathbf{y}+\mathbf{z})]\chi, \quad \cdots\cdots, \end{aligned} \tag{6}$$

where the constants c_1, c_2, $\cdots c_9$ stand for

$$\left.\begin{aligned} c_1 &= (a+e+f)\alpha + e\beta, \\ c_2 &= (b-e-f+2g)\alpha + (a-2e+g)\beta + j\gamma + (d+i)\delta, \\ c_3 &= (c+2h)\alpha + h\beta + a\gamma + (e+f)\delta, \\ c_4 &= (d+i+j)\alpha + i\beta + f\gamma + (a+e)\delta, \\ c_5 &= (b+e-g)\beta - j\gamma - (d+i)\delta, \\ c_6 &= (c-h)\beta + (b+2i)\gamma + (2d-e-f+2g+2j)\delta, \\ c_7 &= (d-i)\beta + (-f+g)\gamma + (b-e+g)\delta, \\ c_8 &= 2(-e+f)\beta + c\gamma + 2h\delta, \\ c_9 &= (-i+j)\beta + (d+e-f+h)\gamma + (c-e+f+h+i+j)\delta. \end{aligned}\right\} \tag{7}$$

Equations (6) may be the most general expressions for the shearing stresses in terms of a single function χ, so long as attention is confined to homogeneous symmetrical partial differential coefficients of the fourth order.

Now we shall find expressions for the three normal stresses. For this purpose it is sufficient to refer to the fundamental stress-equations (1). For instance the first of these equations gives, with equations (6),

$$\frac{\partial X_x}{\partial x} + [c_1(\mathbf{y}+\mathbf{z})\nabla^4 + c_2 \mathbf{yz}(\mathbf{y}+\mathbf{z})\nabla^2 + c_3 \mathbf{x}(\nabla^2 - \mathbf{x}^2)\nabla^2 + c_4 \mathbf{yz}(2\mathbf{x}+\mathbf{y}+\mathbf{z})\nabla^2$$

2) In what follows black letters $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ denote differentiations, that is for instance $\mathbf{x}^2\chi = \frac{\partial^2 \chi}{\partial x^2}$, etc.

$$+ c_5\,\mathbf{yz}(\mathbf{y}+\mathbf{z})(\nabla^2-\mathbf{x}^2-\mathbf{yz}) + 2c_6\,\mathbf{xy}^2\mathbf{z}^2 + c_7\,\mathbf{yz}\{\mathbf{x}(\nabla^2-\mathbf{x}^2)+\mathbf{yz}(\mathbf{y}+\mathbf{z})\}$$
$$+ c_8\,\mathbf{x}^2(\mathbf{y}+\mathbf{z})(\nabla^2-\mathbf{x}^2-\mathbf{yz}) + c_9\,\mathbf{xyz}\{(\nabla^2-\mathbf{x}^2)+\mathbf{x}(\mathbf{y}+\mathbf{z})\}]\,\chi = 0,$$

or, on rearranging,

$$\frac{\partial X_x}{\partial x} + \mathbf{x}[c_3\nabla^4 - c_3\mathbf{x}^2\nabla^2 + (2c_4+c_7+c_9)\mathbf{yz}\nabla^2 + c_8\mathbf{x}(\mathbf{y}+\mathbf{z})\nabla^2$$
$$-(c_7+c_9)\mathbf{x}^2\mathbf{yz} - c_8\mathbf{x}^3(\mathbf{y}+\mathbf{z}) + 2c_6\mathbf{y}^2\mathbf{z}^2 - (c_5+c_8-c_9)\mathbf{xyz}(\mathbf{y}+\mathbf{z})]\,\chi$$
$$+ (\mathbf{y}+\mathbf{z})[c_1\nabla^4 + (c_2+c_4+c_5)\mathbf{yz}\nabla^2 - (c_5-c_7)\mathbf{y}^2\mathbf{z}^2]\,\chi = 0.$$

This equation can be integrated with respect to x, when we assume the third term in the left-hand member to be equal to zero. The reason of this assumption might be justified, for if it were not, the resulting equation for χ would be of higher degree than that which will be presented in equation (24), and an awkward and troublesome equation would result.

Thus we obtain the expression for X_x in the form

$$X_x = -[c_3\nabla^4 - c_3\mathbf{x}^2\nabla^2 + (2c_4+c_7+c_9)\mathbf{yz}\nabla^2 + c_8\mathbf{x}(\mathbf{y}+\mathbf{z})\nabla^2$$
$$-(c_7+c_9)\mathbf{x}^2\mathbf{yz} - c_8\mathbf{x}^3(\mathbf{y}+\mathbf{z}) + 2c_6\mathbf{y}^2\mathbf{z}^2 - (c_5+c_8-c_9)\mathbf{xyz}(\mathbf{y}+\mathbf{z})]\,\chi, \quad (8)$$

provided that

$$c_1 = 0, \qquad c_2+c_4+c_5 = 0, \qquad c_5-c_7 = 0. \quad (9)$$

The other two normal stresses Y_y, Z_z can be obtained by means of cyclical interchange of x, y, z; i. e.

$$Y_y = -[c_3\nabla^4 - c_3\mathbf{y}^2\nabla^2 + (2c_4+c_7+c_9)\mathbf{zx}\nabla^2 + c_8\mathbf{y}(\mathbf{z}+\mathbf{x})\nabla^2$$
$$-(c_7+c_9)\mathbf{xy}^2\mathbf{z} - c_8\mathbf{y}^3(\mathbf{z}+\mathbf{x}) + 2c_6\mathbf{z}^2\mathbf{x}^2 - (c_5+c_8-c_9)\mathbf{xyz}(\mathbf{z}+\mathbf{x})]\,\chi, \quad (10)$$

$$Z_z = -[c_3\nabla^4 - c_3\mathbf{z}^2\nabla^2 + (2c_4+c_7+c_9)\mathbf{xy}\nabla^2 + c_8\mathbf{z}(\mathbf{x}+\mathbf{y})\nabla^2$$
$$-(c_7+c_9)\mathbf{xyz}^2 - c_8\mathbf{z}^3(\mathbf{x}+\mathbf{y}) + 2c_6\mathbf{x}^2\mathbf{y}^2 - (c_5+c_8-c_9)\mathbf{xyz}(\mathbf{x}+\mathbf{y})]\,\chi. \quad (11)$$

We shall next calculate the components of strain. These are readily obtained by the substitution from equations (8), (10), (11), and (6) into equations (2). Thus we first have

$$e_{xx} = \frac{1}{E}[-(1-\sigma)c_3\nabla^4 + \{(1+\sigma)c_3 + 2\sigma c_6\}\mathbf{x}^2\nabla^2 + \{-2c_4-c_7+\sigma c_8-c_9\}\mathbf{yz}\nabla^2$$
$$+ \{2\sigma c_4+\sigma c_7-c_8+\sigma c_9\}\mathbf{x}(\mathbf{y}+\mathbf{z})\nabla^2 - 2\sigma c_6\mathbf{x}^4$$
$$+ \{-2\sigma c_5+c_7-\sigma c_8+(1+2\sigma)c_9\}\mathbf{x}^2\mathbf{yz} + (1+\sigma)c_8\mathbf{x}^3(\mathbf{y}+\mathbf{z})$$
$$- 2c_6\mathbf{y}^2\mathbf{z}^2 + \{(1-\sigma)c_5-\sigma c_7+c_8-c_9\}\mathbf{xyz}(\mathbf{y}+\mathbf{z})]\,\chi. \quad (12)$$

In like manner we obtain for the values of e_{yy} and e_{zz}

$$e_{yy} = \frac{1}{E}[-(1-\sigma)c_3\nabla^4 + \{(1+\sigma)c_3 + 2\sigma c_6\}\mathbf{y}^2\nabla^2 + \{-2c_4-c_7+\sigma c_8-c_9\}\mathbf{zx}\nabla^2$$
$$+ \{2\sigma c_4+\sigma c_7-c_8+\sigma c_9\}\mathbf{y}(\mathbf{z}+\mathbf{x})\nabla^2 - 2\sigma c_6\mathbf{y}^4$$
$$+ \{-2\sigma c_5+c_7-\sigma c_8+(1+2\sigma)c_9\}\mathbf{xy}^2\mathbf{z} + (1+\sigma)c_8\mathbf{y}^3(\mathbf{z}+\mathbf{x})$$
$$- 2c_6\mathbf{z}^2\mathbf{x}^2 + \{(1-\sigma)c_5-\sigma c_7+c_8-c_9\}\mathbf{xyz}(\mathbf{z}+\mathbf{x})]\,\chi, \quad (13)$$

$$
\begin{aligned}
e_{zz} = \frac{1}{E}\,[&-(1-\sigma)c_3\nabla^4 + \{(1+\sigma)c_3+2\sigma c_6\}\mathbf{z}^2\nabla^2 + \{-2c_4-c_7+\sigma c_8-c_9\}\mathbf{xy}\nabla^2 \\
&+ \{2\sigma c_4+\sigma c_7-c_8+\sigma c_9\}\mathbf{z}(\mathbf{x}+\mathbf{y})\nabla^2 - 2\sigma c_6\mathbf{z}^4 \\
&+ \{-2\sigma c_5+c_7-\sigma c_8+(1+2\sigma)c_9\}\mathbf{xyz}^2 + (1+\sigma)c_8\mathbf{z}^3(\mathbf{x}+\mathbf{y}) \\
&-2c_6\mathbf{x}^2\mathbf{y}^2 + \{(1-\sigma)c_5-\sigma c_7+c_8-c_9\}\mathbf{xyz}(\mathbf{x}+\mathbf{y})]\,\chi. \qquad (14)
\end{aligned}
$$

As for the shearing strains these are at once written down, from (6),

$$
\left.
\begin{aligned}
e_{yz} &= \frac{1}{\mu}\,[c_1\nabla^4 + c_2\mathbf{x}^2\nabla^2 + c_3\mathbf{yz}\nabla^2 + c_4\mathbf{x}(\mathbf{y}+\mathbf{z})\nabla^2 + c_5\mathbf{x}^4 \\
&\quad + c_6\mathbf{x}^2\mathbf{yz} + c_7\mathbf{x}^3(\mathbf{y}+\mathbf{z}) + c_8\mathbf{y}^2\mathbf{z}^2 + c_9\mathbf{xyz}(\mathbf{y}+\mathbf{z})]\,\chi, \\
e_{zx} &= \frac{1}{\mu}\,[c_1\nabla^4 + c_2\mathbf{y}^2\nabla^2 + c_3\mathbf{zx}\nabla^2 + c_4\mathbf{y}(\mathbf{z}+\mathbf{x})\nabla^2 + c_5\mathbf{y}^4 \\
&\quad + c_6\mathbf{xy}^2\mathbf{z} + c_7\mathbf{y}^3(\mathbf{z}+\mathbf{x}) + c_8\mathbf{z}^2\mathbf{x}^2 + c_9\mathbf{xyz}(\mathbf{z}+\mathbf{x})]\,\chi, \\
e_{xy} &= \frac{1}{\mu}\,[c_1\nabla^4 + c_2\mathbf{z}^2\nabla^2 + c_3\mathbf{xy}\nabla^2 + c_4\mathbf{z}(\mathbf{x}+\mathbf{y})\nabla^2 + c_5\mathbf{z}^4 \\
&\quad + c_6\mathbf{xyz}^2 + c_7\mathbf{z}^3(\mathbf{x}+\mathbf{y}) + c_8\mathbf{x}^2\mathbf{y}^2 + c_9\mathbf{xyz}(\mathbf{x}+\mathbf{y})]\,\chi.
\end{aligned}
\right\} \qquad (15)
$$

4. We next have to refer to Saint-Venant's identical relations between strains, from which the constants $c_1, c_2, \cdots c_9$ will be determined uniquely so as to obtain a single differential equation for χ. In the first place, let us consider three of the first type of Saint-Venant's relations (3); for instance we have, with (13), (14), and the first of (15),

$$
\begin{aligned}
&\frac{\partial^2 e_{yy}}{\partial z^2} + \frac{\partial^2 e_{zz}}{\partial y^2} - \frac{\partial^2 e_{yz}}{\partial y\,\partial z} = \mathbf{z}^2 e_{yy} + \mathbf{y}^2 e_{zz} - \mathbf{yz}\,e_{yz} \\
&= \frac{1}{E}\Big[-(1-\sigma)c_3\nabla^6 + [(1-\sigma)c_3\mathbf{x} + (-2c_4-c_7+\sigma c_8-c_9)(\mathbf{y}+\mathbf{z})]\,\mathbf{x}\nabla^4 \\
&\quad + [-2(1+\sigma)c_1 + 2\sigma c_4+\sigma c_7-c_8+\sigma c_9]\mathbf{yz}\nabla^4 \\
&\quad + [\{-2(1+\sigma)c_2-2\sigma c_4-\sigma c_7+c_8-\sigma c_9\}\mathbf{yz} + \{2c_4+c_7-\sigma c_8+c_9\}(\mathbf{zx}+\mathbf{xy})]\mathbf{x}^2\nabla^2 \\
&\quad + [\{(1-\sigma)c_5-\sigma c_7+c_8-c_9\}\mathbf{x} + \{(1-\sigma)c_5+c_7-\sigma c_8+\sigma c_9\}(\mathbf{y}+\mathbf{z})]\mathbf{xyz}\nabla^2 \\
&\quad + [\{-(3+\sigma)c_5+\sigma c_7-c_8+c_9\}\mathbf{x} + \{-(1-\sigma)c_5-(2+\sigma)c_7-c_8+c_9\}(\mathbf{y}+\mathbf{z})]\mathbf{x}^3\mathbf{yz} \\
&\quad - (1+\sigma)(c_5-c_7)\mathbf{xy}^2\mathbf{z}^2(\mathbf{y}+\mathbf{z}) + 2c_6\{\sigma\mathbf{y}^2\mathbf{z}^2\nabla^2 - \mathbf{x}^2(\mathbf{y}^4+\mathbf{z}^4+\mathbf{y}^2\mathbf{z}^2)\}\Big]\,\chi; \qquad (16)
\end{aligned}
$$

and two similar equations will be obtained by means of cyclical interchange of **x**, **y**, **z**.

In this connection, on considering the symmetrical character of strain-components, we should infer that the following relations hold, i. e.

$$
\left.
\begin{aligned}
&(1-\sigma)c_3 = -2c_4 - c_7 + \sigma c_8 - c_9, \\
&-2(1+\sigma)c_2 - 2\sigma c_4 - \sigma c_7 + c_8 - \sigma c_9 = 2c_4 + c_7 - \sigma c_8 + c_9, \\
&(1-\sigma)c_5 - \sigma c_7 + c_8 - c_9 = (1-\sigma)c_5 + c_7 - \sigma c_8 + \sigma c_9, \\
&(3+\sigma)c_5 - \sigma c_7 + c_8 - c_9 = (1-\sigma)c_5 + (2+\sigma)c_7 + c_8 - c_9, \\
&c_6 = 0;
\end{aligned}
\right\} \qquad (17)
$$

moreover in equations (9) we have obtained

$$
c_1 = 0, \qquad c_5 = c_7 = -(c_2 + c_4).
$$

In equations (17) any one of the unknown constants which does not vanish may be chosen arbitrarily, since the function χ can be multiplied by any constant without loss of generality, and therefore we for the present take

$$c_2 = 1.$$

Hence, on eliminating c_2, c_5, c_7, equations (17) are rearranged to the following three equations:

$$(1-\sigma)c_3 + c_4 - \sigma c_8 + c_9 = 1, \quad c_4 - c_8 + c_9 = -1, \quad c_4 + c_8 - c_9 = -1.$$

On solving these we have

$$c_4 = -1, \qquad c_8 = c_9 = \frac{2}{1-\sigma} - c_3,$$

of which c_3 remains undetermined yet. In virtue of the values of c_2 and c_4 found above we must obtain

$$c_5 = c_7 = 0.$$

By the above calculations we have seen that

$$\left.\begin{array}{lll} c_1 = 0, & c_2 = 1, & c_3 = \text{unknown}, \\ c_4 = -1, & c_5 = c_6 = c_7 = 0, & \\ c_8 = c_9 = \dfrac{2}{1-\sigma} - c_3. & & \end{array}\right\} \tag{18}$$

Thus our last step to the problem is to determine the value of c_3. In virtue of (18), equation (16) can be reduced to

$$\begin{aligned} \frac{\partial^2 e_{yy}}{\partial z^2} + \frac{\partial^2 e_{zz}}{\partial y^2} - \frac{\partial^2 e_{yz}}{\partial y \partial z} &= \mathbf{z}^2 e_{yy} + y^2 e_{zz} - \mathbf{yz} e_{yz} \\ &= \frac{1}{E}\left[-(1-\sigma)c_3 \nabla^6 + (1-\sigma)c_3(\mathbf{x}+\mathbf{y}+\mathbf{z})\mathbf{x}\nabla^4\right. \\ &\quad \left.+ \{(1-\sigma)c_3 - 2(1+\sigma)\}\mathbf{yz}\nabla^4 - (1-\sigma)c_3(\mathbf{yz}+\mathbf{zx}+\mathbf{xy})\mathbf{x}^2\nabla^2\right]\chi, \end{aligned} \tag{19}$$

and other two similar equations will be obtained by means of cyclical interchange from $\mathbf{x}$ to $\mathbf{y}$ and so on. These three equations must vanish for all values of the variables x, y, z.

In the second place, let us consider three of the second type of Saint-Venant's relations (3); for instance, from the fourth of (3), we may form the following equation in which the values of (18) are taken into account:

$$\begin{aligned} 2\frac{\partial^2 e_{xx}}{\partial y \partial z} + \frac{\partial^2 e_{yz}}{\partial x^2} - \frac{\partial^2 e_{zx}}{\partial x \partial y} - \frac{\partial^2 e_{xy}}{\partial z \partial x} \\ = \frac{1}{E}\left[-(1-\sigma)c_3 \mathbf{yz}\nabla^4 - 2(1+\sigma)\mathbf{x}(-\mathbf{x}+\mathbf{y}+\mathbf{z})\nabla^4\right]\chi, \end{aligned} \tag{20}$$

and other two similar equations will be obtained by means of cyclical interchange from $\mathbf{x}$ to $\mathbf{y}$ and so on.

Equations of the types (19) and (20) must vanish for all values of the variables x, y, z within an elastic solid and over its surface, and therefore we take

$$c_3 = 0.$$

Hence (18) become

$$\left.\begin{array}{llll} c_1 = 0, & c_2 = 1, & c_3 = 0, & c_4 = -1, \\ c_5 = c_6 = c_7 = 0, & & c_8 = c_9 = \dfrac{2}{1-\sigma}. & \end{array}\right\} \quad (21)$$

We then have, from equations of the type (19),

$$\mathbf{y}\mathbf{z}\nabla^4\chi = 0, \qquad \mathbf{z}\mathbf{x}\nabla^4\chi = 0, \qquad \mathbf{x}\mathbf{y}\nabla^4\chi = 0, \quad (22)$$

and hence these equations give

$$\nabla^4\chi = X(x) + Y(y) + Z(z), \quad (23)$$

where X, Y, Z denote arbitrary functions of x, y, z in order.

From three equations of the type (20) we have

$$\mathbf{x}(-\mathbf{x}+\mathbf{y}+\mathbf{z})\nabla^4\chi = 0, \qquad \mathbf{y}(\mathbf{x}-\mathbf{y}+\mathbf{z})\nabla^4\chi = 0, \qquad \mathbf{z}(\mathbf{x}+\mathbf{y}-\mathbf{z})\nabla^4\chi = 0.$$

We then have, with the use of (22),

$$\mathbf{x}^2\nabla^4\chi = 0, \qquad \mathbf{y}^2\nabla^4\chi = 0, \qquad \mathbf{z}^2\nabla^4\chi = 0,$$

or

$$\frac{\partial^2}{\partial x^2}\nabla^4\chi = 0, \qquad \frac{\partial^2}{\partial y^2}\nabla^4\chi = 0, \qquad \frac{\partial^2}{\partial z^2}\nabla^4\chi = 0.$$

These give together with (23)

$$\frac{d^2X}{dx^2} = 0, \qquad \frac{d^2Y}{dy^2} = 0, \qquad \frac{d^2Z}{dz^2} = 0,$$

and therefore

$$X(x) = Ax + A', \qquad Y(y) = By + B', \qquad Z(z) = Cz + C',$$

where A, B, C, A', B', C' are constants.

Thus we have for the present the result

$$\nabla^4\chi = Ax + By + Cz + D, \quad (24)$$

where, for shortness, D is written for $A'+B'+C'$, and, as before,

$$\nabla^4 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)^2.$$

The equation (24) is the fundamental differential equation which must be satisfied by the stress-function χ in three dimensions, provided that

$$A = B = C = D = 0,$$

the reduction of which will be given in Article 6.

5. The six components of stress can be obtained, by substituting from the values of $c_1, c_2, \cdots c_3$, (21), into equations (8), (10), (11), and (6), in the forms

$$\left.\begin{aligned} X_x &= \{(\mathbf{y}^2+\mathbf{z}^2)(\mathbf{yz}+\mathbf{zx}+\mathbf{xy}) - (1-\sigma)\mathbf{yz}\nabla^2\}\,\chi, \\ Y_y &= \{(\mathbf{z}^2+\mathbf{x}^2)(\mathbf{yz}+\mathbf{zx}+\mathbf{xy}) - (1-\sigma)\mathbf{zx}\nabla^2\}\,\chi, \\ Z_z &= \{(\mathbf{x}^2+\mathbf{y}^2)(\mathbf{yz}+\mathbf{zx}+\mathbf{xy}) - (1-\sigma)\mathbf{xy}\nabla^2\}\,\chi, \\ Y_z &= \{-\mathbf{yz}(\mathbf{yz}+\mathbf{zx}+\mathbf{xy}) + \frac{1-\sigma}{2}\mathbf{x}(-\mathbf{x}+\mathbf{y}+\mathbf{z})\nabla^2\}\,\chi, \\ Z_x &= \{-\mathbf{zx}(\mathbf{yz}+\mathbf{zx}+\mathbf{xy}) + \frac{1-\sigma}{2}\mathbf{y}(\mathbf{x}-\mathbf{y}+\mathbf{z})\nabla^2\}\,\chi, \\ X_y &= \{-\mathbf{xy}(\mathbf{yz}+\mathbf{zx}+\mathbf{xy}) + \frac{1-\sigma}{2}\mathbf{z}(\mathbf{x}+\mathbf{y}-\mathbf{z})\nabla^2\}\,\chi, \end{aligned}\right\} \tag{25}$$

or, with the usual notations,

$$\left.\begin{aligned} X_x &= \left\{\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\nabla'^2 - (1-\sigma)\frac{\partial^2}{\partial y\partial z}\nabla^2\right\}\chi, \\ Y_y &= \left\{\left(\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial x^2}\right)\nabla'^2 - (1-\sigma)\frac{\partial^2}{\partial z\partial x}\nabla^2\right\}\chi, \\ Z_z &= \left\{\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\nabla'^2 - (1-\sigma)\frac{\partial^2}{\partial x\partial y}\nabla^2\right\}\chi, \\ Y_z &= \left\{-\frac{\partial^2}{\partial y\partial z}\nabla'^2 + \frac{1-\sigma}{2}\frac{\partial}{\partial x}\left(-\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\nabla^2\right\}\chi, \\ Z_x &= \left\{-\frac{\partial^2}{\partial z\partial x}\nabla'^2 + \frac{1-\sigma}{2}\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\nabla^2\right\}\chi, \\ X_y &= \left\{-\frac{\partial^2}{\partial x\partial y}\nabla'^2 + \frac{1-\sigma}{2}\frac{\partial}{\partial z}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}-\frac{\partial}{\partial z}\right)\nabla^2\right\}\chi, \end{aligned}\right\} \tag{26}$$

where, as usual,

$$\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2},$$

and ∇'^2 represents the operator

$$\nabla'^2 = \frac{\partial^2}{\partial y\partial z}+\frac{\partial^2}{\partial z\partial x}+\frac{\partial^2}{\partial x\partial y}.$$

Here the function χ in (25) or (26) is multiplied by a constant $-2/(1-\sigma)$, as it will be more convenient for our discussion in virtue of simplicity. In what follows we shall exclusively use this new substitution.

It may be seen easily, by substitution, that the above equations (24) and (26) are satisfied by the stress-equations (1), Hooke's law (2), and Saint-Venant's identical relations of strain (3).

In the first place, the substitution of the stress-components obtained in (26) into the left-hand members of equations (1) gives three equations of the type

$$\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial Z_x}{\partial z}=\mathbf{x}X_x+\mathbf{y}X_y+\mathbf{z}Z_x$$
$$=[\mathbf{x}(\mathbf{y}^2+\mathbf{z}^2)\nabla^2-(1-\sigma)\mathbf{xyz}\nabla^2-\mathbf{xy}^2\nabla^2+\frac{1-\sigma}{2}\mathbf{yz}(\mathbf{x}+\mathbf{y}-\mathbf{z})\nabla^2$$
$$-\mathbf{xz}^2\nabla^2+\frac{1-\sigma}{2}\mathbf{yz}(\mathbf{x}-\mathbf{y}+\mathbf{z})\nabla^2]\chi,$$

which vanishes identically. It has thus been proved that the stress-equations (1) are satisfied by our equations (26).

In the second place, we shall calculate the strain-components in terms of χ. We obtain three equations of the type

$$e_{xx}=\frac{1}{E}\{X_x-\sigma(Y_y+Z_z)\}$$
$$=\frac{1}{E}\{\nabla^2(\nabla^2-\mathbf{x}^2)-(1-\sigma)\mathbf{yz}\nabla^2\}\chi-\frac{\sigma}{E}\{\nabla^2(\nabla^2+\mathbf{x}^2)-(1-\sigma)\mathbf{x}(\mathbf{y}+\mathbf{z})\nabla^2\}\chi;$$

and this can be rearranged into the first of the three following equations, and the other two are obtained by cyclical interchange from $\mathbf{x}$ to $\mathbf{y}$ and so on.

$$\left.\begin{aligned}e_{xx}&=\frac{1}{2\mu}\mathbf{x}\{-\mathbf{x}\nabla^2+(1-\sigma)(\mathbf{y}+\mathbf{z})\nabla^2\}\chi,\\ e_{yy}&=\frac{1}{2\mu}\mathbf{y}\{-\mathbf{y}\nabla^2+(1-\sigma)(\mathbf{z}+\mathbf{x})\nabla^2\}\chi,\\ e_{zz}&=\frac{1}{2\mu}\mathbf{z}\{-\mathbf{z}\nabla^2+(1-\sigma)(\mathbf{x}+\mathbf{y})\nabla^2\}\chi.\end{aligned}\right\}\tag{27}$$

We next obtain for the three shearing strains the equations

$$\left.\begin{aligned}e_{yz}&=\frac{1}{\mu}\left\{-\mathbf{yz}\nabla^2+\frac{1-\sigma}{2}\mathbf{x}(-\mathbf{x}+\mathbf{y}+\mathbf{z})\nabla^2\right\}\chi,\\ e_{zx}&=\frac{1}{\mu}\left\{-\mathbf{zx}\nabla^2+\frac{1-\sigma}{2}\mathbf{y}(\mathbf{x}-\mathbf{y}+\mathbf{z})\nabla^2\right\}\chi,\\ e_{xy}&=\frac{1}{\mu}\left\{-\mathbf{xy}\nabla^2+\frac{1-\sigma}{2}\mathbf{z}(\mathbf{x}+\mathbf{y}-\mathbf{z})\nabla^2\right\}\chi.\end{aligned}\right\}\tag{28}$$

In the third place, we shall examine Saint-Venant's relations (3). First we obtain three equations of the type

$$\frac{\partial^2 e_{yy}}{\partial z^2}+\frac{\partial^2 e_{zz}}{\partial y^2}-\frac{\partial^2 e_{yz}}{\partial y\,\partial z}$$
$$=\frac{1}{2\mu}\mathbf{yz}[-\mathbf{yz}\nabla^2+(1-\sigma)\mathbf{z}(\mathbf{z}+\mathbf{x})\nabla^2-\mathbf{yz}\nabla^2$$
$$+(1-\sigma)\mathbf{y}(\mathbf{x}+\mathbf{y})\nabla^2+2\mathbf{yz}\nabla^2-(1-\sigma)\mathbf{x}(-\mathbf{x}+\mathbf{y}+\mathbf{z})\nabla^2]\chi$$
$$=\frac{1-\sigma}{2\mu}\mathbf{yz}\nabla^4\chi,$$

and also three equations of the type

$$2\frac{\partial^2 e_{xx}}{\partial y\,\partial z}-\frac{\partial}{\partial x}\quad-\frac{\partial e_{yz}}{\partial x}+\frac{\partial e_{zx}}{\partial y}+\frac{\partial e_{xy}}{\partial z})$$

$$= \frac{1}{\mu}\mathbf{x}\Big[-\mathbf{xyz}\nabla^2 + (1-\sigma)\mathbf{yz}(\mathbf{y}+\mathbf{z})\nabla^2 - \mathbf{xyz}\nabla^2 + \frac{1-\sigma}{2}\mathbf{x}^2(-\mathbf{x}+\mathbf{y}+\mathbf{z})\nabla^2$$
$$+ \mathbf{xyz}\nabla^2 - \frac{1-\sigma}{2}\mathbf{y}^2(\mathbf{x}-\mathbf{y}+\mathbf{z})\nabla^2 + \mathbf{xyz}\nabla^2 - \frac{1-\sigma}{2}\mathbf{z}^2(\mathbf{x}+\mathbf{y}-\mathbf{z})\nabla^2\Big]\chi$$
$$= \frac{1-\sigma}{2\mu}\mathbf{x}(-\mathbf{x}+\mathbf{y}+\mathbf{z})\nabla^4\chi.$$

But, as we have obtained in (24) the equation

$$\nabla^4\chi = Ax + By + Cz + D,$$

we easily see that the above two types of Saint-Venant's relations are satisfied.

Thus it can be concluded that our equations (24) and (26) are satisfied by the three kinds of original equations, that is the stress-equations (1), Hooke's law (2), and Saint-Venant's relations (3). The last, however, is not sufficient for securing the condition of compatibility, that is the definition equations of strain.

6. We may easily find the displacement-components in terms of χ, by integrating the three equations (27) with respect to x, y, z respectively, that is

$$\left.\begin{aligned} u &= \frac{1}{2\mu}\{-\mathbf{x}\nabla^2 + (1-\sigma)(\mathbf{y}+\mathbf{z})\nabla^2\}\chi, \\ v &= \frac{1}{2\mu}\{-\mathbf{y}\nabla^2 + (1-\sigma)(\mathbf{z}+\mathbf{x})\nabla^2\}\chi, \\ w &= \frac{1}{2\mu}\{-\mathbf{z}\nabla^2 + (1-\sigma)(\mathbf{x}+\mathbf{y})\nabla^2\}\chi, \end{aligned}\right\} \qquad (29)$$

where no arbitrary function need be added, for any such function can be included in χ. These displacements should give rise to the three shearing strains obtained in (28). For instance e_{yz} is derived from (29) as follows:

$$e_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \frac{1}{\mu}\left\{-\mathbf{yz}\nabla^2 + \frac{1-\sigma}{2}\mathbf{x}(-\mathbf{x}+\mathbf{y}+\mathbf{z})\nabla^2 + \frac{1-\sigma}{2}\nabla^4\right\}\chi,$$

and this must be in accordance with the first of (28), from which we at once obtain

$$\nabla^4\chi = 0. \qquad (30)$$

This is the required differential equation for χ, and that obtained in (24) is by no means the case; but the constants A, B, C, D in (24) need be brought to zero.

It will be seen easily that the displacement-equations

$$(\lambda+\mu)\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\Big)\Delta + \mu\nabla^2(u, v, w) = 0 \qquad (31)$$

are satisfied by equations (29) and (30). To see this we form

$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{1-2\sigma}{2\mu} \nabla^2 \nabla^2 \chi, \tag{32}$$

and hence on substitution from (32) and (29) into the left-hand members of equations (31) we have

$$(\lambda+\mu)\frac{1-2\sigma}{2\mu}(\mathbf{x}, \mathbf{y}, \mathbf{z})\nabla^2\nabla^2\chi + \frac{1}{2}\nabla^2(-\mathbf{x}, -\mathbf{y}, -\mathbf{z})\nabla^2\chi$$
$$+\frac{1}{2}\nabla^2(1-\sigma)(\mathbf{y}+\mathbf{z},\ \mathbf{z}+\mathbf{x},\ \mathbf{x}+\mathbf{y})\nabla^2\chi$$
$$= \frac{1-\sigma}{2}(\mathbf{y}+\mathbf{z},\ \mathbf{z}+\mathbf{x},\ \mathbf{x}+\mathbf{y})\nabla^4\chi, \ \cdots\cdots,$$

and these vanish in virtue of (30).

7. Singular potential. The general potential stated above is not sufficient for any boundary conditions, for the equation $\nabla^4\chi = 0$ has only four degrees of freedom. Now we take the relation

$$u = \frac{1}{2\mu}\{-\mathbf{x}\nabla^2 + (1-\sigma)(\mathbf{y}+\mathbf{z})\nabla^2\}\chi,$$

which is one of (29). If we consider this equation to be a differential equation with respect to the function χ, and u some known function, then this equation will have a particular integral, which is the required solution. u may take various forms of functions for respective boundary-value problems, and is always restricted within the domain of biharmonic functions. In this connection we may put

$$\left.\begin{aligned} u &= \frac{1}{2\mu}\{-\mathbf{x}\nabla^2 + (1-\sigma)(\mathbf{y}+\mathbf{z})\nabla^2\}\chi, \\ v &= \frac{1}{2\mu}\{-\mathbf{y}\nabla^2 + (1-\sigma)(\mathbf{z}+\mathbf{x})\nabla^2\}\chi + v', \\ w &= \frac{1}{2\mu}\{-\mathbf{z}\nabla^2 + (1-\sigma)(\mathbf{x}+\mathbf{y})\nabla^2\}\chi + w', \end{aligned}\right\} \tag{33}$$

in which v' and w' are some functions of x, y, and z, which are not included in χ.

It can be verified easily that, introducing a function, ϕ say, the displacement-equations (31) are satisfied by

$$u' = 0, \qquad v' = \frac{1}{2\mu}\frac{\partial\phi}{\partial z}, \qquad w' = -\frac{1}{2\mu}\frac{\partial\phi}{\partial y}, \tag{34}$$

provided that ϕ satisfies the harmonic equation

$$\nabla^2\phi = 0; \tag{35}$$

the constant $1/2\mu$ being multiplied for convenience.

The cyclical interchanges of letters in (34) may also be solutions of the displacement-equations (31); that is we obtain the two sets of functions

$$u'=-\frac{1}{2\mu}\frac{\partial\psi}{\partial z},\qquad v'=0,\qquad w'=\frac{1}{2\mu}\frac{\partial\psi}{\partial x},\tag{36}$$

$$u'=\frac{1}{2\mu}\frac{\partial\psi}{\partial y},\qquad v'=-\frac{1}{2\mu}\frac{\partial\psi}{\partial x},\qquad w'=0,\tag{37}$$

ψ satisfying (35) as before.

The sum of (34), (36), and (37) are also a solution of the displacement-equations (31); that is

$$u'=\frac{1}{2\mu}\Big(\frac{\partial}{\partial y}-\frac{\partial}{\partial z}\Big)\psi,\qquad v'=\frac{1}{2\mu}\Big(\frac{\partial}{\partial z}-\frac{\partial}{\partial x}\Big)\psi,\qquad w'=\frac{1}{2\mu}\Big(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\Big)\psi,\tag{38}$$

where, as before,

$$\nabla^2\psi=0.\tag{39}$$

Typical solutions of (39) are

$$\psi=(C_1\cos\alpha x\cos\beta y+C_2\cos\alpha x\sin\beta y+C_3\sin\alpha x\cos\beta y+C_4\sin\alpha x\sin\beta y)\,e^{-\gamma z},\tag{40}$$

α, β, γ being parameters, provided $\alpha^2+\beta^2=\gamma^2$.

Stress-components derived from (38), provided Hooke's law (2) is referred to, become

$$\left.\begin{aligned}
X'_x&=\Big(\frac{\partial^2}{\partial x\partial y}-\frac{\partial^2}{\partial z\partial x}\Big)\psi, & Y'_z&=\frac{1}{2}\Big(\frac{\partial^2}{\partial x\partial y}-\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial z\partial x}\Big)\psi,\\
Y'_y&=\Big(\frac{\partial^2}{\partial y\partial z}-\frac{\partial^2}{\partial x\partial y}\Big)\psi, & Z'_x&=\frac{1}{2}\Big(\frac{\partial^2}{\partial y\partial z}-\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial x\partial y}\Big)\psi,\\
Z'_z&=\Big(\frac{\partial^2}{\partial z\partial x}-\frac{\partial^2}{\partial y\partial z}\Big)\psi, & X'_y&=\frac{1}{2}\Big(\frac{\partial^2}{\partial z\partial x}-\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial y\partial z}\Big)\psi.
\end{aligned}\right\}\tag{41}$$

This system of functions consisting of (38), (39), and (41) may be called 'singular potential' in three-dimensional elasticity by reason of the substitution made in (33). In contrast with this, the preceding function χ may reasonably be called 'general potential'; and the aggregate of the general potential and the singular potential constitutes the proposed three-dimensional stress-functions.

8. We shall see that the substitution (33) enables us to find particular solutions of the displacement-equations (31) in the forms of product of functions, and that they will be the same as those derived from (38) in the preceding Aritle, § 7.

Substituting (33) into the displacement-equations

$$\Big(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\Big)\Delta+(1-2\sigma)\nabla^2(u,v,w)=0,$$

we obtain

$$\frac{\partial}{\partial x}\Big(\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\Big)=0,$$

$$\left.\begin{aligned}\frac{\partial}{\partial y}\left(\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right)+(1-2\sigma)\nabla^2 v'=0,\\ \frac{\partial}{\partial z}\left(\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right)+(1-2\sigma)\nabla^2 w'=0.\end{aligned}\right\} \tag{42}$$

If we assume

$$v'=F_1(y,z)\cdot X_1(x), \qquad w'=F_2(y,z)\cdot X_2(x), \tag{43}$$

then the above three equations become

$$\frac{\partial F_1}{\partial y}\frac{X_1'}{X_2'}+\frac{\partial F_2}{\partial z}=0, \tag{44}$$

$$2(1-\sigma)\frac{\partial^2 F_1}{\partial y^2}+(1-2\sigma)\frac{\partial^2 F_1}{\partial z^2}+\frac{\partial^2 F_2}{\partial y\,\partial z}\frac{X_2}{X_1}+(1-2\sigma)F_1\frac{X_1''}{X_1}=0, \tag{45}$$

$$2(1-\sigma)\frac{\partial^2 F_2}{\partial z^2}+(1-2\sigma)\frac{\partial^2 F_2}{\partial y^2}+\frac{\partial^2 F_1}{\partial y\,\partial z}\frac{X_1}{X_2}+(1-2\sigma)F_2\frac{X_2''}{X_2}=0. \tag{46}$$

Equation (44) holds only when

$$X_2=aX_1+b,$$

and further we may, without any loss of generality, take

$$a=1, \qquad b=0,$$

since the latter of (43) might have been assumed to be of the form

$$w'=F_2(y,z)\{aX_2(x)+b\}.$$

We therefore have

$$X_2=X_1; \tag{47}$$

and on substituting this into (45) and (46) we obtain

$$\left\{2(1-\sigma)\frac{\partial^2 F_1}{\partial y^2}+(1-2\sigma)\frac{\partial^2 F_1}{\partial z^2}+\frac{\partial^2 F_2}{\partial y\,\partial z}\right\}+(1-2\sigma)F_1\frac{X_1''}{X_1}=0, \tag{48}$$

$$\left\{2(1-\sigma)\frac{\partial^2 F_2}{\partial z^2}+(1-2\sigma)\frac{\partial^2 F_2}{\partial y^2}+\frac{\partial^2 F_1}{\partial y\,\partial z}\right\}+(1-2\sigma)F_2\frac{X_1''}{X_1}=0. \tag{49}$$

These equations hold only when

$$\frac{X_1''}{X_1}=\text{const.}; \tag{50}$$

and, for our present purpose, we may confine our attention only to the case when

$$\frac{X_1''}{X_1}=-\alpha^2, \quad (\alpha>0) \tag{51}$$

Equations (48) and (49) then reduce to

$$2(1-\sigma)\frac{\partial^2 F_1}{\partial y^2}+(1-2\sigma)\frac{\partial^2 F_1}{\partial z^2}+\frac{\partial^2 F_2}{\partial y\,\partial z}-(1-2\sigma)\alpha^2 F_1=0, \tag{52}$$

$$2(1-\sigma)\frac{\partial^2 F_2}{\partial z^2}+(1-2\sigma)\frac{\partial^2 F_2}{\partial y^2}+\frac{\partial^2 F_1}{\partial y\partial z}-(1-2\sigma)\alpha^2 F_2=0. \tag{53}$$

Now we assume

$$F_1=\begin{matrix}\cos\\ \sin\end{matrix}\beta y\cdot Z_1(z), \qquad F_2=\begin{matrix}\sin\\ \cos\end{matrix}\beta y\cdot Z_2(z), \tag{54}$$

and then equations (52) and (53) become, dividing out $\begin{matrix}\cos\\ \sin\end{matrix}\beta y$,

$$(1-2\sigma)Z_1''\pm\beta Z_2'-\{(1-2\sigma)\alpha^2+2(1-\sigma)\beta^2\}Z_1=0, \tag{55}$$

$$2(1-\sigma)Z_2''\mp\beta Z_1'-(1-2\sigma)(\alpha^2+\beta^2)Z_2=0. \tag{56}$$

To solve these simultaneous equations, we first differentiate (55) with respect to z, and substitute the result into (56), we obtain

$$\mp 2(1-\sigma)Z_1'''\pm\{2(1-\sigma)\alpha^2+(3-2\sigma)\beta^2\}Z_1'-(\alpha^2+\beta^2)\beta Z_2=0;$$

and furthermore on differentiating this and substituting the result into (55), we have

$$\begin{aligned}2(1-\sigma)Z_1^{IV}&-\{(3-4\sigma)\alpha^2+4(1-\sigma)\beta^2\}Z_1''\\ &+(\alpha^2+\beta^2)\{(1-2\sigma)\alpha^2+2(1-\sigma)\beta^2\}Z_1=0.\end{aligned} \tag{57}$$

On making the substitution $Z=e^{\nu z}$, we obtain the indicial equation

$$[\nu^2-(\alpha^2+\beta^2)][2(1-\sigma)\nu^2-\{(1-2\sigma)\alpha^2+2(1-\sigma)\beta^2\}]=0,$$

from which we have

$$\nu=\pm\sqrt{\alpha^2+\beta^2}, \qquad \pm\sqrt{\frac{1-2\sigma}{2(1-\sigma)}\alpha^2+\beta^2}.$$

i) If we take

$$Z_1=Ae^{-\sqrt{\alpha^2+\beta^2}\,z}, \tag{58}$$

equations (55) and (56) afford

$$Z_2'=\pm A\beta e^{-\sqrt{\alpha^2+\beta^2}\,z},$$

$$2(1-\sigma)Z_2''-(1-2\sigma)(\alpha^2+\beta^2)Z_2\pm A\beta\sqrt{\alpha^2+\beta^2}\,e^{-\sqrt{\alpha^2+\beta^2}\,z}=0,$$

from which we obtain

$$Z_2=\mp A\frac{\beta}{\sqrt{\alpha^2+\beta^2}}e^{-\sqrt{\alpha^2+\beta^2}\,z}. \tag{59}$$

ii) If we take

$$Z_1=B\exp\left(-\sqrt{\frac{1-2\sigma}{2(1-\sigma)}\alpha^2+\beta^2}\cdot z\right), \tag{60}$$

equations (55) and (56) afford

$$Z_2'=\pm\frac{B}{\beta}\left\{\frac{1-2\sigma}{2(1-\sigma)}\alpha^2+\beta^2\right\}\exp\left(-\sqrt{\frac{1-2\sigma}{2(1-\sigma)}\alpha^2+\beta^2}\cdot z\right)$$

or, on integrating,

$$Z_2 = \mp \frac{B}{\beta}\sqrt{\frac{1-2\sigma}{2(1-\sigma)}\alpha^2+\beta^2}\cdot \exp\left(-\sqrt{\frac{1-2\sigma}{2(1-\sigma)}\alpha^2+\beta^2}\cdot z\right) \tag{61}$$

This system of solutions (60) and (61) does not satisfy the first of (42), and therefore is not solution of (42).

In this way we may arrive at the results

$$\left.\begin{aligned} &u'=0, && v'=\frac{\gamma^2}{\mu}\frac{\partial\phi_1}{\partial y}, && w'=\frac{\beta^2}{\mu}\frac{\partial\phi_1}{\partial z},\\ &X_x=0, && Y_y=2\gamma^2\frac{\partial^2\phi_1}{\partial y^2}, && Z_z=2\beta^2\frac{\partial^2\phi_1}{\partial z^2},\\ &Y_z=(\beta^2+\gamma^2)\frac{\partial^2\phi_1}{\partial y\partial z}, && Z_x=\beta^2\frac{\partial^2\phi_1}{\partial z\partial x}, && X_y=\gamma^2\frac{\partial^2\phi_1}{\partial x\partial y},\\ & && && (\gamma^2=\alpha^2+\beta^2) \end{aligned}\right\} \tag{62}$$

where ϕ_1 is represented by

$$\phi_1=(C_1'\cos\alpha x\cos\beta y+C_2'\cos\alpha x\sin\beta y+C_3'\sin\alpha x\cos\beta y+C_4'\sin\alpha x\sin\beta y)\,e^{-\gamma z}. \tag{63}$$

9. The particular solutions found in (63) in the preceding Article, § 8, will be coincident with those derived from the singular potential (38) in § 7. As particular solutions for ψ in (34), we take the functions given in (40), and then with (34)

$$u'=0,$$

$$v'=-\frac{\gamma}{2\mu}(C_1\cos\alpha x\cos\beta y+C_2\cos\alpha x\sin\beta y+C_3\sin\alpha x\cos\beta y+C_4\sin\alpha x\sin\beta y)e^{-\gamma z},$$

$$w'=-\frac{\beta}{2\mu}(-C_1\cos\alpha x\sin\beta y+C_2\cos\alpha x\cos\beta y-C_3\sin\alpha x\sin\beta y+C_4\sin\alpha x\cos\beta y)e^{-\gamma z}.$$

On the other hand equations (62) give, on referring to (63),

$$u'=0,$$

$$v'=-\frac{\beta\gamma^2}{\mu}(C_1'\cos\alpha x\sin\beta y-C_2'\cos\alpha x\cos\beta y+C_3'\sin\alpha x\sin\beta y-C_4'\sin\alpha x\cos\beta y)e^{-\gamma z},$$

$$w'=-\frac{\beta^2\gamma}{\mu}(C_1'\cos\alpha x\cos\beta y+C_2'\cos\alpha x\sin\beta y+C_3'\sin\alpha x\cos\beta y+C_4'\sin\alpha x\sin\beta y)e^{-\gamma z}.$$

The two systems of functions above are identical with each other, if we put

$$C_1=-2\beta\gamma C_2', \qquad C_2=2\beta\gamma C_1', \qquad C_3=-2\beta\gamma C_4', \qquad C_4=2\beta\gamma C_3'.$$

Thus we have seen the coincidence of the two systems of functions, and hence may conclude that, so long as the product form of functions is concerned, the singular potential (38) has the complete generality.

10. The aggregate of the general potential and the singular potential constitutes the proposed stress-functions in three dimensions. That is, with

the aggregation of (29) and (38), we write down

$$u = \frac{1}{2\mu}\left\{-\frac{\partial}{\partial x}\overline{\nabla}^2 + (1-\sigma)\left(\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\nabla^2\right\}\chi + \frac{1}{2\mu}\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial z}\right)\psi, \quad \cdots\cdots, \tag{64}$$

where, as before, μ is the rigidity and σ Poisson's ratio, and

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \qquad \overline{\nabla}^2 = \frac{\partial^2}{\partial y\partial z} + \frac{\partial^2}{\partial z\partial x} + \frac{\partial^2}{\partial x\partial y},$$

the remaining displacement-components v and w being given by cyclical interchange of letters. χ and ψ satisfy respectively the equations

$$\nabla^4\chi = 0, \qquad \nabla^2\psi = 0. \tag{65}$$

The stress-components derived from the above displacement-components become

$$\left.\begin{aligned} X_x &= \left\{\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\overline{\nabla}^2 - (1-\sigma)\frac{\partial^2}{\partial y\partial z}\nabla^2\right\}\chi + \left(\frac{\partial^2}{\partial x\partial y} - \frac{\partial^2}{\partial z\partial x}\right)\psi, \quad \cdots\cdots, \\ Y_z &= \left\{-\frac{\partial^2}{\partial y\partial z}\overline{\nabla}^2 + \frac{1-\sigma}{2}\frac{\partial}{\partial x}\left(-\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\nabla^2\right\}\chi \\ &\qquad + \frac{1}{2}\left(\frac{\partial^2}{\partial x\partial y} - \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial z\partial x}\right)\psi, \quad \cdots\cdots, \end{aligned}\right\} \tag{66}$$

the remaining stress-components being given by cyclical interchange of letters.

It will sometimes be convenient to extract harmonic functions, ϕ' say, from the χ-function. In this case $\nabla^2\phi' = 0$. Then $\overline{\nabla}^2\phi'$, which is always common in all expressions in (64) and (66) and is harmonic as well, can be replaced by a new harmonic, ϕ say. Thus we have

$$\left.\begin{aligned} u &= \frac{1}{2\mu}\frac{\partial\phi}{\partial x} + \frac{1}{2\mu}\left\{-\frac{\partial}{\partial x}\overline{\nabla}^2 + (1-\sigma)\left(\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\nabla^2\right\}\chi' + \frac{1}{2\mu}\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial z}\right)\psi, \quad \cdots, \\ X_x &= \frac{\partial^2\phi}{\partial x^2} + \left\{\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\overline{\nabla}^2 - (1-\sigma)\frac{\partial^2}{\partial y\partial z}\nabla^2\right\}\chi' + \left(\frac{\partial^2}{\partial x\partial y} - \frac{\partial^2}{\partial z\partial x}\right)\psi, \quad \cdots, \\ Y_z &= \frac{\partial^2\phi}{\partial y\partial z} + \left\{-\frac{\partial^2}{\partial y\partial z}\overline{\nabla}^2 + \frac{1-\sigma}{2}\frac{\partial}{\partial x}\left(-\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\nabla^2\right\}\chi' \\ &\qquad + \frac{1}{2}\left(\frac{\partial^2}{\partial x\partial y} - \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial z\partial x}\right)\psi, \quad \cdots\cdots, \end{aligned}\right\} \tag{67}$$

in which ϕ, ψ, and χ' satisfy respectively the equations

$$\nabla^2\phi = 0, \qquad \nabla^2\psi = 0, \qquad \nabla^4\chi' = 0, \tag{68}$$

χ' being 'proper' biharmonics.

The above system of equations consisting of (67) and (68) has been effectively employed in solving the generalized Boussinesq's problem of elastic foundation, which is compatible with any distribution of shearing forces, as well as with of normal pressure. Complete description of this work will appear in the Transactions of the Japan Society of Civil Engineers in the near future.

Acknowledgment. I should like to express my sincere thanks to Dr. Yutaka TANAKA, member of the Japan Academy, professor emeritus of Tokyo University, and director of the Yokogawa Bridge Works, Inc., for his enduring encouragement and kind supervision.

Epilogue. The present work, so far as the fundamental equations (67) and (68) are concerned, has been done since 1948, together with the solving of a certain simple Boussinesq's problem. At that time the late Dr. H. M. Westergaard, then professor of Civil Engineering in Harvard University, Mass., gave me a letter in reply of my manuscript on the work through Dr. H. C. Kelly, then Acting Chief at Scientific & Technical Division, GHQ. In that letter he stated :

"......The paper contains statements of general procedures which I assume are applied in the larger work. ...... I have reason to assume the particular procedures would work out well in applications.

"I shall appreciate if you will convey to Professor Tanimoto my best wishes for continued success in his work and my thanks for having caused his paper to be sent to me."

Also, Professor H. Neuber, Dresden, gave me a reply in which he stated:

"......Allerdings glaube ich, dass sich der von Ihnen benutzte Ansatz für den Übergang auf beliebige krummlinige Koordinaten nicht so gut eignet wie der von mir im Jahre 1934 in der ZAMM veröffentlichte Dreifunktionenansatz......"

In fact, I must say that I have not yet worked out the transformation to curvilinear coordinates, except for that to cylindrical coordinates only, its applicability to technological boundary-value problems being set about as a trial.

In conclusion, it is added that a rather brief description of the generalized Boussinesq's problem cited has appeared in the Proceedings of the Japan Academy, November, 1955, and that a succeeding boundary-value problem has just been solved, which is that a thick plate is subjected to three pairs of external forces on its bounding planes.