NEW APPROXIMATION FOR TIME-DELAYED LINEAR SYSTEM

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1. INTRODUCTION

It has long been considered how to approximate the time delay element in the control systems with it. Many kinds of approximate method for time delay element in the control system has been presented. Above all, the Pade's method which approximates the time delay with a rational function using complex variable "s" is typical one. In this method, the larger the lag time is, the higher the order of the function becomes, and the order of this function sometimes increases over the order of original system.

It has been known that the Pade's approach is less valid for transiential region. Then, the paper gives a new approximate technique to improve this drawbacks. This method is not approximation for delay element itself, but for the system with time-delayed state.

2. STATEMENT OF SYSTEM

In this paper, we consider the system to be described by first order linear differential-difference equation with constant coefficients, such as

$$\frac{dx(t)}{dt} = ax(t) + bx(t-l) + m(t) \qquad \cdots\cdots\cdots\cdots (1)$$

$$\{x(\tau) = \phi(\tau);\ \tau \in [-l,\ 0]\},\ t \geqq 0$$

where

a, b : coefficients
l : time delay (constant)
m : input variable
t : time
ϕ : initial condition.

The general solution for equation (1) is expressed as follows,

$$x(t)=\mathrm{K}(t,\ 0)\phi(0)+\int_{-l}^{0}\mathrm{K}(t,\ \tau+l)\,b\phi(\tau)\,d\tau$$

$$+\int_{0}^{t}\mathrm{K}(t,\ \tau)\,m(\tau)d\tau \qquad \cdots\cdots\cdots\cdots (2)$$

a kernel $\mathrm{K}(t,\ \tau)$ satisifies the following condition and adjoint equations,

$$\mathrm{K}(t,\ \tau)=\begin{cases}1 & \tau=t\\ 0 & \tau>t\end{cases} \qquad \cdots\cdots\cdots\cdots (3)$$

$$\frac{\partial\mathrm{K}(t,\ \tau)}{\partial\tau}=-a\mathrm{K}(t,\ \tau)+b\mathrm{K}(t,\ \tau+l) \qquad \cdots\cdots\cdots\cdots (4)$$

$$\frac{\partial\mathrm{K}(t,\ \tau)}{\partial t}=a\mathrm{K}(t,\ \tau)+b\mathrm{K}(t-l,\ \tau). \qquad \cdots\cdots\cdots\cdots (5)$$

It is very difficult however to obtain the analytical solution of $\mathrm{K}(t,\ \tau)$.

Then, this paper outlines a graphical technique to obtain an approximate solution for $\mathrm{K}(t,\ \tau)$.

3. PADE'S APPROXIMATION

We obtain the following formula by the Laplace transform of eq. (1) under zero state condition i. e. $\{\phi(\tau)=0;\ \tau\in[-l,\ 0]\}$

$$(s-a-be^{-ls})\ X(s)=M(s) \qquad \cdots\cdots\cdots\cdots (6)$$

where $X(s)$ and $M(s)$ are Laplace transfoms of $x(t)$ and $m(t)$, respectively.

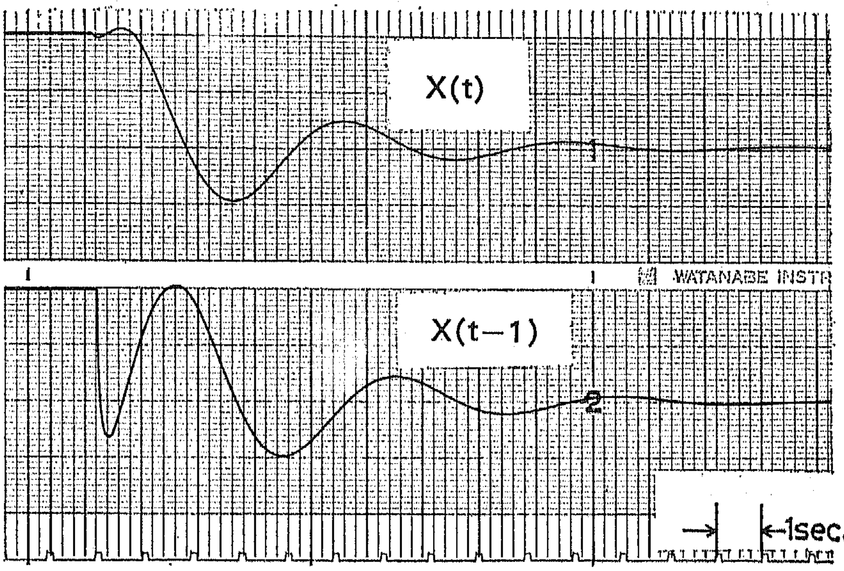
Pade approximated e^{-ls} in eq. (6) with a rational function by the Maclaurin expansion and by the method of indeterminate coefficients. For instance, using second order denominator and third order numerator, we get

$$e^{-ls}\cong\frac{1+a_1s+a_2s^2+a_3s^3}{b_0+b_1s+b_2s^2} \qquad \cdots\cdots\cdots\cdots (7)$$

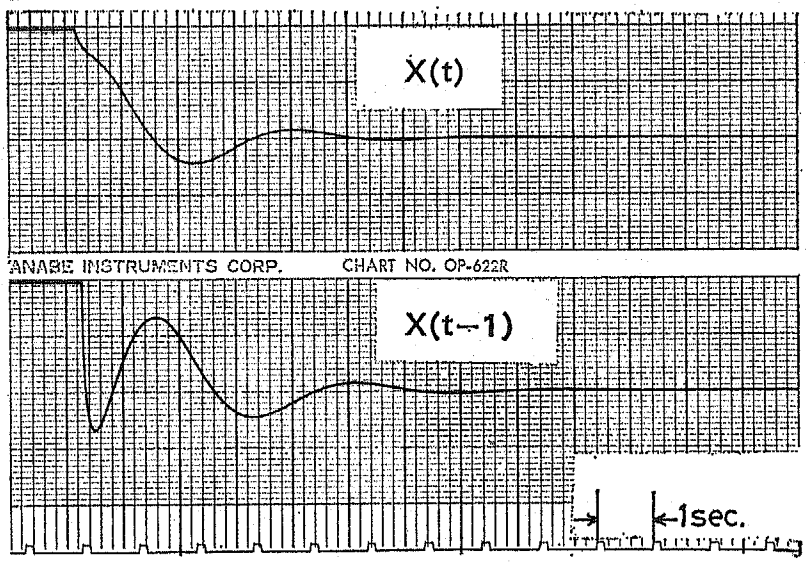
These six coefficients are determined by the above described approach. Appling this method, we have to consider the setting of initial condition. Figure 1 shows two solutions with Pade's approximation (by analog computer). The initial conditions are $\phi(0)=1$, $\{\phi(\tau)=0;\ \tau\in[-1,\ 0)]\}$. We can recognize the larger discrepancy on interval $0\leqq t\leqq 1$.

4. NEW APPROXIMATION

4-1 Purpose of Approximation.



(A) $\frac{dX(t)}{dt} = -X(t-1)$; $\phi(0)=1$, $\{\phi(\tau)=0;\ \tau\in[-1,\ 0)\}$



(B) $\frac{dX(t)}{dt} = -X(t)-X(t-1)$; $\phi(0)=1$, $\{\phi(\tau)=0;\ \tau\in[-1,\ 0)\}$

Fig. 1 Solutions with Pade's Approximation

The purpose of this paper is to improve of Pade's approximation provided that

1) Setting up of arbitrary function $\phi(\tau)$ is possible,
2) Approximation on transiential state is improved,
3) Order of approximated system is less than 3.

For these objects, the author does not approximate the time delay element, but approximates the integral kernel, $K(t, \tau)$ in an expression of eq. (2), provided that the original system is stable.

4-2 Solution by Continuation Process.

In section 2, the author described that an integral kernel $K(t, \tau)$ was not derived analytically. However the continuation method provides a solution. By this method, the solution is extended forward that is, in the direction of increasing t from interval to interval.

For instance, suppose that

$$\phi(0)=1, \quad \{\phi(\tau)=0 : \tau \in [-l, 0)\}$$
$$m(t)=0 \quad t \in [0, \infty)$$

then eq. (1) determines $x(t)$ as following;

$$x(t)=e^{at}\sum_{j=0}^{i-1}\frac{b^j}{j!}e^{jal}(t-jl)^j \qquad \cdots\cdots\cdots\cdots (8)$$
$$t \in [(i-1)l,\ il] \quad (i=1,\ 2,\ 3,\cdots\cdots)$$

We shall call this solution "solution in i-th interval". It is evident that solution (8) is equivalent to $K(t, 0)$ in eq. (2).

And since the original system is steady system, we can replace $K(t, \tau)$ with $K(t-\tau)$ in equation (2).

If it is supposed that

$$al=k_a \quad bl=k_b \ (k_a,\ k_b;\ \text{constant})$$

then the solution in i-th interval is as follows,

$$x(t)\ (=K(t)) = \begin{cases} e^{k_a t}\sum_{j=0}^{i-1}\frac{k_b{}^j}{j!}e^{-jk_a}(t-j)^j \ \ t\in[i-1,\ i] \\ \text{for } l=1 & \cdots\cdots\cdots\cdots (9) \\ e^{(k_a/l)t}\sum_{j=0}^{i-1}\frac{(k_b/l)^j}{j!}e^{-jk_a}(t-jl)^j \\ \text{for } l\neq 1 \quad t\in[(i-1)l,\ il] & \cdots\cdots\cdots\cdots (10) \end{cases}$$

If we put $\tilde{t}=t/l$, then the solution (10) is rewritten as follows,

$$e^{k_a\tilde{t}}\sum_{j=0}^{i-1}\frac{k_b{}^j}{j!}e^{-jk_a}(\tilde{t}-j)^j \quad \tilde{t}\in[i-1,\ i] \qquad \cdots\cdots\cdots\cdots (11)$$

From solutions (9), (10) and (11), it is concluded that if we exchange time t for nondimensional time $\tilde{t}$, then these solutions are equivalent to each other, provided that both products al and bl are constant.

4-3 Solution by New Approximation

In section 4-1, the author described already that the system was assumed to be stable linear.

The stable region of first order steady state linear system with time delayed state is shown in Figure 2.[2] In this stable region, the system exhibits as following,

(i) in area A……oscillation with damping
(ii) in area B……uniformly damping
(iii) in area C……summing of unifomly damping and oscillation with damping.

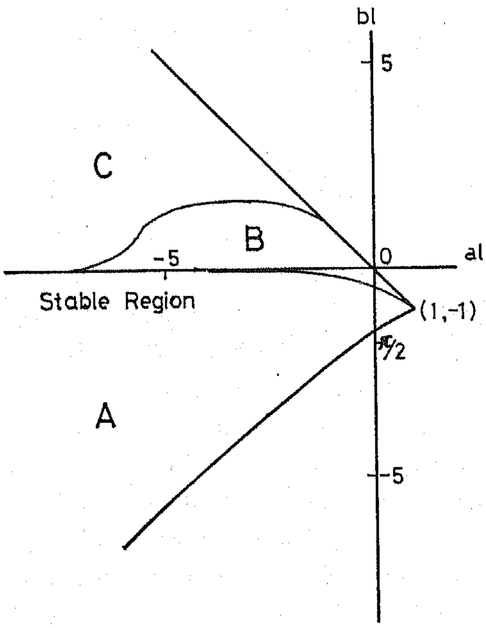


Fig. 2 Stable Region

These responses are verified by numerical analysis. But it is noted that two boundary curves are rough outlines. From the above mentioned, it is used *al* axis and *bl* axis as abscisa and ordinate, respectively.

In this report, we shall derive the approximate solution $K_a(\tilde{t})$ on area A in Figure 2. For example, the numerical solution which is calculated under the condition that $al=-1$, $bl=-2$ and $\phi(0)=1$, $\{\phi(\tau)=0;\ \tau\in[-l,\ 0)\}$, is shown in Figure 3.

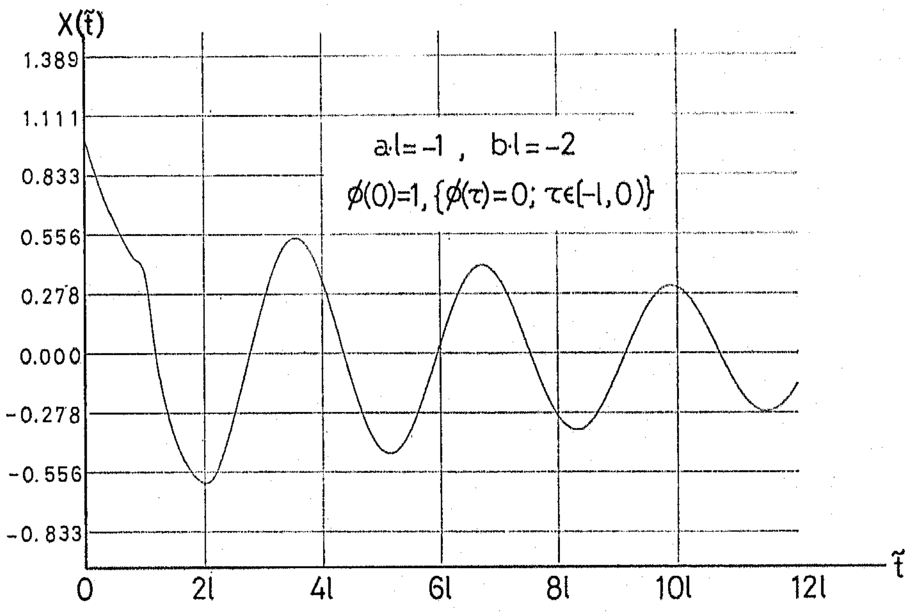


Fig. 3 Numerical Solution

It is evident that the solution curve in Fig. 3 exhibits a normal oscillation with damping on $\tilde{t}\geqq 3$ and exhibits abnormal oscillation on $\tilde{t}<3$. For this reason, it is impossible to approximate the solution such as Fig. 3 with only one equation. Therefor the author divided the abscisa into three intervals $\tilde{t}\in[0,\ 1)$, $\tilde{t}\in[1,\ 2)$ and $\tilde{t}\in[2,\ \infty)$ and derived a solution in each interval. This is also performed to achieve the purpose above mentioned. We take the solutions by the continuation process, for first and second intervals.

For the interval $\tilde{t}\in[2,\ \infty)$, it is found that if $al=k_a$(constant) and $bl=k_b$ (constant), then the damping ratio is inversely proportional to time delay and the period is proportional to it, in other words, product of the damping ratio and the period is constant for some cases.

As a consequence, it follows that

$$\mathrm{K}_a(\tilde{t}) = \begin{cases} e^{k_a \tilde{t}} & \tilde{t} \in [0,\ 1) \quad (12-1) \\ e^{k_a \tilde{t}}\,(1 + k_b\ e^{-k_a}\ (\tilde{t} - 1)) & \tilde{t} \in [1,\ 2) \quad (12-2) \\ \alpha e^{-\xi_1 \tilde{t}}\ \sin\dfrac{2\pi}{T_1}(\tilde{t} - 1 + \dfrac{e^{k_a}}{k_b}) & \tilde{t} \in [2,\ \infty) \quad (12-3) \end{cases}$$

k_a : al

k_a : bl

$\tilde{t}$: nondimensional time ($=t/l$)

ξ_1 : damping ratio for $l=1$

T_1 : period for $l=1$.

In the expression (12−3), $(1 - \dfrac{e^{k_a}}{k_b})$ is the time on which formula (12−2) equals to zero.

4-4 Determination of Period T.

Period T is determined so that some maximals (or minimals) and their time

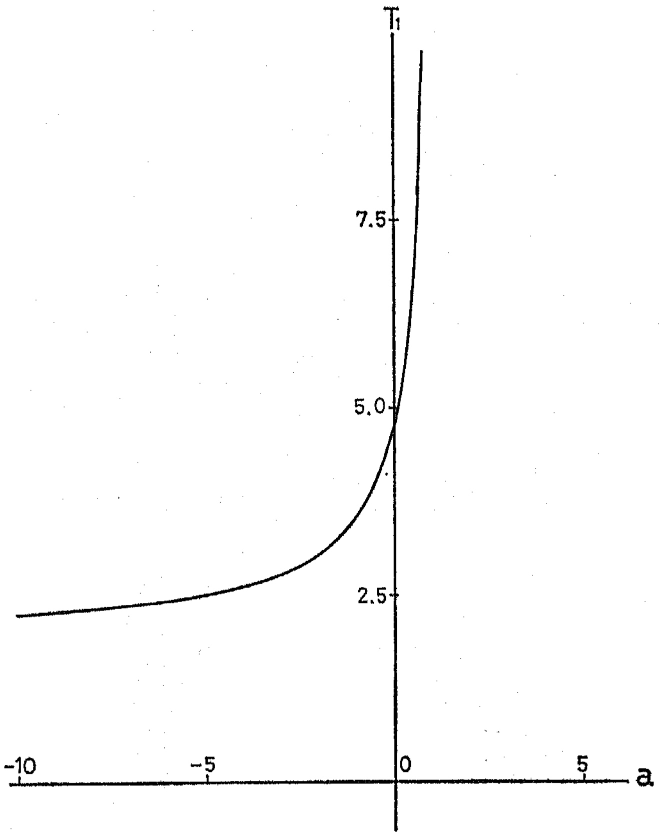


Fig. 4 Period T_1 vesus a ($b=-1$, $l=1$)

are sought by numerical analysis and next the period is decided by those datas.

If both al and bl are constant, then period T is proportional to l. (section 4-3)

As a consequence, it was evident that if al was proportional to $\ell n(-bl)$, then the period was almost constant. In other words, if the period of the system is T, then

$$al = \ell n(-bl) + c$$
$$a<0,\ b<0,\ c\text{: constant.} \qquad \cdots\cdots\cdots\cdots (13)$$

And the relationship between a and period T_1 is illustrated on $b=-1$ and $l=1$ in Figure 4.

For example, three systems are supposed as follows,

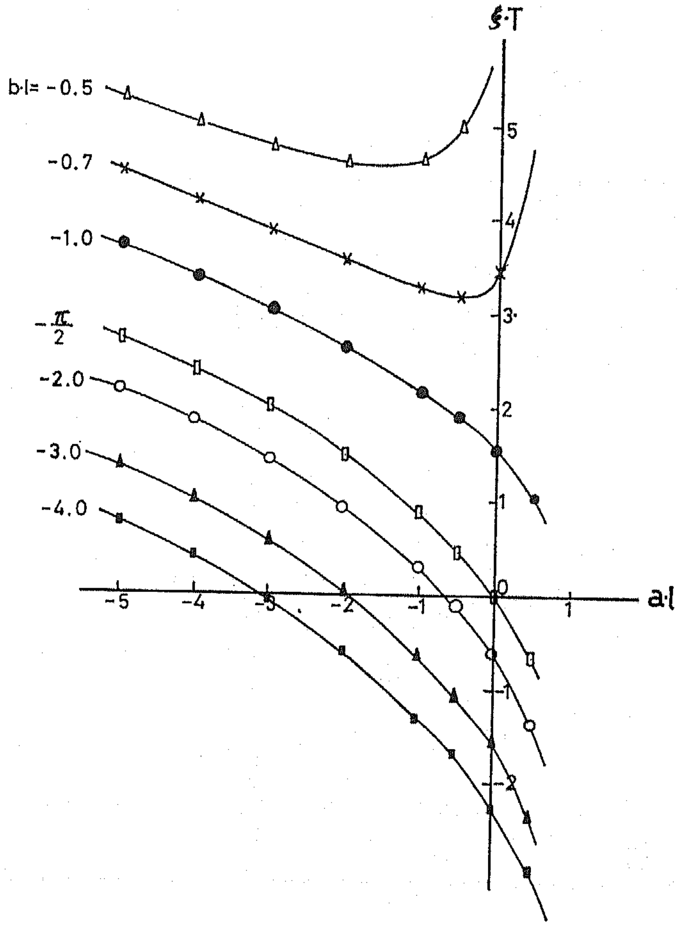


Fig. 5 $\xi \cdot T$ versus $a \cdot l$

system A : $al=-k_a$, $bl=-k_b$, $l=l(\neq 1)$
system B : $a=-k_a$, $b=-k_b$, $l=1$
system C : $a=?$, $b=-1$, $l=1$

and if system C has same period as system B, then the value a of system C is given by the value c which is determined by expression (13) about system A or B. The period T_1 of system C is decided by Figure 4. This T_1 is also the period of system B, then the period of system A is given by the product $T_1 \cdot \bar{l}$.

4-5 Decision on damping Ratio ξ.

Numerical analysis to decide the damping ratio ξ using maximals in section 4-4 results in that if al and bl are constant, then the product ξT is almost constant for $t \geqq 3l$. This fact is verified by substituting (12−3) in expression (1). In Figure 5, the relationship between ξT, al and bl is shown. Then the damping ratio ξ is determined by T (to be determined previously) and Fig. 5.

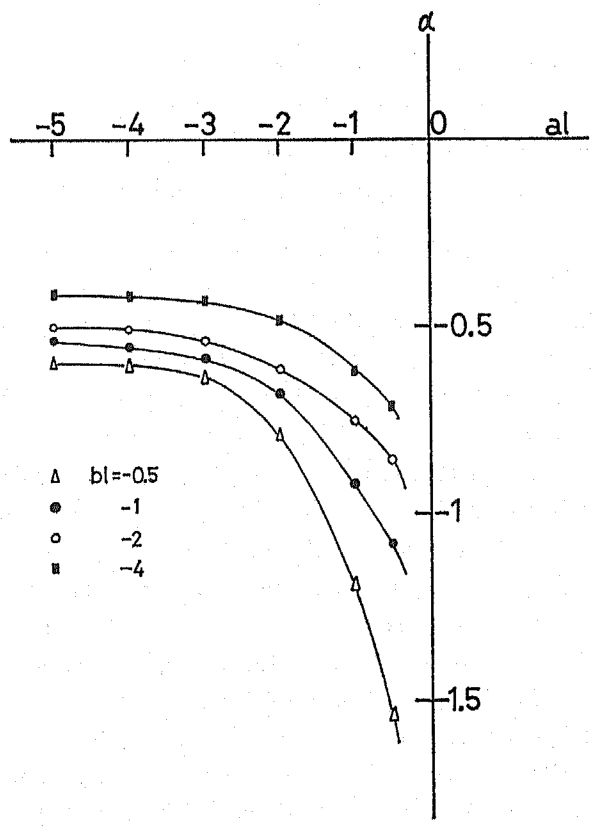


Fig. 6 α versus $a \cdot l$

4-6 Decision on α.

α is determined by extrapolation This method also results in that α becomes constant under constant al, bl.

Relationship between α, al and bl, is represented in Figure 6.

5. APPLICATIONS AND CONCLUSIONS

To represent the new approximation compared with the numerical solution, the results of several cases are illustrated in Figure 7. ~Figure 10.

Figure 7 illustrates the solution obtained for $al=-1$,

$$bl=-2,\quad \{\phi(\tau)=e^{0.3\tau};\ \tau\in-[1,\ 0]\}\ \text{and}\quad m(\tilde{t})=0.5+0.3\cos(1.5\tilde{t})$$

Figure 8 illustrates the solution obtained for $al=-3$,

$$bl=-1,\quad \{\phi(\tau)=e^{0.1\tau};\ \tau\in[-1,\ 0]\}\ \text{and}\quad m(\tilde{t})=1+0.5\cos(5\tilde{t})$$

Figure 9 illustrates the solution obtained for $al=-3$,

$$bl=-2,\quad \{\phi(\tau)=1;\ \tau\in[-1,\ 0]\}\ \text{and}\ m(\tilde{t})=-1$$

Figure 10 illustrates the solution obtained for $al=-5$,

$$bl=-4,\quad \phi(0)=1,\quad \{\phi(\tau)=0;\ \tau\in[-1,\ 0)\}\ \text{and}\ m(\tilde{t})=0$$

Figure 10 represents larger discrepancy than others. This approximation error is due to time shift, $1-e^{k_a/k_b}$, at $\tilde{t}=2$.

The time shift means, as described in 4-3, a value on which the integral kernel $K(\tilde{t})$ crosses $\tilde{t}$ axis in second interval. If values k_a and k_b increase, then the time derivative, $dK(\tilde{t})/d\tilde{t}$, becomes large in above interval. Therefore it is difficult to approximate the kernel with a normal damping oscillations. In order to obtain better approximation with above method, the time shift is should be modify by any other way.

This new approach, however, can be applied to the system with any initial function $\phi(\cdot)$, so as the Pade's approximation can not be applied. This approximate method is not the approximation of the time delay element itself, but of total system including the delay. Namely, the approximation is not expressed by time delay, but by the product of coefficients and the delay. Once the approximation $K_a(\tilde{t})$ is expressed, the solution $x(t)$ of eq. (1) is calculated by manual ——— may be slight trouble ——— even for arbitrary function $\phi(\cdot)$ and input $m(t)$. The solution is determined, of course, for any $\phi(\cdot)$ and $m(t)$ by the continuation process, but it becomes high order polynomials with t.

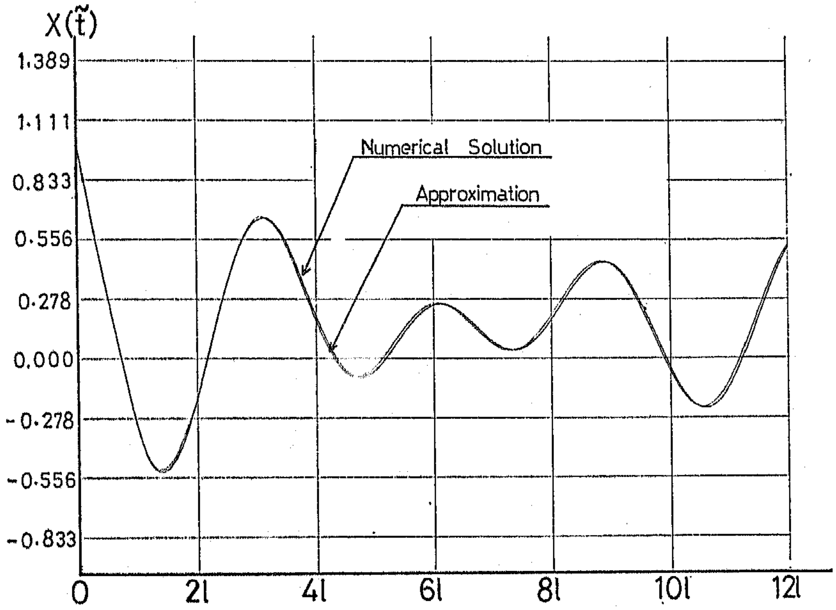


Fig. 7 Application No. 1

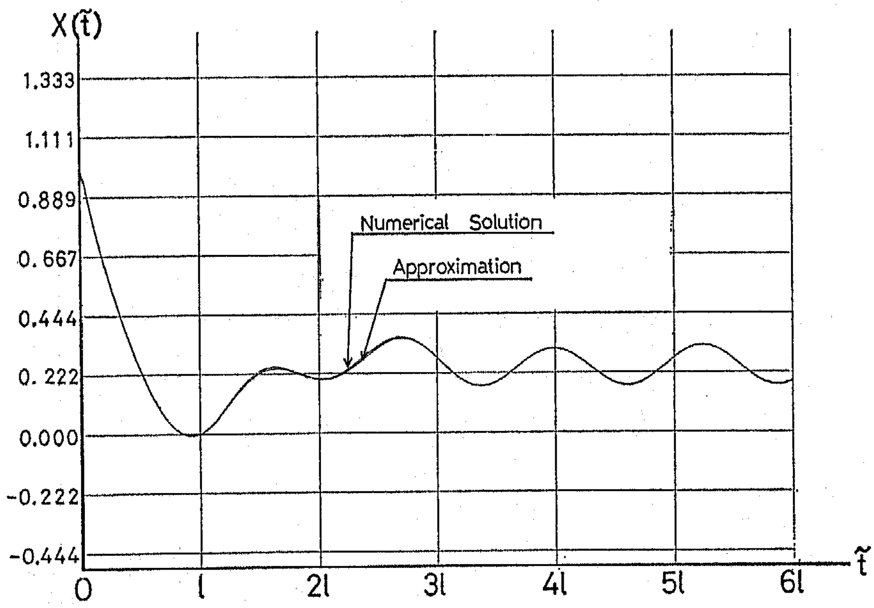


Fig. 8 Application No. 2

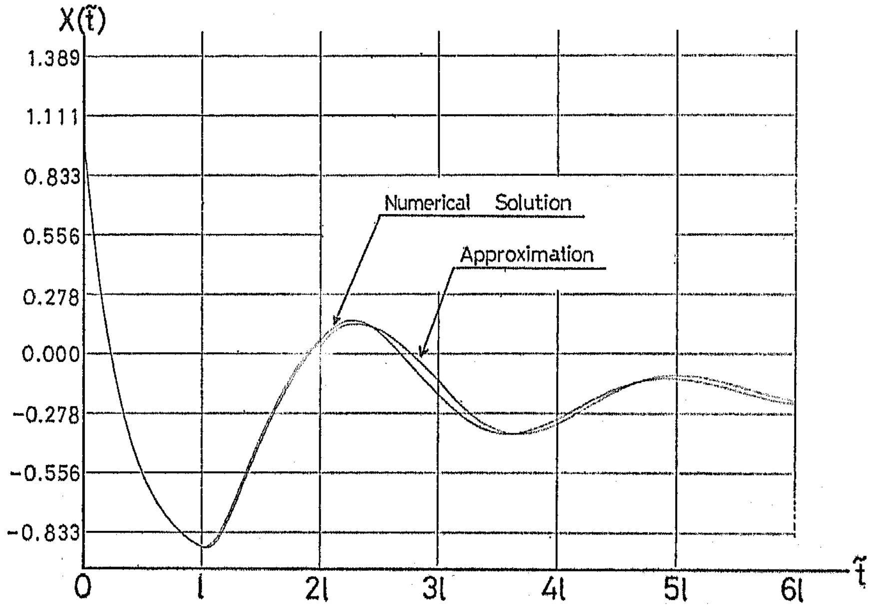


Fig. 9 Application No. 3

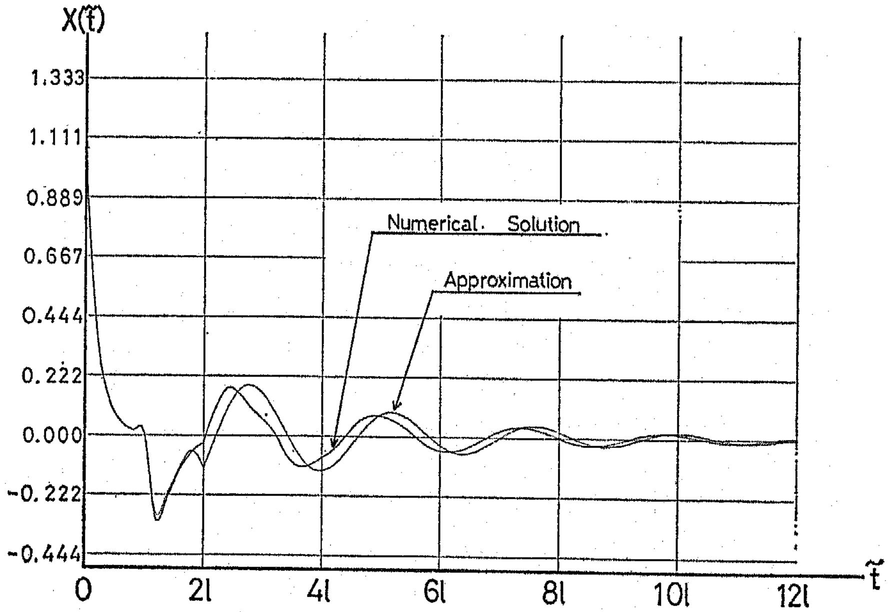


Fig. 10 Application No. 4

In this paper, the approximate method is restricted for area A in Figure 2. However, the author will report the results for the cases in area B and C some other day. All numerical and graphical results are obtained by the personal computer OKI-if-800.

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