ON THE MECHANICAL CUBATURE

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Synopsis. From interpolation formulas for the function of two arguments, general expressions for the double definite integral or mechanical cubature are derived. Various rules for the mechanical cubature can be formulated from these expressions, and they will be useful for practical evaluation of the double integral even when this is difficult or impossible to evaluate. The procedure can easily be extended to the evaluation of triple integral and so on.

§ 1. We have sometimes met with the evaluation of the definite integral

$$I = \int_a^{a'} f(x)\ dx ,$$

a and a' being constants. When this integral is difficult to evaluate by means of analytical process, we have to rely on the numerical method which is known as mechanical quadrature.

Elementary rules for the mechanical quadrature that have widely been used are:

Trapezoidal rule, of the form: $I = \frac{h}{2}(y_0 + y_1)$,

Simpson's rule, of the form: $I = \frac{h}{3}(y_{-1} + 4y_0 + y_1)$,

and so on.

§ 2. We also encounter with the double definite integral

$$I = \int_a^{a'} \int_b^{b'} f(x, y)\, dx\, dy ,$$

where a, a', b and b' are constants, and the integrand $f(x, y)$ is for the time being supposed to have no singularities within the domain considered. Numerical mathematical method for the integral last written is called mechanical cubature. In this regard, duplicate use of known rules in the mechanical quadrature have sometimes been employed, almost no further developments have been made, and no systematic process of

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1) An elementary treatment of the problem may be found for instance, in Steffenson's 'Interpolation.'

the mechanical cubature seems to have been proposed.

The present work is in deducing general expressions of the mechanical cubature, by beginning with Stirling interpolation formula, and Bessel interpolation formula for the function $f(x, y)$.

Furthermore, general expressions for the 'mechanical biquadrature' can also be written down; and in addition, general expressions for multiple integral of higher order may be written down if required.

My temporary necessity to work out general deduction in the mechanical cubature has its origin in the evaluation of a series of double definite integrals such as

$$\int_0^\infty \int_0^\infty (1+\gamma z) \cos \alpha x \cos \beta y \sin \alpha a \sin \beta b \, e^{-\gamma z} \frac{d\alpha d\beta}{\alpha\beta},$$

$$\int_0^\infty \int_0^\infty \{2(1-\sigma) + \gamma z\} \cos \alpha x \cos \beta y \sin \alpha a \sin \beta b \, e^{-\gamma z} \frac{d\alpha d\beta}{\alpha\beta\gamma}, \quad \text{etc.,}$$

where

$$\gamma^2 = \alpha^2 + \beta^2;$$

and a, b, σ, x, y and z are here supposed to be constant. These integrals presented themselves as the solution of a Boussinesq's problem of simple character[2]. It would not be so easy to evaluate these integrals by

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[2] The problem treated was that a semi-infinite elastic solid is pressed by a uniform load, Q, within a rectangular area $2a \times 2b$. The procedure of solving it has been proposed by me, and it is

$$u = \frac{1}{2\mu}\frac{\partial \phi}{\partial x} + \frac{1}{2\mu}\left\{-\frac{\partial}{\partial x}\nabla^2 + (1-\sigma)\left(\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\nabla^2\right\}\chi + \frac{1}{2\mu}\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial z}\right)\psi, \; \cdots\cdots,$$

$$X_x = \frac{\partial^2 \phi}{\partial x^2} + \left\{\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\nabla^2 - (1-\sigma)\frac{\partial^2}{\partial y \partial z}\bar{\nabla}^2\right\}\chi + \left(\frac{\partial^2}{\partial x \partial y}-\frac{\partial^2}{\partial z \partial x}\right)\psi, \; \cdots\cdots,$$

$$Y_z = \frac{\partial^2 \phi}{\partial y \partial z} + \left\{-\frac{\partial^2}{\partial y \partial z}\bar{\nabla}^2 + \frac{1-\sigma}{2}\frac{\partial}{\partial x}\left(-\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\nabla^2\right\}\chi + \frac{1}{2}\left(\frac{\partial^2}{\partial x \partial y}-\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial x \partial z}\right)\psi, \cdots\cdots,$$

other components of displacement and stress being given by means of cyclical interchange of x, y, z; and ϕ, χ and ψ satisfy the equations

$$\nabla^2\phi = 0, \; \nabla^4\chi = 0 \text{ and } \nabla^2\psi = 0,$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}, \qquad \bar{\nabla}^2 = \frac{\partial^2}{\partial y \partial z}+\frac{\partial^2}{\partial z \partial x}+\frac{\partial^2}{\partial x \partial y}.$$

The boundary condition may be expressed

$$(Z_z)_{z=0} = -\frac{4Q}{\pi^2}\int_0^\infty \int_0^\infty \cos \alpha x \cos \beta y \sin \alpha a \sin \beta b \frac{d\alpha d\beta}{\alpha\beta};$$

and the stress distribution results in

$$Z_z = -\frac{4Q}{\pi^2}\int_0^\infty \int_0^\infty (1+\gamma z) \cos \alpha x \cos \beta y \sin \alpha x \sin \beta b \, e^{-\gamma z}\frac{d\alpha d\beta}{\alpha\beta},$$

$$Y_z = -\frac{4Qz}{\pi^2}\int_0^\infty \int_0^\infty \cos \alpha x \sin \beta y \sin \alpha a \sin \beta b \; e^{-\gamma z}\frac{d\alpha}{\alpha} d\beta,$$

$$Z_x = -\frac{4Qz}{\pi^2}\int_0^\infty \int_0^\infty \sin \alpha x \cos \beta y \sin \alpha a \sin \beta b \; e^{-\gamma z} d\alpha \frac{d\beta}{\beta},$$

means of analytical method. To such cases the theory here developed will serve effectively. In fact, it is not so laborious to secure two effective figures of results by means of mechanical cubature.

Torsion-problem of a prism also requires similar evaluation as to the torsion-function, an example of which will be given elsewhere[3].

A. ON THE MECHANICAL CUBATURE OF STIRLING TYPE

§ 3. We take the double definite integral

$$I=\int_a^{a'}\int_b^{b'} f(x,\ y)\,dx\ dy, \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(1)$$

where a, a', b and b' are constants, and

$$a \leqq x \leqq a', \qquad b \leqq y \leqq b'.$$

For convenience sake, the integrand $f(x,\ y)$ is here supposed to have no singularities within the domain considered.

Now the central interpolation formula of Stirling type for the function $f(x,\ y)$ may be written in the form

$$\begin{aligned} f(x,\ y) &= \sum_{r+s=0}^{n}\sum_{s=0}^{r+s}\Big[\varphi(u,\ r)\ \varphi(v,\ s)\ \varDelta_{2r\ 2s}+\psi(u,\ r)\ \varphi(v,\ s)\frac{1}{2}\varDelta_{(2r+1)\ 2s} \\ &\quad +\varphi(u,\ r)\ \psi(v,\ s)\frac{1}{2}\varDelta_{2r\ (2s+1)}+\psi(u,\ r)\ \psi(v,\ s)\frac{1}{4}\varDelta_{(2r+1)(2s+1)}\Big], \end{aligned}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(2)$$

where

$$\left.\begin{aligned} &u=\frac{x-x_0}{h}, \qquad\qquad v=\frac{y-y_0}{k}, \\ &\varphi(\theta,\ \nu)=\frac{\theta^2(\theta^2-1)(\theta^2-4)\cdots(\theta^2-\overline{\nu-1}^2)}{\underline{|2\nu}}, \quad \psi(\theta,\ \nu)=\frac{\theta(\theta^2-1)(\theta^2-4)\cdots(\theta^2-\nu^2)}{\underline{|2\nu+1}}, \end{aligned}\right\}$$

$$X_x=-\frac{4Q}{\pi^2}\int_0^\infty\int_0^\infty\left(1+2\sigma\frac{\beta^2}{\alpha^2}-\gamma z\right)\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b\ e^{-\gamma z}\frac{\alpha}{\beta\gamma^2}\ d\alpha d\beta,$$

$$Y_y=-\frac{4Q}{\pi^2}\int_0^\infty\int_0^\infty\left(1+2\sigma\frac{\alpha^2}{\beta^2}-\gamma z\right)\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b\ e^{-\gamma z}\frac{\beta}{\alpha\gamma^2}\ d\alpha d\beta,$$

$$X_y=\frac{4Q}{\pi^2}\int_0^\infty\int_0^\infty(1-2\sigma-\gamma z)\sin\alpha x\sin\beta y\sin\alpha a\sin\beta b\ e^{-\gamma z}\frac{1}{\gamma^2}\ d\alpha d\beta,$$

and it would be of some interest to note that the appearance of these expressions is much similar to that derived from Boussinesq's potentials.

The vertical displacement is given by

$$w=\frac{2Q}{\pi^2\mu}\int_0^\infty\int_0^\infty\{2(1-\sigma)+\gamma z\}\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b\ e^{-\gamma z}\frac{d\alpha d\beta}{\alpha\beta\gamma}.$$

Recently, with the aid of two students at the Shinshu University, I have worked out the general solution of Boussinesq's problem, in which any distributions of the normal pressure and the shearing forces within the rectangular form of loaded area are assumed on the top surface of the semi-infinite elastic solid. The solution is again made up of a considerable number of double Fourier's integrals.

[3] B. Tanimoto, 'Difference Method for Partial Differential Equations, Part II,' Transactions of the Japan Society of Civil Engineers, September 1953.

and differences are here written in brief form.

Let the intervals $(a'-a)$ and $(b'-b)$ of the integral (1) be both divided into $2\nu h \times 2\nu k$ -small divisions of equal distance, so that

$$a - a' = 2\nu h, \qquad b' - b = 2\nu k.$$

We then have

$$a = x_0 - \nu h, \qquad a' = x_0 + \nu h, \qquad b = y_0 - \nu k, \qquad b' = y_0 + \nu k.$$

Then the integral (1) in question transforms to

$$I = \int_a^{a'} \int_b^{b'} f(x, y) dx dy = hk \int_{-\nu}^{\nu} \int_{-\nu}^{\nu} F(u, v) du\, dv,$$

where $F(u, v)$ is represented by the right side of (2). Since the ψ-functions are polynomials of odd powers so that integrations of them between $-\nu$ and ν vanish, the equation last written simply becomes

$$I = hk \sum_{r+s=o}^{n} \sum_{s=o}^{r+s} \Delta_{2r\,2s} \int_{-\nu}^{\nu} \varphi(u, r) du \int_{-\nu}^{\nu} \varphi(v, s) dv. \quad \cdots\cdots\cdots\cdots\cdots\cdots(3)$$

We take the function $\varphi(\theta, \nu)$, and on expanding this we may write

$$\varphi(\theta, \nu) = \frac{1}{\underline{|2\nu}} \sum_{t=o}^{\nu-1} (-)^t A_t (\nu-1) \theta^{2\nu-2t}, \cdots\cdots\cdots\cdots\cdots\cdots(4)$$

where

$$\left.\begin{aligned} &A_0(\nu)=1, \qquad A_1(\nu)=1^2+2^2+3^2+\cdots+\nu^2, \\ &A_2(\nu)=1^2(2^2+3^2+\cdots+\nu^2)+2^2(3^2+4^2+\cdots+\nu^2)+\cdots+(\nu-1)^2\nu^2, \\ &A_3(\nu)=1^2\{2^2(3^2+4^2+\cdots+\nu^2)+3^2(4^2+\cdots+\nu^2)+\cdots+(\nu-1)^2\nu^2\} \\ &\qquad +2^2\{3^2(4^2+\cdots+\nu^2)+4^2(5^2+\cdots+\nu^2)+\cdots+(\nu-1)^2\nu^2\} \\ &\qquad +\cdots\cdots\cdots+(\nu-2)^2(\nu-1)^2\nu^2, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ &A_\nu(\nu)=1^2 2^2 3^2 \cdots \nu^2. \end{aligned}\right\} \cdots\cdots(5)$$

§ 4. For actual evaluation of these coefficients, the following recurrence formula is useful:

$$A_t(\nu) = A_t(\nu-1) + \nu^2 A_{t-1}(\nu-1). \quad \cdots\cdots\cdots\cdots\cdots\cdots(6)$$

On integrating both sides of (4) between $-\nu$ and ν, and substituting the result into (3), we have

$$\begin{aligned} I = 4hk \sum_{r+s=0}^{n} \sum_{s=0}^{r+s} & \frac{\Delta_{2r\,2s}}{\underline{|2r}\ \underline{|2s}} \\ & \times \sum_{t=0}^{r-1} (-)^t \frac{A_t(r-1)}{2r-2t+1} \nu^{2r-2t+1} \sum_{t=0}^{s-1} (-)^t \frac{A_t(s-1)}{2s-2t+1} \nu^{2s-2t+1}, \cdots\cdots(7) \end{aligned}$$

which is the required general expression. It is to be noted, in the right side of this result, that, when t formally takes -1 which occurs in cases of $r=0$ and $s=0$, we understand that

$$\left[(-)^t \frac{A_t(r-1)}{2r-2t+1} \nu^{2r-2t}\right]_{r=0} = 1, \quad \left[(-)^t \frac{A_t(s-1)}{2s-2t+1} \nu^{2s-2t}\right]_{s=0} = 1;$$

for in this case we have $\varphi(\theta, 0)=1$, and then

$$\frac{1}{2} \int_{-\nu}^{\nu} \varphi(\theta, 0)\, d\theta = \frac{1}{2} \Big[\theta \Big]_{-\nu}^{\nu} = \nu.$$

For the convenience of practical use, some numerical values of the coefficient $A_t(\nu)$ are given here (Table 1):

Table 1 Numerical values of coefficient $A_t(\nu)$.

$\nu \backslash t$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	5	4					
3	1	14	49	36				
4	1	30	273	820	576			
5	1	55	1,023	7,645	21,076	14,400		
6	1	91	3,003	44,473	296,296	773,136	518,400	
7	1	140	7,462	191,620	2,475,473	15,291,640	38,402,064	25,401,600

§ 5. As a simple example of the general expression (7), we take

$\nu=1$ and retain terms in Δ^2;

and shall obtain a rule, which may be called Simpson's rule of the first kind in the mechanical cubature.

In this case we first have

$$I = 4hk\left[(00) + \frac{1}{\underline{|2}}\Delta_{20} \times \frac{1}{3} + \frac{1}{\underline{|2}}\Delta_{02} \times \frac{1}{3}\right],$$

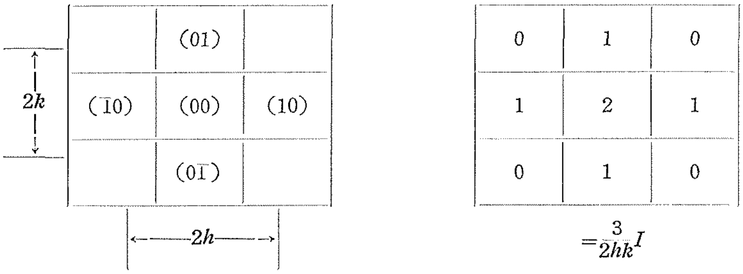
or on rearranging

$$I = 4hk\frac{1}{6}\{2(00) + (10) + (\bar{1}0) + (01) + (0\bar{1})\}. \cdots\cdots\cdots\cdots\cdots\cdots (8)$$

This rule is illustrated in the following figures (Fig. 1).

Fig. 1 Simpson's rule of the first kind.
(Domain of integration$=2h\times 2k$.)

1a. Lattice arrangement. 1b. Weight table.



0	1	0
1	2	1
0	1	0

$=\frac{3}{2hk}I$

This rule is in accordance with that which is obtained by means of a more elementary method, in which a quadric surface of the form

$$f(x, y) = a + bx + cy + dx^2 + exy + fy^2$$

is assumed. In fact, on the assumption that the quadric surface, having six unknown coefficients, can be determined so as to pass six points which are marked ∘ in the following figure (Fig. 2).

Fig. 2

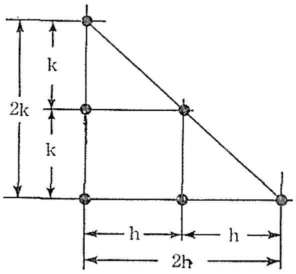
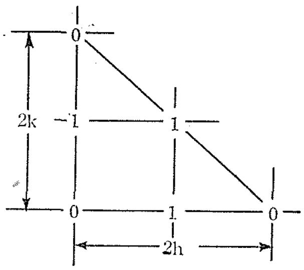


Fig. 3



In this way we can obtain the weight table given in Fig. 3. Then a duplicate use of this weight table at once gives us the weight table of Fig. 1 (right, 1b).

When the above rule (8) tends to one dimension, it will reduce to the well-known Simpson's rule in one dimension. In fact, if we take the limit $\lim_{b'\to b} I/(b'-b)$, we then have

$$\lim_{b'\to b}\frac{I}{b'-b}=\lim_{b'\to b}\frac{1}{b'-b}\int_a^{a'}\int_b^{b'} f(x,\ y)\,dx\ \ dy\ =\int_a^{a'} f(x)\,dx.$$

On the other hand, the rule (8) in this case becomes

$$\lim_{k\to o}\frac{1}{2k}I=\lim_{k\to o}\frac{1}{2k}4hk\frac{1}{6}\{2(00)+(10)+(\bar{1}0)+(01)+(0\bar{1})\}=\frac{h}{3}\{(1)+4(0)+(\bar{1})\},$$

which is the one-dimensional Simpson's rule.

§ 6. If in addition to the above rule (8) we retain term in Δ_{22}, then (7) becomes

$$I=4hk\frac{1}{36}[16(00)+4\{(10)+(\bar{1}0)+(01)+(0\bar{1})\}+\{(11)+(1\bar{1})+(\bar{1}1)+(\bar{1}\bar{1})\}],\ \cdots(9)$$

or briefly

$$I=\frac{hk}{9}\{16(00)+4\Sigma(10)+\Sigma(11)\}.$$

This rule is illustrated in the following figure (Fig. 4).

Fig. 4 Weight table.
(Domain of integration $=2h\times 2k$.)

1	4	1
4	16	4
1	4	1

$$=\frac{9}{hk}I$$

It can easily be proved that this rule (9) may also be derived by duplicate use of the known Simpson's rule in mechanical quadrature. The rule is customarily used in naval engineering in computing the displacement due to a vessel floating on the water.

§ 7. If in addition we take terms in Δ_{40} and Δ_{04} into account, we obtain

$$I = \frac{hk}{45}\{68(00) + 24\Sigma(10) + 5\Sigma(11) - \Sigma(20)\}. \quad \cdots\cdots\cdots\cdots\cdots(10)$$

This rule is illustrated in the following figure (Fig. 5).

Fig. 5 Weight table for mechanical cubature. (Domain of integration $= 2h\times 2k$.)

		−1			
	5	24	5		
−1	24	68	24	−1	$= \frac{45}{hk}I$
	5	24	5		
		−1			

As a simple example of the above rule (10), let us take the elementary integral

$$I = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \cos x \cos y \, dx \, dy;$$

the result of which is at once evaluated as unity. From the integrand $f(x,y) = \cos x \cos y$, we have lattice values as follows (Fig. 6):

Fig. 6 Lattice values of integrand $f(x, y) = \cos x \cos y$.

		−.500,000		
	0	0	0	
.500,000	.707,107	.500,000	0	−.500,000
	1.000,000	.707,107	0	
		.500,000		

Here we have to take

$$h = k = \frac{\pi}{4}.$$

We thus obtain

$$I=\left(\frac{\pi}{4}\right)^2\frac{1}{45}\left\{68\times0.500,000+24\times1.414,214+5\times1.100,000-0\right\}$$
$$=0.999,860,$$

the error entailed being

$$E = \frac{0.999,860-1}{1} = -0.000,140 = -0.014\ \%.$$

§ 8. In the second place, we take

$\nu=2$ and retain terms up to Δ^8;

and shall obtain a more accurate rule for the mechanical cubature. From the general expression (7), we then can obtain, after rearrangement,

$$I=16hk\frac{1}{56,700}\Big\{1,228\Sigma(00)+4,160\Sigma(10)+4,768\Sigma(11)+712\Sigma(20)$$
$$+2,048\Sigma(21)-64\Sigma(30)+343\Sigma(22)-80\Sigma(31)+13\Sigma(40)\Big\}.\cdots(11)$$

This rule may be expressed in the schematic from as follows (Fig. 7).

Fig. 7 Weight table in case of $\nu=2$, $n=4$.
(Domain of integration $= 4h\times4k$.)

				13				
			−80	−64	−80			
		343	2,048	712	2,048	343		
	−80	2,048	4,768	4,160	4,768	2,048	−80	
13	−64	712	4,160	1,228	4,160	712	−64	13
	−80	2,048	4,768	4,160	4,768	2,048	−80	
		343	2,048	712	2,048	343		
			−80	−64	−80			
				13				

$$=\frac{56,700}{16hk}I$$

§ 9. As a simple numerical example of the above rule (11), we take, as before,

$$I = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \cos x \cos y \, dx \, dy.$$

Then we have to take $h=k=\pi/8$, and the corresponding lattice values become as follows (Fig. 8).

Fig. 8 Lattice values of integrand $f(x, y)=\cos x \cos y$.

				-.500,000				
			-.353,553	-.270,598	-.146,446			
		0	0	0	0	0		
	.353,553	.382,683	.353,553	.270,598	.146,446	0	-.146,446	
.500,000	.653,282	.707,107	.653,282	.500,000	.270,598	0	-.270,598	-.500,000
	.853,554	.923,880	.853,554	.653,282	.353,553	0	-.353,553	
		1.000,000	.923,880	.707,107	.382,683	0		
			.853,554	.653,282	.353,553			
				.500,000				

Then we have the result

$$I=\left(\frac{\pi}{2}\right)^2 \frac{1}{56,700}\Big[1,228\times 0.500,000+4,160\times 1.847,760+4,768\times 1.707,106$$
$$+712\times 1.414,214+2,048\times 2.613,126+343\times 1.000,000-64\times 0.765,368$$
$$-80\times 1.414,216+13\times 0\Big]=1.000,000,$$

which shows a perfect accordance with the true value.

An alternative form of the present rule, which is written in the form of respective differences, will give the numerical result

$$I=\left(\frac{\pi}{2}\right)^2 \times\Big[0.500,000-0.101,493+0.006,952-0.000,175+0.000,001\Big]$$
$$=1.000,001,$$

in which we clearly see the convergence of respective differences. From the standpoint of practical calculation, the discrepancy between the above two results would be of no importance, since the significant figures here employed are six in number throughout the calculation.

§ 10. In what follows, attention will be confined to the case

$$\nu = \frac{1}{2}. \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(11)$$

From theoretical standpoint, it is widely accepted to be desirable that the arguments u and v in the interpolation formula should be as small as possible, and that they do not exceed unity. The least value of ν in this case is 1/2, so that the integration may take the form

$$I_0 = \int_{-\frac{k}{2}}^{\frac{k}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} f(x_0+\xi,\ y_0+\eta)\, d\xi d\eta. \cdots\cdots\cdots\cdots\cdots\cdots(12)$$

This integration affords the result extending over one division whose area is hk, and the sum of such elementary integrations will afford the required result of the integration in question.

§ 11. The general expression (7) is then written

$$\frac{I}{hk} = \sum_{r+s=0}^{n} \sum_{s=0}^{r+s} \frac{\Delta_{2r\ 2s}}{\underline{|2r}\ \underline{|2s}} \sum_{t=0}^{r-1} (-)^t \frac{A_t(r-1)}{2r-2t+1} \frac{1}{2^{2r-2t}} \sum_{t=0}^{s-1} (-)^t \frac{A_t(s-1)}{2s-2t+1} \frac{1}{2^{2s-2t}}. \cdots(13)$$

If we write

$$C_p = \sum_{t=0}^{p-1} (-)^t \frac{A_t(p-1)}{2p-2t+1} \frac{1}{2^{2p-2t}},$$

then the equation just written takes the simple form

$$\frac{I}{hk} = \sum_{r+s=0}^{n} \sum_{s=0}^{r+s} C_r C_s \frac{\Delta_{2r\ 2s}}{\underline{|2r}\ \underline{|2s}}.$$

For first values of p, the coefficients C_p's are computed in the following. We first can write down

$$\left.\begin{aligned}
C_0 &= 1,\\
C_1 &= \frac{A_0(0)}{3}\frac{1}{2^2},\\
C_2 &= \frac{A_0(1)}{5}\frac{1}{2^4} - \frac{A_1(1)}{3}\frac{1}{2^2},\\
C_3 &= \frac{A_0(2)}{7}\frac{1}{2^6} - \frac{A_1(2)}{5}\frac{1}{2^4} + \frac{A_2(2)}{3}\frac{1}{2^2},\\
C_4 &= \frac{A_0(3)}{9}\frac{1}{2^8} - \frac{A_1(3)}{7}\frac{1}{2^6} + \frac{A_2(3)}{5}\frac{1}{2^4} - \frac{A_3(3)}{3}\frac{1}{2^2},\\
C_5 &= \frac{A_0(4)}{11}\frac{1}{2^{10}} - \frac{A_1(4)}{9}\frac{1}{2^8} + \frac{A_2(4)}{7}\frac{1}{2^6} - \frac{A_3(4)}{5}\frac{1}{2^4} + \frac{A_4(4)}{3}\frac{1}{2^2},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots.
\end{aligned}\right\}$$

Then we have, on rearranging,

$$\left.\begin{aligned}
&C_0 = 1, \quad C_1 = \frac{1}{12}, \quad C_2 = -\frac{17}{240}, \quad C_3 = \frac{367}{1,344},\\
&C_4 = -\frac{27,859}{11,520}, \quad C_5 = \frac{1,295,803}{33,792}, \quad \cdots\cdots\cdots\cdots.
\end{aligned}\right\}$$

Now the above equation is written

$$\frac{I}{hk} = C_0C_0\Delta_{00} + \frac{C_1C_0}{\underline{|2}\ \underline{|0}}(\Delta_{20}+\Delta_{02}) + \frac{C_2C_0}{\underline{|4}\ \underline{|0}}(\Delta_{40}+\Delta_{04}) + \frac{C_1C_1}{\underline{|2}\ \underline{|2}}\Delta_{22}$$

$$+\frac{C_3C_0}{\underline{|6}\,\underline{|0}}(\Delta_{60}+\Delta_{06})+\frac{C_2C_1}{\underline{|4}\,\underline{|2}}(\Delta_{42}+\Delta_{24})+\frac{C_4C_0}{\underline{|8}\,\underline{|0}}(\Delta_{80}+\Delta_{08})$$
$$+\frac{C_3C_1}{\underline{|6}\,\underline{|2}}(\Delta_{62}+\Delta_{26})+\frac{C_2C_2}{\underline{|4}\,\underline{|4}}\Delta_{44}+\frac{C_5C_0}{\underline{|10}\,\underline{|0}}(\Delta_{10\,0}+\Delta_{0\,10})$$
$$+\frac{C_4C_1}{\underline{|8}\,\underline{|2}}(\Delta_{82}+\Delta_{28})+\frac{C_3C_2}{\underline{|6}\,\underline{|4}}(\Delta_{64}+\Delta_{46})+\cdots\cdots\cdots\cdots.$$

Then the substitution of C_p's into the above equation gives us

$$\frac{I}{hk}=\Delta_{00}+\frac{1}{24}(\Delta_{20}+\Delta_{02})-\frac{17}{5,760}(\Delta_{40}+\Delta_{04})+\frac{1}{576}\Delta_{22}$$
$$+\frac{367}{967,680}(\Delta_{60}+\Delta_{06})-\frac{17}{138,240}(\Delta_{42}+\Delta_{24})-\frac{27,859}{464,486,400}(\Delta_{80}+\Delta_{08})$$
$$+\frac{367}{23,224,320}(\Delta_{62}+\Delta_{26})+\frac{289}{33,177,600}\Delta_{44}$$
$$+\frac{1,295,803}{122,624,409,600}(\Delta_{10\,0}+\Delta_{0\,10})-\frac{27,859}{11,147,673,600}(\Delta_{82}+\Delta_{28})$$
$$-\frac{6,239}{5,573,836,800}(\Delta_{64}+\Delta_{46})+\cdots\cdots\cdots\cdots. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(14)$$

§ 12. As a simple result from (14), we take differences up to the second order into account and neglect higher differences. We then can write down

$$\frac{I}{hk}=\Delta_{00}+\frac{1}{24}(\Delta_{20}+\Delta_{02})+O(\Delta^4),$$

or

$$24\frac{I}{hk}=24\,\Delta_{00}+(\Delta_{20}+\Delta_{02})+O(\Delta^4). \quad \cdots\cdots\cdots\cdots(15)$$

This equation can be expressed in the schmatic form (Fig. 9):

Fig. 9 Weight table for mechanical cubature $\left(\nu=\frac{1}{2}\right)$.

(Domain of integration $=hk$.)

$$\begin{array}{|c|c|c|} \hline & 1 & \\ \hline 1 & 20 & 1 \\ \hline & 1 & \\ \hline \end{array} = 24\frac{I}{hk}+O(\Delta^4)$$

It will appear that the rule just written is recommended for ordinary use.

A criterion for the error of this rule can be obtained as follows. The above rule, Fig. 9, or equation (15), is written

$$\frac{I}{hk}=(00)+\frac{1}{24}(h^2D_{20}+k^2D_{02})f+\frac{1}{288}(h^4D_{40}+k^4D_{04})f+\cdots\cdots\cdots,$$

where

$$D_{rs}f = \frac{\partial^{r+s}f}{\partial x^r\,\partial y^s} \quad \text{at (00) lattice-point.}$$

On the other hand the true integral (12) gives the expansion

$$I_0 = \int_{-\frac{k}{2}}^{\frac{k}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} f(x_0+\xi,\ y_0+\eta)\,d\xi\,d\eta$$

$$= \int_{-\frac{k}{2}}^{\frac{k}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \Big\{ f(x_0, y_0) + \frac{1}{2}(\xi^2 D_{20} + \eta^2 D_{02}) f + \frac{1}{24}(\xi^4 D_{04} + 6\xi^2\eta^2 D_{22} + \eta^4 D_{04}) f + \cdots\cdots\cdots\cdots \Big\}\,d\xi\,d\eta$$

$$= hk\Big[(00) + \frac{1}{24}(h^2D_{20} + k^2D_{02})f + \frac{1}{1,920}(h^4D_{40} + k^4D_{04})f + \frac{1}{576}h^2k^2D_{22}f + \cdots\cdots\Big].$$

On subtracting we have

$$E = I - I_0 = hk\Big\{\frac{17}{5,760}(h^4D_{40} + k^4D_{04})f - \frac{1}{576}h^2k^2D_{22}f + \cdots\cdots\Big\}$$

$$= hk\Big\{0.002,951\cdots \times (h^4D_{40} + k^4D_{04})f - 0.001,736\cdots \times h^2k^2D_{22}f + \cdots\cdots\Big\},$$

which will afford a criterion for the evaluation of error. In comparison with this, the rule (8), which is a Simpson's rule, has as its error

$$E' = \frac{I}{4} - I_0 = hk\Big\{0.011,805\cdots \times (h^4D_{40} + k^4D_{04})f + \cdots\cdots\Big\},$$

so that we might see the superiority of the rule given in Fig. 9.

§ 13. Furthermore, if in equation (14) differences of the fourth order are taken into account, we may obtain

$$5,760\frac{I}{hk} = 5,760\,\Delta_{00} + 240\,(\Delta_{20} + \Delta_{02}) - 17\,(\Delta_{40} + \Delta_{04}) + 10\,\Delta_{22} + O\,(\Delta^6).$$

This equation can be expressed in the schematic form (Fig. 10):

Fig. 10 Weight table for mechanical cubature ($\nu=1/2$).
(Domain of integration $=hk$.)

		−17		
	10	288	10	
−17	288	4,636	288	−17
	10	288	10	
		−17		

$$= 5,760\,\frac{I}{hk} + O(\Delta^6)$$

§ 14. Sometimes lattice values are connected one another with a certain differential equation. In such a case higher differences in equation (14) can be lowered by taking advantage of the differential equation considered. The device has been originated by Mr. Nishimura, assistant professor of Nagoya University, in case of the solving of partial differential equations.

In the following we assume that lattice values are related with the differential equation

$$\nabla^2 f + c = 0. \quad (c \text{ being constant.}) \quad \cdots\cdots\cdots\cdots\cdots\cdots(16)$$

In the subsequent calculation, frequent references may be made to 'Relations between Differences and Derivatives——Derivatives in terms of Differences, and vice versa'[4].

§ 15. From (14), we take

$$\frac{I}{hk} = \Delta_{00} + \frac{1}{24}(\Delta_{20} + \Delta_{02}) - \frac{17}{5,760}(\Delta_{40} + \Delta_{04}) + \frac{1}{576}\Delta_{22}$$
$$+ \frac{367}{967,680}(\Delta_{60} + \Delta_{06}) - \frac{17}{138,240}(\Delta_{42} + \Delta_{24}) + O(\Delta^8). \quad \cdots\cdots\cdots(17)$$

This equation, as it stands, extends over seven lattices in both directions of x and y. But the extension can be reduced to 3×3-lattice distribution by taking advantage of the differential equation (16). We now have, with $h=k$,

$$(\Delta_{40} + \Delta_{04}) = h^4 (D_{40} + D_{04}) f + \frac{h^6}{6}(D_{60} + D_{06}) f + O\,(h^8),$$
$$(\Delta_{60} + \Delta_{06}) = h^6 (D_{60} + D_{06}) f + O\,(h^8),$$
$$(\Delta_{42} + \Delta_{24}) = h^6 (D_{42} + D_{24}) f + O\,(h^8),$$

but we have, with the differential equation (16),

$$(D_{40} + D_{04}) f = \nabla^4 f - 2D_{22} f = -2D_{22} f,$$
$$(D_{60} + D_{06}) f = \nabla^6 f - 3D_{22} \nabla^2 f = 0,$$
$$(D_{42} + D_{24}) f = D_{22} \nabla^2 f = 0\,.$$

Then the differences just written become respectively

$$(\Delta_{40} + \Delta_{04}) = -2h^4 D_{22} f + O(h^8), \quad (\Delta_{60} + \Delta_{06}) = (\Delta_{42} + \Delta_{24}) = O(h^8)\,;$$

and furthermore

$$h^4 D_{22} f = \Delta_{22} - \frac{1}{12}(\Delta_{42} + \Delta_{24}) + O\,(h^8) = \Delta_{22} + O\,(h^8),$$

so that we have

$$\Delta_{40} + \Delta_{04} = -2\,\Delta_{22} + O\,(h^8).$$

Thus (17) becomes

$$1,440\,\frac{I}{hk} = 1,440\,\Delta_{00} + 60\,(\Delta_{20} + \Delta_{02}) + 11\,\Delta_{22} + O\,(h^8). \quad \cdots\cdots\cdots\cdots(18)$$

This equation can be expressed in the schematic from (Fig. 11):

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[4] These tables will be published elsewhere.

Fig. 11 Weight table for mechanical cubature, provided $\nabla^2 f + c = 0$.
(Domain of integration $= hk$.)

11	38	11
38	1,244	38
11	38	11

$$= 1{,}440 \, \frac{I}{hk} + \mathrm{O}\,(h^8)$$

If the whole domain of integration consists of 2×2-divisions, we aggregate weights of the four adjacent schemes, each of which is due to Fig. 11. We then obtain the following scheme (Fig. 12):

Fig. 12 Weight table for mechanical cubature, provided $\nabla^2 f + c = 0$.
(Domain of integration $= 4hk$.)

11	49	49	11
49	1,331	1,331	49
49	1,331	1,331	49
11	49	49	11

$$= 1{,}440 \, \frac{I}{hk} + \mathrm{O}\,(h^8)$$

It is noted that a seeming restriction that $h = k$ is imposed on the process of the above reduction, but this is in reality not the case, the reason of which will be clear from the above reduction.

§ 16. When in equation (14) differences up to the tenth order are taken into account, the lattice distribution will be of 5×5-extent with reference to the differential equation (16), and a calculation similar to §15 will afford the following result (Fig. 13):

Fig. 13 Weight table for mechanical cubature, provided $\nabla^2 f + c = 0$.
(Domain of integration $= hk$.)

21,728	−144,032	−1,126,272	−144,032	21,728
−144,032	1,611,008	21,903,168	1,611,008	−144,032
−1,126,272	21,903,168	376,000,128	21,903,168	−1,126,272
−144,032	1,611,008	21,903,168	1,611,008	−144,032
21,728	−144,032	−1,126,272	−144,032	21,728

$$= 464{,}486{,}400 \, \frac{I}{hk} + \mathrm{O}\,(h^{12})$$

B. ON THE MECHANICAL CUBATURE OF MODIFIED BESSEL TYPE

§ 17. We shall derive anotner general expression for the mechanical cubature of the defihite integral

$$I = \int_a^{a'} \int_b^{b'} f\,(x,\ y)\ dx\ dy \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(19)$$

by using the interpolation formula of modified Bessel type, where

$$a \leqq x \leqq a', \qquad b \leqq y \leqq b';$$

and shall obtain some of useful rules for the mechanical cubature.

§ 18. The interpolation formula is written in the form

$$f(x,\ y) = \sum_{r+s=0}^{n} \sum_{s=0}^{r+s} \Big[\rho(u,\ r)\ \rho(v,\ s)\frac{1}{4}\varDelta_{2r\ 2s} + \sigma(u,\ r)\ \rho(v,\ s)\frac{1}{2}\varDelta_{(2r+1)2s}$$

$$+\rho(u,\ r)\,\sigma(v,\ s)\frac{1}{2}\varDelta_{2r\ (2s+1)} + \sigma(u,\ r)\ \sigma(v,\ s)\varDelta_{(2r+1)\,(2s+1)}\Big], \cdots(20)$$

where

$$\left.\begin{aligned} u &= \frac{x-x_0}{h} - \frac{1}{2}, \qquad v = \frac{y-y_0}{k} - \frac{1}{2}, \\ \rho(\theta,\ \nu) &= \frac{1}{\underline{|2\nu}}\Big(\theta^2 - \frac{1}{4}\Big)\Big(\theta^2 - \frac{9}{4}\Big)\cdots\Big(\theta^2 - \frac{\overline{2\nu-1}^2}{4}\Big), \\ \sigma(\theta,\ \nu) &= \frac{1}{\underline{|2\nu+1}}\,\theta\,\Big(\theta^2 - \frac{1}{4}\Big)\Big(\theta^2 - \frac{9}{4}\Big)\cdots\Big(\theta^2 - \frac{\overline{2\nu-1}^2}{4}\Big); \end{aligned}\right\}\cdots\cdots\ (21)$$

and differences, which are written in brief form, may be converted to linear combination of lattice values multiplied by weights.

§ 19. Now let the intervals $(a'-a)$ and $(b'-b)$ of the integration (19) be both divided into $2n$ small divisions of equal distance. We then have

$$\left.\begin{aligned} a &= x_0 - \Big(n - \frac{1}{2}\Big)h, \qquad a' = x_0 + \Big(n + \frac{1}{2}\Big)h, \\ b &= y_0 - \Big(n - \frac{1}{2}\Big)k, \qquad b' = y_0 + \Big(n + \frac{1}{2}\Big)k. \end{aligned}\right\}$$

On transforming the variables x and y to another set of variables u and v, we can write the integral (19) in the form

$$I = \int_a^{a'} \int_b^{b'} f\,(x, y)\ dx\ dy = hk \int_{-n}^{n} \int_{-n}^{n} F\,(u, v)\ du\ dv, \cdots\cdots\cdots\cdots(22)$$

where $F\,(u, v)$ is represented by the right side of (20). Since we in general have

$$\int_{-n}^{n} \theta^{2\nu}\ d\theta = \frac{2}{2\nu+1}\,n^{2\nu+1}, \qquad \int_{-n}^{n} \theta^{2\nu+1}\ d\theta = 0,$$

and since the σ-function defined in (21) is polynomials of odd powers, the above integral (22) simply reduces to

$$I = \frac{hk}{4} \sum_{r+s=0}^{m} \sum_{s=0}^{r+s} \varDelta_{2r\,2s} \int_{-n}^{n} \rho\,(u,\ r)\,du \int_{-n}^{n} \rho\,(v,\ s)\,dv. \quad \cdots\cdots\cdots\cdots(23)$$

The function $\rho\,(\theta, \nu)$ is expanded in the form

$$\underline{|2\nu}\ \rho\,(\theta,\ \nu) = \sum_{t=0}^{\nu} (-)^t B_t\,(\nu)\theta^{2\nu-2t} \quad (\nu=0,\ 1,\ 2,\ \cdots;\ t \leqq \nu), \quad \cdots\cdots\cdots(24)$$

where

$$\left.\begin{aligned}
&B_0(\nu)=1,\\
&B_1(\nu)=\frac{1}{2^2}\Big[1^2+3^2+5^2+\cdots+(2\nu-1)^2\Big],\\
&B_2(\nu)=\frac{1}{2^4}\Big[1^2\{3^2+5^2+\cdots+(2\nu-1)^2\} + 3^2\{5^2+7^2+\cdots+(2\nu-1)^2\}\\
&\qquad\qquad + \cdots + (2\nu-3)^2(2\nu-1)^2\Big],\\
&B_3(\nu)=\frac{1}{2^6}\Big[1^2\Big(3^2\{5^2+7^2+\cdots+(2\nu-1)^2\} + 5^2\{7^2+9^2+\cdots+(2\nu-1)^2\}\\
&\qquad\qquad +\cdots+(2\nu-3)^2(2\nu-1)^2(2\nu-1)^2\Big)+3^2\Big(5^2\{7^2+9^2+\cdots\\
&\qquad\qquad +(2\nu-1)^2\}+7^2\{9^2+\cdots+(2\nu-1)^2\}+\cdots+(2\nu-3)^2\\
&\qquad\qquad (2\nu-1)^2\Big)+\cdots\cdots\cdots\cdots+(2\nu-5)^2(2\nu-3)^2(2\nu-1)^2\Big],\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
&B_\nu\,(\nu)=\frac{1}{2^{2\nu}}1^2 3^2 5^2 \cdots (2\nu-1)^2.
\end{aligned}\right\}$$

For the actual evaluation of coefficients $B_t\,(\nu)$'s, it is to be noted that the following recurrence formula holds:

$$B_t\,(\nu) = B_t\,(\nu-1) + \frac{(2\nu-1)^2}{4} B_{t-1}\,(\nu-1). \quad \cdots\cdots\cdots\cdots\cdots(25)$$

Now on integrating both members of (24) between the limits $-n$ and n, we have

$$\frac{\underline{|2\nu}}{2}\int_{-n}^{n} \rho(\theta,\ \nu)\,d\theta = \sum_{t=0}^{\nu} (-)^t \frac{B_t\,(\nu)}{2\nu-2t+1} n^{2\nu-2t+1}.$$

Then on taking $\nu=r$ and $\nu=s$, and substituting the results into (23), we obtain

$$I = hk \sum_{r+s=0}^{m} \sum_{s=0}^{r+s} \frac{\varDelta_{2r\,2s}}{\underline{|2r}\ \underline{|2s}} \sum_{t=0}^{r} (-)^t \frac{B_t\,(r)}{2r-2t+1} n^{2r-2t+1} \sum_{t=o}^{s} (-)^t \frac{B_t\,(s)}{2s-2t+1} n^{2s-2t+1}, \quad \cdots\cdots\cdots\cdots\cdots\cdots(26)$$

which is the required general expression.

From (26) we can get the simplest rule, which is of the form

$$I = \frac{hk}{4}\Big\{(00) + (10) + (01) + (11)\Big\}. \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots(27)$$

This is sometimes employed in civil engineering in computing cut or bank extending over a broad area.

§ **20.** With the aid of the recurrence formula (25), numerical values of the coefficients $B_t(\nu)$'s are tabulated thus (Table 2):

Table 2. Numerical values of coefficient $B_t(\nu)$ $(t \leqq \nu)$.

ν \ t	0	1	2	3	4	5	6	7
0	1							
1	1	$\frac{1}{4}$						
2	1	$\frac{10}{4}$	$\frac{9}{16}$					
3	1	$\frac{35}{4}$	$\frac{259}{16}$	$\frac{225}{64}$				
4	1	$\frac{84}{4}$	$\frac{1,974}{16}$	$\frac{12,916}{64}$	$\frac{11,025}{256}$			
5	1	$\frac{165}{4}$	$\frac{8,778}{16}$	$\frac{172,810}{64}$	$\frac{1,057,221}{256}$	$\frac{893,025}{1,024}$		
6	1	$\frac{286}{4}$	$\frac{28,743}{16}$	$\frac{1,234,948}{64}$	$\frac{21,967,231}{256}$	$\frac{128,816,766}{1,024}$	$\frac{108,056,025}{4,096}$	
7	1	$\frac{455}{4}$	$\frac{77,077}{16}$	$\frac{6,092,515}{64}$	$\frac{230,673,443}{256}$	$\frac{3,841,278,805}{1,024}$	$\frac{21,878,089,479}{4,096}$	$\frac{18,261,468,225}{16,384}$

§ 21. The least value of n in the general expression (26) must be $n=\frac{1}{2}$, so that the integral (19) takes the form

$$I = \int_{x_0}^{x_0+h} \int_{y_0}^{y_0+k} f(x, y)\, dx\, dy,$$

and the general expression (26) is written

$$I = hk \sum_{r+s=0}^{m} \sum_{s=0}^{r+s} C_r\, C_s \frac{\Delta_{2r\,2s}}{\underline{\lfloor 2r}\ \underline{\lfloor 2s}}, \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots(28)$$

where

$$C_p = \sum_{t=0}^{p} (-)^t \frac{B_t(p)}{2p-2t+1} \frac{1}{2^{2p-2t+1}}.$$

Some of first values of C_p are written down as follows:

$$\left.\begin{aligned} C_0 &= \frac{B_0(0)}{1}\frac{1}{2}, \\ C_1 &= \frac{B_0(1)}{3}\frac{1}{2^3} - \frac{B_1(1)}{1}\frac{1}{2}, \\ C_2 &= \frac{B_0(2)}{5}\frac{1}{2^5} - \frac{B_1(2)}{3}\frac{1}{2^3} + \frac{B_2(2)}{1}\frac{1}{2}, \\ C_3 &= \frac{B_0(3)}{7}\frac{1}{2^7} - \frac{B_1(3)}{5}\frac{1}{2^5} + \frac{B_2(3)}{3}\frac{1}{2^3} - \frac{B_3(3)}{1}\frac{1}{2}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots. \end{aligned}\right\}$$

or, on rearranging,

$$C_0 = \frac{1}{2}, \quad C_1 = -\frac{1}{12}, \quad C_2 = \frac{11}{60}, \quad C_3 = -\frac{191}{168}, \quad \cdots\cdots\cdots\cdots.$$

Thus equation (28) becomes

$$\frac{I}{hk} = \frac{1}{4}\Delta_{00} - \frac{1}{48}(\Delta_{20} + \Delta_{02}) + \frac{11}{2{,}880}(\Delta_{40} + \Delta_{04}) + \frac{1}{576}\Delta_{22}$$
$$- \frac{191}{241{,}920}(\Delta_{60} + \Delta_{06}) - \frac{11}{34{,}560}(\Delta_{42} + \Delta_{24}) + \cdots\cdots\cdots. \qquad \cdots\cdots\cdots(29)$$

§ 22. From equation (29), we can successively obtain the following (Fig. 14).

Fig. 14 Weight table for mechanical cubature.
(Domain of integration $=hk$.)

14a. O(Δ^2)-type.

1	1
1	1

$$= \frac{4}{hk}I + \mathrm{O}(\Delta^2)$$

14b. O(Δ^4)-type.

	−1	−1	
−1	14	14	−1
−1	14	14	−1
	−1	−1	

$$= \frac{48}{hk}I + \mathrm{O}(\Delta^4)$$

14c. $O(\Delta^6)$-type.

		11	11		
	5	−98	−98	5	
11	−98	889	889	−98	11
11	−98	889	889	−98	11
	5	−98	−98	5	
		11	11		

$$= 2,880 \frac{I}{hk} + O(\Delta^6)$$

14d. $O(\Delta^8)$-type.

			−191	−191			
		−77	1,956	1,956	−77		
	−77	882	−10,336	−10,336	882	−77	
−191	1,956	−10,336	76,894	76,894	−10,336	1,956	−191
−191	1,956	−10,336	76,894	76,894	−10,336	1,956	−191
	−77	882	−10,336	−10,336	882	−77	
		−77	1,956	1,956	−77		
			−191	−191			

$$= 241,920 \frac{I}{hk} + O(\Delta^8)$$

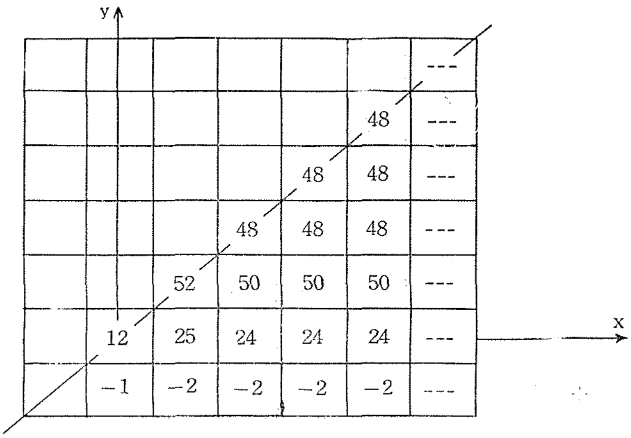
§ 23. If we consider the integral of the form

$$I = \int_0^\infty \int_0^\infty f(x, y)\, dx\, dy,$$

in which the integrand $f(x,y)$ is supposed to be of rapid convergence, then the evaluation of this integral may be performed by repeated application of the above rules.

For instance, '14b' is adopted, and then the aggregate rule for the integration becomes (Fig. 15):

Fig. 15 Aggregate rule for mechanical cubature of O(Δ^4)-type.
(Domain of integration$=\infty\times\infty$.)



					48	---
				48	48	---
			48	48	48	---
		52	50	50	50	---
	12	25	24	24	24	---
	−1	−2	−2	−2	−2	---

$$=48\frac{I}{hk}+O(\Delta^4)$$

In this diagram, lattice weights in the upper half are not written, and these are at once written down from their symmetrical distribution with respect to the diagonal.

§ 24. A simple application of one of rules obtained above will be given to the evaluation of the integral

$$-\frac{\pi^2}{4Q}Z_z=\int_0^\infty\int_0^\infty f(\alpha,\ \beta\,;\ x,\ y,\ z)\ d\alpha\, d\beta, \quad\cdots\cdots\cdots\cdots\cdots\cdots(30)$$

where

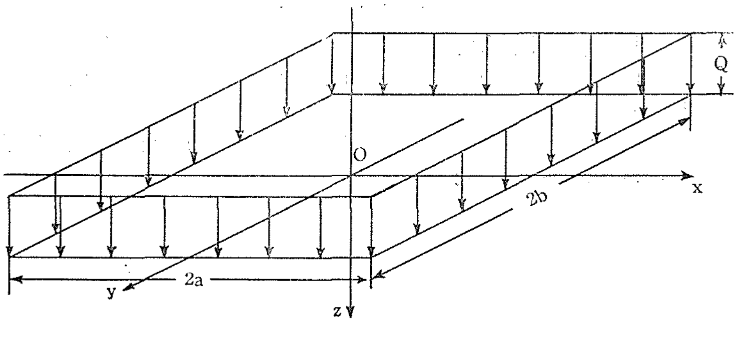
$$f\,(\alpha,\ \beta\,;\ x, y, z, a, b)=\frac{1+\gamma z}{\alpha\beta}\cos\alpha x\cos\beta y\sin\alpha a\sin\beta b\ e^{-\gamma z}, \quad\cdots\cdots\cdots(31)$$

provided

$$\alpha^2+\beta^2=\gamma^2.$$

This is a stress-component in the solution of a Boussinesq's problem, in which a uniform load, its intensity per unit of area being Q, is applied over a rectangular area $2a\times 2b$, which is shown in Fig. 16.

Fig. 16 Configuration of Boussinesq's problem.



We suppose that

$$a = b = 1;$$

and shall find the value of Z_z/Q at a coordinate point where $x=0$, $y=0$, $z=1$. Then the integrand (31) reduces to

$$f(\alpha, \beta;\, 0,\, 0,\, 1) = \frac{1+\gamma}{\alpha\beta} \sin\alpha \sin\beta\, e^{-\gamma}.$$

The following lattice values of the integrand will then be obtained (Fig. 17).

Fig. 17 Lattice values of the integrand $f(\alpha, \beta) = \frac{1+\gamma}{\alpha\beta} \sin\alpha \sin\beta\, e^{-\gamma}$.

$\left(\text{Pitch } h = k = \frac{\pi}{4}.\right)$

								0	0
							0	0	0
						1	1	0	0
					0	0	0	0	0
				14	0	−3	−2	−1	0
			141	43	0	−9	−6	−2	0
		568	272	78	0	−15	−9	−3	0
	1, 000	732	340	96	0	−18	−11	−4	0
	732	568	272	78	0	−15	−9	−3	0

In this diagram also, lattice values in the upper half are not written; but these at once follow from symmetry, since we have

$$f(\alpha, \beta) = f(\beta, \alpha).$$

Thus by multiplying corresponding values in Figs. 15 and 17, and adding the results, we have

$$\frac{Z_z}{Q} = -\frac{4}{\pi^2}\frac{hk}{48}\Sigma\kappa(rs) = -\frac{134.256}{192} = -0.699 = -0.70.$$

This value is in accordance with that from the alternative solution, whose mathematical reduction was due to Love[5], its numerical calculation being due to the late Jiro Kimura.

§ 25. To save labours in calculation, we may take

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5) A. E. H. Love, Phil. Trans., A, vol. 228 (1929).

$$h = k = \frac{\pi}{2}.$$

In this case lattices to be computed reduce a great deal in number, and the following lattice values are sufficient (Fig. 18).

Fig. 18 Lattice values of the integrand $f\,(\alpha,\ \beta) = \frac{1+\gamma}{\alpha\beta}\sin\alpha\sin\beta\, e^{-\gamma}$.

$\left(\text{Pitch}\ \ h = k = \frac{\pi}{2}.\right)$

				0	0
			0	0	0
		141	0	−6	0
	1, 000	340	0	−11	0
	(340)	(141)	0	(−6)	0

Then the mechanical cubature in question amounts, with Figs. 15 and 17, to

$$-\frac{Z_z}{Q} = \frac{4}{\pi^2} I = \frac{4}{\pi^2}\frac{hk}{48}\Sigma\,\kappa\,(rs) = \frac{33.984}{48} = 0.708 = 0.71.$$

We see that the result obtained is fairly good, in spite of the computation at only four lattices.

In this way, we can obtain the following result (Fig. 19).

This result is in accordance with that of Love's solution, the numerical evaluation of which was worked out by the late J. Kimura[6].

§ 26. It can be inferred from the foregoing results that the general expression for the mechanical biquadrature of the form

$$I = \int_a^{a'}\int_b^{b'}\int_c^{c'} f\,(x,\ y,\ z)\,dx\,dy\,dz$$

in terms of the interpolation formula of Stirling type for the function $f(x,\ y,\ z)$, will take the form

$$I = 8\ hkl \sum_{r+s=0}^{n}\ \sum_{s+t=0}^{r+s}\ \sum_{t=0}^{s+t} \frac{C_r\,C_s\,C_t}{\underline{|2r}\ \underline{|2s}\ \underline{|2t}}\,\Delta_{2r\ 2s\ 2t},$$

in which

$$C_p = \sum_{t=0}^{p-1} (-)^t\,\frac{A_t\,(p-1)}{2p-2t+1}\,\nu^{2p-2t+1}.$$

The above expression may be developed as was done in the mechan-

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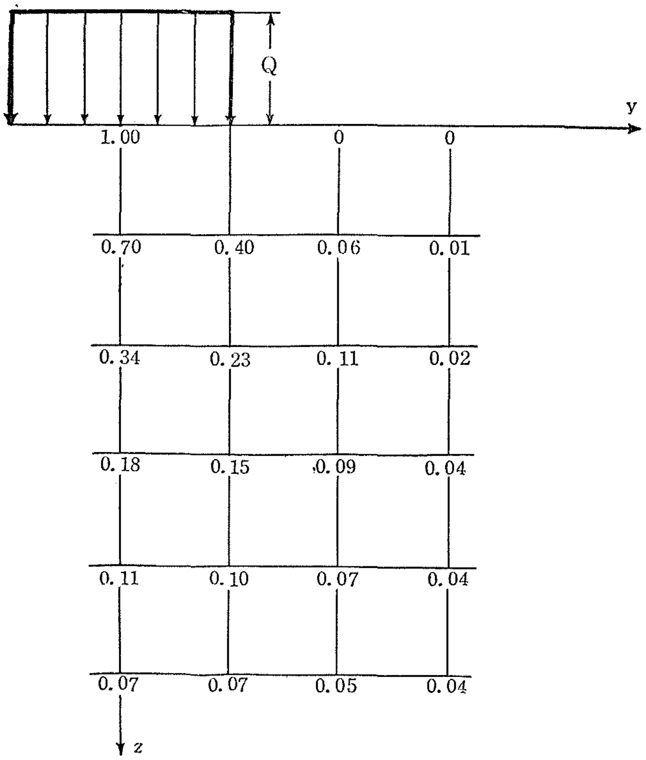
6) J. Kimura, 'Stresses in Soil Loaded with a Square Block on its Surface,' Bulletin of the Geotechnical Committee, Government Railways of Japan, June, 1931.

Fig. 19 Values of $-Z_z/Q$ in the plane $x = 0$.



ical cubature. General expressions of different type may also be derived from other interpolation formulas for the function $f(x, y, z)$. Furthermore, general expressions for the integration of higher order can also be written down if wanted.

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