

# A NOTE ON THE CONVERGENCE OF NON- CONFORMING FINITE ELEMENT SOLUTIONS IN PLATE BENDING

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## INTRODUCTION

As is well known, the application of finite element method to plate bending problems encounters special difficulties caused by the conformity (continuities of deflections and slopes across element boundary) of shape functions.<sup>1)</sup> Many numerical examinations<sup>2)3)4)5)</sup> show that some of the shape functions ever proposed do not yield such solutions that would converge to the exact values with successive divisions of plate region into elements. Some investigators state that the finite element method is equivalent in its mathematical principle to the Ritz method so that the shape functions should be constructed to make displacements and slopes continuous between the adjacent elements.<sup>6)</sup> In other words, they assert that to make the shape functions conforming is the necessary conditions for the correct convergence of the finite element solutions. Complying with this conformity requirement, TONG,<sup>7)</sup> et al., STRANG<sup>8)</sup> and ZLÁMAL<sup>9)</sup> studied the orders of convergence of the solutions. LYNN and BORESI<sup>10)</sup> derived the convergence criteria of the solutions from the variational principle. In spite of the importance of the conformity condition, it is difficult to construct the shape function conforming in a concrete form.<sup>1)</sup> Many authors have investigated the convergence properties of various kinds of shape functions with numerical calculations.<sup>2)3)4)5)</sup> They revealed that some of the solutions of the conforming shape functions do not seem to converge to the exact solutions, and on the other hand some of the solutions of the non-conforming shape functions seem to converge to exact solutions. BAZELEY<sup>11)</sup> et al. and IRON<sup>12)</sup> et al. proposed a semi-empirical method called "Patch Test" to deduce the convergence of the solutions by examining the stiffness matrix of an element. This method is characterized by the calculations of the nodal forces for displacements which give a certain curvature. Afterward STRANG<sup>13)</sup> et al. CIARLET,<sup>14)</sup> and LASCAUX<sup>15)</sup> et al. proved the mathematical meaning of the patch test and pointed out that

the concept of patch test plays essential role in the mathematical theory of the convergence. WALTZ<sup>16)17)</sup> et al. derived the equilibrium equation at a node from the stiffness matrix equation, and compared it with the equation of the classical thin plate theory. They revealed the convergence properties of several kinds of stiffness matrix. MIYOSHI<sup>18)</sup> proved theoretically the convergence properties of the famous shape function (A. C. M)<sup>19)</sup> by the energy principle. He used skillfully the symmetrical nature of the shape function. Recently CIARLET,<sup>14)</sup> LASCAUX<sup>15)</sup> et al. and KIKUCHI<sup>20)21)</sup> investigated the convergence of the finite element solutions by Pythagorean theorem on strains and Bramble-Hilbert theorem. In this paper the author derived theoretically the convergence criterion of the non-conforming finite element solutions for plate bending. He used the energy relations derived by the variational principle. The criterion obtained is for non-conforming rectangular and right-angled triangular elements and is expressed in the form of discriminants for the shape function in terms of its weighting functions. Further he examined the influences of the singularities of the exact solutions to the corresponding finite element solutions.

#### DEFINITION OF DEFLECTION AND RELATED FUNCTIONAL

Let transverse force acting on an elastic thin plate be  $f(x, y)$  and transverse deflection caused by  $f(x, y)$  be  $w^*(x, y)$ ,  $x, y$  being Cartesian coordinates in the plate domain  $\Omega$ . As  $w^*(x, y)$  is considered mathematically the exact solution for  $f(x, y)$ , it satisfies the following equilibrium equation of the classical thin plate theory and certain boundary conditions on the plate boundary  $I$ .

$$D\nabla^4 w^* = f \quad (1)$$

in which  $\nabla^4$  is the biharmonic operator in  $x, y$  coordinates and  $D = Eh'^3/[12(1 - \nu^2)]$ , the flexural rigidity of the plate, where  $E, \nu$  and  $h'$  denote the longitudinal elastic modulus, Poisson's ratio and the plate thickness respectively. In this paper  $w^*(x, y)$  is assumed to satisfy the following conditions.

**Condition [I]** :  $w^*(x, y)$  is expansible in finite Taylor series at almost everywhere in  $\Omega$ .

**Condition [II]** : The number of singular points in  $\Omega$  are finite, and the following inequalities hold everywhere in  $\Omega$ .

$$\left. \begin{aligned} |w_{,\mu_1\mu_2}^*| < Kr^{-\rho}, |w_{,\mu_1\mu_2\mu_3}^*| < Kr^{-\rho-1}, |w_{,\mu_1\mu_2\mu_3\mu_4}^*| < Kr^{-\rho-2} \\ 0 \leq \rho < 1 \end{aligned} \right\} \quad (2)$$

where subscripts  $\mu_1, \mu_2, \dots$  after the comma denote partial differentiations with respect to  $x, y$  or other coordinates in  $\Omega$ , and  $r, \rho$  and  $K$  denote the distance

from the singular point, the index of singularity, and a constant respectively. In some cases equation (2) are replaced by the condensed form

$$|w_{,\mu_1\mu_2\cdots\mu_s}^*| < Kr^{-\rho-s+2}, \quad 0 \leq \rho < 1 \tag{2'}$$

From the physical view point the inequality  $0 \leq \rho < 1$  implies that the displacement and slope of the plate are bounded and generalized strains are singular at some points in  $\Omega$ . Now let us divide the domain  $\Omega$  into small sub-regions (elements)  $\omega_1, \omega_2, \dots, \omega_p, \dots, \omega_n$  with their boundaries (internal boundaries)  $\gamma_1, \gamma_2, \dots, \gamma_p, \dots, \gamma_n$  where  $n$  is the number of elements. We denote the multiply connected domain which is obtained by joining  $\omega_1, \omega_2, \dots, \omega_n$  at the nodes by  $\Omega_n$ . Let us define the admissible function  $w_n(x, y)$  which satisfies the following conditions in  $\Omega_n$ .

**Condition [i]** :  $w_n(x, y)$  is the function of class  $C^{(1)}$  at the nodes namely,

$$w_n(x_h, y_h) \in C^{(1)} \quad (h = 1, 2, \dots, k) \tag{3}$$

**Condition [ii]** :  $w_n(x, y)$  is the function of class  $C^{(2)}$  in  $\Omega_n$  except at the nodes, namely,

$$w_n(x, y) \in C^{(2)}, \quad (x, y) \neq (x_h, y_h), \quad (h = 1, 2, \dots, k) \tag{4}$$

where  $h$  denotes the running number of the node,  $k$  the total number of them and  $(x_h, y_h)$  denotes the coordinate of the nodes. The finite element method with non-conforming shape function can be regarded as the method in which the exact solution  $w^*(x, y)$  defined on  $\Omega$  is approximated with the admissible function  $w_n(x, y)$  defined on  $\Omega_n$ . In almost every part of this report for the sake of brevity, the domain  $\Omega$  is assumed to be a rectangle with side dimension

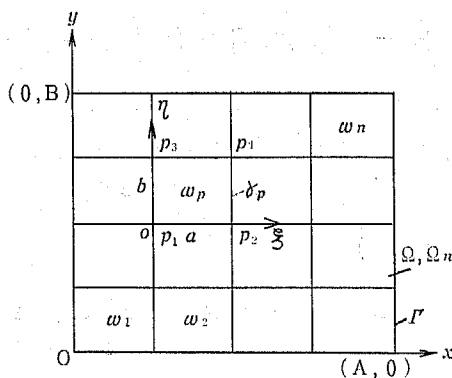


Fig. 1 Rectangular plate and its sub-division

$A$  and  $B$ , and also  $\omega_1, \omega_2, \dots, \omega_n$  to be equally divided rectangles with side dimension  $a$  and  $b$  as shown in figure 1. Without loss of generality we can assume

$$A \geq B, \quad a \geq b$$

The nodes (the joining points) are selected at the grid points in  $\Omega_n$  (or in other words at the four corners of every elements in  $\Omega_n$ ). Let us define the functionals in terms of the deflection functions  $u(x, y)$  and  $v(x, y)$  which represent  $w^*(x, y)$  or  $w_n(x, y)$  as follows :

$$\Psi_e(u, v) \equiv \iint_e uv dx dy \quad (5)$$

$$\begin{aligned} \Phi_e(u, v) \equiv & \frac{D}{2} [\Psi_e(u, xx, v, xx) + \nu \Psi_e(u, xx, v, yy) + \nu \Psi_e(u, yy, v, xx) \\ & + 2(1-\nu) \Psi_e(u, xy, v, xy) + \Psi_e(u, yy, v, yy)] \end{aligned} \quad (6)$$

$$\begin{aligned} A_c(u, v) \equiv & D \iint_e [(u, xx + \nu u, yy) v, x]_x dx dy + D \iint_e [(u, yy + \nu u, xx) v, y]_y dx dy \\ & - D \iint_e [(u, xx + (2-\nu)u, xy) v]_x dx dy - D \iint_e [(u, yy + (2-\nu)u, xy) \\ & \cdot v]_y dy + 2(1-\nu) D \iint_e (u, xy)_{,xy} dx dy \end{aligned} \quad (7)$$

where  $e$  and  $c$  are the domain and its boundary for  $u(x, y)$  and  $v(x, y)$  respectively. For the case when the domain  $e$  is assumed to be rectangle with side dimension  $a$  and  $b$  as shown in figure 2, the above equation (7) becomes

$$\begin{aligned} A_c(u, v) \equiv & [D \int_0^a (u, yy + \nu u, xx) v, y dx]_{y=0}^b + [D \int_0^b (u, xx + \nu u, yy) v, x dy]_{x=0}^a \\ & - [D \int_0^a (u, yy + (2-\nu)u, xy) v dx]_{y=0}^b - [D \int_0^b (u, xx + (2-\nu)u, xy) v dy]_{x=0}^a \\ & + 2(1-\nu) D [u, xy v]_1 - u, xy v|_2 - u, xy v|_3 + u, xy v|_4 \end{aligned} \quad (7')$$

in which subscripts 1, 2, 3 and 4 denote the nodes as shown in figure 2. The elastic energy of plate bending in  $e$  is describable as

$$|u|_e^2 \equiv \Phi_e(u, u) \quad (8)$$

$|u|_e$  may be considered as the square mean norm of generalized strains. The total potential energy can be expressed as follows :

$$\Pi[u] \equiv |u|_e^2 - \Psi_e(f, u) \quad (9)$$

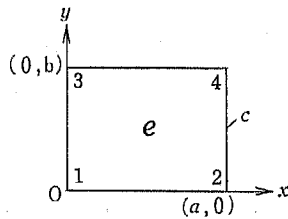


Fig. 2 Rectangular element

**INEQUALITIES CONCERNING WITH DISPLACEMENT, ELASTIC ENERGY AND TOTAL POTENTIAL ENERGY**

The following theorem exists concerning with the displacement, the elastic energy and the total potential energy.

[Theorem 1] For any  $w_n(x, y)$

(i) there exists a positive constant ( $0 < \theta < \infty$ ) which satisfies the following inequality :

$$\theta \max_{x, y \in \Omega_n} |w_n| \leq |w_n|_{\Omega_n} \tag{10}$$

$$(ii) \quad \Pi[w_n] \geq -AB \|f\|_{\Omega}^2 / (4\theta^2) \tag{11}$$

$$(iii) \quad \max_{x, y \in \Omega_n} |w_n| \leq \sqrt{AB} \|f\|_{\Omega} / (2\theta^2) + [\Pi(w_n) / \theta^2 + AB \|f\|_{\Omega}^2 / (4\theta^4)]^{\frac{1}{2}} \tag{12}$$

where  $\|\cdot\|$  denotes the square mean norm, namely,

$$\|f\|_{\Omega} = \left[ \iint_{\Omega} f^2 dx dy \right]^{\frac{1}{2}}$$

and  $w^*$  is assumed always to involve displacements caused by elastic deformation.

The above inequalities (i), (ii) and (iii) exist also for  $w^*(x, y)$  in  $\Omega$ .

The statement (i) means that for any number of the divisions the displacement is bounded only if the elastic energy is bounded. As the finite element method is a method based on the minimum principle of potential energy, it is necessary that  $\Pi[w_n]$  is bounded from below for any number of the divisions. The statement (ii) shows that if  $\|f\|_{\Omega}$  is bounded  $\Pi[w_n]$  is also bounded from below for any division number  $n$ . The statement (iii) shows that the finite element solution is bounded.

[Proof of theorem 1] Considering conditions [i] and [ii] in the previous section and equations (6) and (8), we obtain the following relation between the

displacements at the nodes and the elastic energy in  $\Omega_n$ .

$$\begin{aligned} & |w_n(x_{h_1}, y_{h_1}) - w_n(x_{h_1}, y_{h_0}) - w_n(x_{h_0}, y_{h_1}) + w_n(x_{h_0}, y_{h_0})| \\ &= \left| \int_{y_{h_0}}^{y_{h_1}} \int_{x_{h_0}}^{x_{h_1}} w_{n,xy} dx dy \right| \leq \sqrt{AB} \|w_{n,xy}\|_{\Omega_n} \leq \frac{\sqrt{AB}}{\sqrt{D(1-\nu)}} |w_n|_{\Omega_n} \end{aligned}$$

where  $(x_{h_i}, y_{h_j})$  denote the coordinates of the node. As  $w_n$  is assumed not to be mere rigid body displacement, the following relation holds in  $\Omega_n$  :

$$\max |w_n(x_{h_1}, y_{h_1}) - w_n(x_{h_1}, y_{h_0}) - w_n(x_{h_0}, y_{h_1}) + w_n(x_{h_0}, y_{h_0})| \neq 0$$

From the above two equations we have

$$\text{const} \cdot \max_{h=1,2,\dots,k} |w_n(x_h, y_h)| \leq \frac{\sqrt{AB}}{\sqrt{D(1-\nu)}} |w_n|_{\Omega_n}$$

A short consideration on the above equation and **conditions** [i] and [ii] on continuities of  $w_n$  proves the existence of a positive constant  $\Theta$  for equation (10). Substituting equation (10) in the total potential energy equations obtained by replacing  $u$  of equation (9) by  $w_n$ , we have

$$\Pi[w_n] \geq [|w_n|_{\Omega_n} - \sqrt{AB} \|f\|_{\Omega} / (2\Theta)]^2 - AB \|f\|_{\Omega}^2 / (4\Theta^2)$$

The above equation leads to equation (11) and (12) using equation (10). In the next place we examine the theorem for  $w^*$ . From equation (2) we obtain the following inequality.

$$\begin{aligned} \iint_{\Omega} w_{,\mu_1\mu_2}^* dx dy &\leq \lim_{\varepsilon \rightarrow 0} \int_0^{2\pi} \int_{\varepsilon}^{(\text{dia. of } \Omega)} K^2 r^{-2\rho} r dr d\theta \\ &= \frac{\pi K^2}{(1-\rho)} [(\text{dia. of } \Omega)^{2(1-\rho)} - \lim_{\varepsilon \rightarrow 0} \varepsilon^{2(1-\rho)}] < \infty \end{aligned} \quad (13)$$

This inequality leads to :

$$|w^*|_{\Omega} < \infty$$

Since  $w^*(x, y)$  satisfies **conditions** [I] and [II] in the previous section,

$$\max_{x,y \in \Omega} |w^*(x, y)| < \infty$$

The above two inequalities mean the existence of a positive constant  $\Theta$  which satisfies equation (10). The items [ii] and [iii] for  $w^*$  are proved in the similar manner as the case for  $w_n$ .

### SHAPE FUNCTIONS FOR FINITE ELEMENT METHOD

In the finite element method the admissible function  $w_n(x, y)$  is constructed by joining the shape functions  $w_{n_1}(x, y), w_{n_2}(x, y), \dots, w_{n_p}(x, y), \dots, w_{n_n}(x, y)$  at the nodes in  $\Omega_n$  so as to keep continuities of displacements and slopes. The shape function  $w_{np}(x, y)$  is a kind of local function defined in  $\omega_p$ , and is expressed by the following form for almost all cases of a rectangular element as shown in figure 3 :

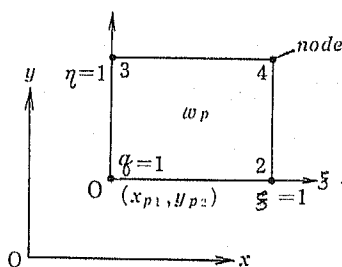


Fig. 3 Coordinate systems of plate

$$w_{np}(x, y) = \sum_{q=1}^4 [G_q^{(0)}(\xi, \eta)w_{pq}^{(0)} + aG_q^{(1)}(\xi, \eta)w_{pq}^{(1)} + bG_q^{(2)}(\xi, \eta)w_{pq}^{(2)}] \quad (14)$$

where  $q$  denotes the element corner number, and  $w_{pq}^{(0)}, w_{pq}^{(1)}, w_{pq}^{(2)}$  denote the displacement, the slope for  $x$  and  $y$  direction respectively at the nodes indicated by  $p$  and  $q$ .  $G_q^{(e)}(\xi, \eta)$ , ( $q=1, 2, 3, 4, l=0, 1, 2$ ) are weighting functions for  $w_{pq}^{(e)}$ .  $\xi, \eta$  are local non-dimensional coordinates as

$$\xi = (x - x_{p1})/a, \quad \eta = (y - y_{p1})/b$$

where  $(x_{p1}, y_{p1})$  is  $x, y$  coordinate of the origin of  $\xi, \eta$  coordinate system.  $w_{np}(x, y), w_{np,x}(x, y)$  and  $w_{np,y}(x, y)$  should take the value  $w_{pq}^{(0)}, w_{pq}^{(1)}$  and  $w_{pq}^{(2)}$  respectively at the corner  $q$ , namely,

$$w_{np}(x_{pq}, y_{pq}) = w_{pq}^{(0)}, \quad w_{np,x}(x_{pq}, y_{pq}) = w_{pq}^{(1)}, \quad w_{np,y}(x_{pq}, y_{pq}) = w_{pq}^{(2)}$$

These conditions are expressed as the conditions of  $G_q^{(0)}(\xi, \eta), G_q^{(1)}(\xi, \eta)$  and  $G_q^{(2)}(\xi, \eta)$  as follows :

$$\left. \begin{aligned}
 G_q^{(0)} = G_{q,\xi}^{(1)} = G_{q,\eta}^{(2)} = 1 \\
 G_{q,\xi}^{(0)} = G_{q,\eta}^{(0)} = G_q^{(1)} = G_{q,\eta}^{(1)} = G_q^{(2)} = G_{q,\xi}^{(2)} = 0 \\
 G_q^{(0)} = G_{q,\xi}^{(0)} = G_{q,\eta}^{(0)} = G_q^{(1)} = G_{q,\xi}^{(1)} = G_{q,\eta}^{(1)} = G_q^{(2)} = G_{q,\xi}^{(2)} = G_{q,\eta}^{(2)} = 0
 \end{aligned} \right\} \begin{array}{l} \text{at the corner } q \\ \\ \text{at the corner } q' (\neq q) \end{array} \quad (15)$$

Substitution of equation (14) into equation (9) gives

$$II[w_n] = \sum_{p=1}^n |w_{np}|_{\omega_p}^2 - \sum_{p=1}^n \Psi_{\omega_p}(f, w_{np})$$

In the finite element method, to get an approximate solution means to determine the nodal values  $w_{pq}^{(0)}$ ,  $w_{pq}^{(1)}$  and  $w_{pq}^{(2)}$  ( $p = 1, 2, \dots, n$ ,  $q = 1, 2, 3, 4$ ) (or with another notations  $w_h^{(0)}$ ,  $w_h^{(1)}$  and  $w_h^{(2)}$  ( $h = 1, 2, \dots, k$ ) which make  $II[w_n]$  minimum. The deflections in  $\Omega_n$  except at the nodes are obtained by substituting the obtained nodal values in equation (14). Here we express the obtained solution by notation  $\tilde{w}_n(x, y)$ . The relation between  $w_n$  and  $\tilde{w}_n$  is given by

$$II[\tilde{w}_n] \leq II[w_n] \quad (16)$$

As was already mentioned some of the shape functions do not give the solutions  $\tilde{w}_n$  which converge to the exact value  $w^*$  with increase of  $n$ . The above fact implies that the weighting functions  $G_q^{(0)}$ ,  $G_q^{(1)}$  and  $G_q^{(2)}$  must satisfy not only equation (15) but also some other conditions. This report intend to derive theoretically these condition, namely, the convergence criterion as the discriminants in terms of  $G_q^{(l)}$  ( $l = 0, 1, 2$ ,  $q = 1, 2, 3, 4$ ). Now let us define the local function  $\bar{w}_{np}(x, y)$  in  $\omega_p$  as follows, replacing  $w_{pq}^{(0)}$ ,  $w_{pq}^{(1)}$  and  $w_{pq}^{(2)}$  ( $q = 1, 2, 3, 4$ ) of equation (14) by  $w^*(x_{pq}, y_{pq})$ ,  $w_{,x}^*(x_{pq}, y_{pq})$  and  $w_{,y}^*(x_{pq}, y_{pq})$ ,

$$\begin{aligned}
 \bar{w}_{np}(x, y) = \sum_{q=1}^4 [G_q^{(0)}(\xi, \eta)w^*(x_{pq}, y_{pq}) + aG_q^{(1)}(\xi, \eta)w_{,x}^*(x_{pq}, y_{pq}) \\
 + bG_q^{(2)}(\xi, \eta)w_{,y}^*(x_{pq}, y_{pq})] \quad (17)
 \end{aligned}$$

Here we denote the assembly of  $\bar{w}_{np}(x, y)$  ( $p = 1, 2, \dots, n$ ) as  $\bar{w}_n(x, y)$  in  $\Omega_n$ . Accordingly  $\bar{w}_n(x, y)$  is a kind of interpolation function which gives the same displacement and slopes with  $w^*$  at the nodes and is interpolated by equation (17) at the other points in  $\Omega_n$ . We do not distinguish  $\Omega_n$  from  $\Omega$  and use the notation  $\Omega$  both for  $\Omega$  and  $\Omega_n$  here after, if nothing interferes.



**INEQUALITIES FOR BENDING ENERGY OF  $w^*$ ,  $\tilde{w}_n$  AND  $\bar{w}_n$**

[Theorem 2] For any  $w^*$ ,  $\tilde{w}_n$  and  $\bar{w}_n$  the following inequality holds :

$$|\tilde{w}_n - w^*|_g^2 \leq |\bar{w}_n - w^*|_g^2 + \left| \sum_{p=1}^n A_{T_p}[w^*, \bar{w}_{np} - w^*] \right| + \left| \sum_{p=1}^n A_{T_p}[w^*, \tilde{w}_{np} - w^*] \right| \quad (18)$$

The above theorem means that the finite element solution  $\tilde{w}_n$  converges to  $w^*$  in the sense of energy norm (or the norm of generalized strains), if the interpolating function  $\bar{w}_n$  can interpolate  $w^*$  as precise as desired in the sense of energy norm, and further discontinuities of the strain energy between the adjacent elements due to non-conformities of the shape function converge to zero as  $n$  approaches to infinity.

[Proof of theorem 2] Integrating equations (6) and (7) by parts repeatedly, we obtain the following identity :

$$\Psi_e(D\nabla^4 u, v) = 2\Phi_e(u, v) - A_c(u, v)$$

Equation (9) is transformed as follows : substituting  $f$  of equation (1) and taking into account the above relation,

$$\Pi[u] = |u - w^*|_g^2 - |w^*|_g^2 + \sum_{p=1}^n A_{T_p}[w^*, u]$$

Replacing  $u$  by  $\tilde{w}_n$ ,  $\bar{w}_n$  and  $w^*$ , and combining these equations we obtain the following two equation :

$$\Pi[\tilde{w}_n] - \Pi[w^*] = |\tilde{w}_n - w^*|_g^2 + \sum_{p=1}^n A_{T_p}[w^*, \tilde{w}_{np} - w^*] \quad (19-a)$$

$$\Pi[\bar{w}_n] - \Pi[w^*] = |\bar{w}_n - w^*|_g^2 + \sum_{p=1}^n A_{T_p}[w^*, \bar{w}_{np} - w^*] \quad (19-b)$$

From equation (16) we get

$$\Pi[\bar{w}_n] - \Pi[w^*] \leq \Pi[\tilde{w}_n] - \Pi[w^*]$$

Substituting equations (19-a) and (19-b) in the above equation we get

$$|\tilde{w}_n - w^*|_g^2 + \sum_{p=1}^n A_{T_p}[w^*, \tilde{w}_{np} - w^*] \leq |\bar{w}_n - w^*|_g^2 + \sum_{p=1}^n A_{T_p}[w^*, \bar{w}_{np} - w^*]$$

The above equation leads to equation (18).

**TAYLOR SERIES EXPRESSIONS OF  $w^*$  AND  $\bar{w}_{up}$ ,  
AND RELATED EQUATIONS**

It is convenient to express  $w^*(x, y)$  in Taylor series expanded about an arbitrary point  $(x_{p_0}, y_{p_0})$  ( $(\xi_0 = (x_{p_0} - x_{p_1})/a, \eta_0 = (y_{p_0} - y_{p_1})/b)$ ) as follows :

$$w^*(x, y) = W^*(x, y) + R_t(x, y) \quad (20)$$

where

$$W^*(x, y) = \sum_{s=0}^t \sum_{m=0}^s \sum_{i=0}^m \sum_{j=0}^{s-m} \frac{(-1)^{s+i+j} \xi_0^{m-i} \eta_0^{s-m-j}}{(m-i)! i! (s-m-j)! j!} \xi^i \eta^j \kappa^{s-m} a^s \left[ \frac{\partial^s w^*}{\partial x^m \partial y^{s-m}} \right]_{\theta_p} \quad (21)$$

$$R_t(x, y) = \sum_{m=0}^{t+1} I_m^*(\xi, \eta) \left[ \frac{\partial^{t+1} w^*}{\partial x^m \partial y^{t-m+1}} \right]_{\theta_p} a^{t+1} \quad (22)$$

$$I_m^*(\xi, \eta) = \sum_{i=0}^m \sum_{j=0}^{t-m+1} \frac{(-1)^{t+1-i-j} \xi_0^{m-i} \eta_0^{t+1-m-j}}{(m-i)! i! (t-m-j+1)! j!} \xi^i \eta^j \kappa^{t-m+1} \quad (23)$$

Notations  $\left[ \frac{\partial^s w^*}{\partial x^m \partial y^{s-m}} \right]_{\theta_p}$  and  $\left[ \frac{\partial^{t+1} w^*}{\partial x^m \partial y^{t-m+1}} \right]_{\theta_p^{(0)}}$  mean  $\left[ \frac{\partial^s w^*(x_{p_0}, y_{p_0})}{\partial x^m \partial y^{s-m}} \right]$  and  $\left[ \frac{\partial^{t+1} w^*(x_{p_0} + \theta_p(x - x_{p_0}), y_{p_0} + \theta_p(y - y_{p_0}))}{\partial x^m \partial y^{t-m+1}} \right]$ , ( $0 < \theta_p < 1$ ) respectively.  $R_t$  represents  $(t+1)$  degree remainder. Generally  $t$  is arbitrary but for the elements which involve singular points  $t$  should be confined less than or equal to unity as there is a possibility that equation (21) becomes infinite for  $s \geq 2$ . In this paper for simplicity sake we suppose that the singular point is limited only at the origin of the  $x, y$  coordinate system, and sub-division (element) involving singular point is confined to  $\omega_1$  as shown in figure 1. In the same way as equation (20) we obtain the corner deformations  $w^*(x_{p_q}, y_{p_q})$ ,  $w_{,x}^*(x_{p_q}, y_{p_q})$  and  $w_{,y}^*(x_{p_q}, y_{p_q})$  in Fourier series as follows :

$$\left. \begin{aligned} w^*(x_{p_q}, y_{p_q}) &= W^*(x_{p_q}, y_{p_q}) + R_t^{(0)}(x_{p_q}, y_{p_q}) \\ w_{,x}^*(x_{p_q}, y_{p_q}) &= W_{,x}^*(x_{p_q}, y_{p_q}) + R_t^{(1)}(x_{p_q}, y_{p_q}) \\ w_{,y}^*(x_{p_q}, y_{p_q}) &= W_{,y}^*(x_{p_q}, y_{p_q}) + R_t^{(2)}(x_{p_q}, y_{p_q}) \end{aligned} \right\} \quad (24)$$

where

$$\left. \begin{aligned} R_t^{(0)}(x_{p_q}, y_{p_q}) &= \sum_{m=0}^{t+1} I_m^*(\xi_q, \eta_q) \left[ \frac{\partial^{t+1} w^*}{\partial x^m \partial y^{t-m+1}} \right]_{\theta_{p_q}^{(0)}} a^{t+1} \\ R_t^{(1)}(x_{p_q}, y_{p_q}) &= \sum_{m=0}^{t+1} I_{m,\xi}^*(\xi_q, \eta_q) \left[ \frac{\partial^{t+1} w^*}{\partial x^m \partial y^{t-m+1}} \right]_{\theta_{p_q}^{(1)}} a^t \end{aligned} \right\} \quad (25)$$

$$R_t^{(2)}(x_{pq}, y_{pq}) = \sum_{m=0}^{t+1} I_{m,\eta}^*(\xi_a, \eta_a) \left[ \frac{\partial^{t+1} w^*}{\partial x^m \partial y^{t-m+1}} \right]_{\theta_{pq}^{(2)}} \kappa^{-1} a^t \quad (25 \text{ cont.})$$

$$\begin{pmatrix} \xi_1=0 \\ \eta_1=0 \end{pmatrix} \quad \begin{pmatrix} \xi_2=1 \\ \eta_2=0 \end{pmatrix} \quad \begin{pmatrix} \xi_3=0 \\ \eta_3=1 \end{pmatrix} \quad \begin{pmatrix} \xi_4=1 \\ \eta_4=1 \end{pmatrix}$$

$$0 < \theta_{pq}^{(l)} < 1 \quad (l=0, 1, 2)$$

Substituting equation (24) in equation (17),  $\bar{w}_{np}(x, y)$  is expressed in a power series of  $a$  as follows :

$$\begin{aligned} \bar{w}_{np}(x, y) = & \sum_{s=0}^i \sum_{m=0}^s \sum_{i=0}^m \sum_{j=0}^{s-m} J_{ij}(\xi, \eta) \frac{(-1)^{s+i+j} \xi_0^{m-i} \eta_0^{s-m-j}}{(m-i)! i! (s-m-j)! j!} \kappa^{s-m} \\ & \cdot a^s \left[ \frac{\partial^s w^*}{\partial x^m \partial y^{s-m}} \right]_{\theta_{pq}} + \bar{R}_{ntp} \end{aligned} \quad (26)$$

where

$$\bar{R}_{ntp} = \sum_{m=0}^{t+1} \left\{ \sum_{q=1}^4 \sum_{l=0}^2 I_{mq}^{(l)}(\xi, \eta) \left[ \frac{\partial^{t+1} w^*}{\partial x^m \partial y^{t-m+1}} \right]_{\theta_{pq}^{(l)}} a^{t+1} \right\} \quad (27)$$

$$\left. \begin{aligned} I_{mq}^{(0)}(\xi, \eta) &= G_q^{(0)}(\xi, \eta) I_m^*(\xi_a, \eta_a) \\ I_{mq}^{(1)}(\xi, \eta) &= G_q^{(1)}(\xi, \eta) I_{m,\xi}^*(\xi_a, \eta_a) \\ I_{mq}^{(2)}(\xi, \eta) &= G_q^{(2)}(\xi, \eta) I_{m,\eta}^*(\xi_a, \eta_a) \\ J_{00}(\xi, \eta) &= G_1^{(0)} + G_2^{(0)} + G_3^{(0)} + G_4^{(0)} \\ J_{10}(\xi, \eta) &= G_2^{(0)} + G_4^{(0)} + G_1^{(1)} + G_2^{(1)} + G_3^{(1)} + G_4^{(1)} \\ J_{01}(\xi, \eta) &= G_3^{(0)} + G_4^{(0)} + G_1^{(2)} + G_2^{(2)} + G_3^{(2)} + G_4^{(2)} \\ J_{11}(\xi, \eta) &= G_4^{(0)} + G_3^{(1)} + G_4^{(1)} + G_2^{(2)} + G_4^{(2)} \\ J_{i0}(\xi, \eta) &= G_2^{(0)} + G_4^{(0)} + i G_2^{(1)} + i G_4^{(1)} \quad (i \geq 2) \\ J_{0j}(\xi, \eta) &= G_3^{(0)} + G_4^{(0)} + j G_3^{(2)} + j G_4^{(2)} \quad (j \geq 2) \\ J_{i1}(\xi, \eta) &= G_4^{(0)} + G_2^{(2)} + G_4^{(2)} + i G_4^{(1)} \quad (i \geq 2) \\ J_{1j}(\xi, \eta) &= G_4^{(0)} + G_4^{(1)} + G_4^{(1)} + j G_4^{(2)} \quad (j \geq 2) \\ J_{ij}(\xi, \eta) &= G_4^{(0)} + i G_4^{(1)} + j G_4^{(2)} \quad (i \geq 2, j \geq 2) \end{aligned} \right\} \quad (28)$$

From equations (20) and (26), we get

$$\bar{w}_{np}(x, y) - w^*(x, y) = \sum_{s=0}^t \sum_{m=0}^s L_{sm}(\xi, \eta) \kappa^{s-m} a^s \left[ \frac{\partial^s w^*}{\partial x^m \partial y^{s-m}} \right]_{0p} + \bar{R}_{ntp} - R_t \quad (29)$$

where

$$L_{sm}(\xi, \eta) = \sum_{i=0}^m \sum_{j=0}^{s-m} (J_{ij} - \xi^i \eta^j) \frac{(-1)^{s+i+j} \xi_0^{m-i} \eta_0^{s-m-j}}{(m-i)! i! (s-m-j)! j!} \quad (30)$$

There exists the following lemma concerning with  $L_{sm}(\xi, \eta)$  and  $J_{ij}(\xi, \eta)$  :

[Lemma 1] If  $G_q^{(l)}(\xi, \eta)$  satisfies the condition of equation (15), the following statements (i) and (ii) are equivalent for an arbitrary integer  $t$ .

(i) For any combination of integer  $s$  and  $m$  which satisfy  $0 \leq m \leq s \leq t$ , there exist the following equations :

$$L_{sm, \xi\xi} = 0, \quad L_{sm, \xi\eta} = 0, \quad L_{sm, \eta\eta} = 0 \quad (31)$$

(ii) For any combination of integer  $i$  and  $j$  which satisfy  $0 \leq i + j \leq t$ , there exists the following equation.

$$J_{ij}(\xi, \eta) = \xi^i \eta^j \quad (32)$$

[Proof of lemma 1] Integrating equation (31) we get,

$$L_{sm}(\xi, \eta) = C_{sm}^{(\xi)} \xi + C_{sm}^{(\eta)} \eta + C_{sm} \quad (33)$$

where  $C_{sm}^{(\xi)}$ ,  $C_{sm}^{(\eta)}$  and  $C_{sm}$  are constants. Solving equation (30) for  $J_{ij}(\xi, \eta)$  we obtain,

$$\left. \begin{aligned} J_{00}(\xi, \eta) &= 1 + L_{00} \\ J_{10}(\xi, \eta) &= \xi + L_{11} + L_{00} \xi_0 \\ J_{01}(\xi, \eta) &= \eta + L_{10} + L_{00} \eta_0 \\ J_{20}(\xi, \eta) &= \xi^2 + 2L_{22} + 2\xi_0 L_{11} + \xi_0^2 L_{00} \\ J_{11}(\xi, \eta) &= \xi\eta + L_{21} + \eta_0 L_{11} + \xi_0 L_{10} + \xi_0 \eta_0 L_{00} \\ J_{02}(\xi, \eta) &= \eta^2 + 2L_{20} + 2\eta_0 L_{10} + \eta_0^2 L_{00} \\ &\dots\dots\dots \end{aligned} \right\}$$

In view of the above equations and (33), we find that  $J_{ij}(\xi, \eta)$  must have the following form :

$$J_{ij}(\xi, \eta) = \xi^i \eta^j + d_{ij}^{(\xi)} \xi + d_{ij}^{(\eta)} \eta + d_{ij} \quad (34)$$

where  $d_{ij}^{(\xi)}$ ,  $d_{ij}^{(\eta)}$  and  $d_{ij}$  are constants. The condition for the shape function (equation (15)) is describable in terms of  $J_{ij}(\xi, \eta)$  as follows using equation (28) :

$$\left. \begin{aligned}
 J_{00}(0, 0) &= J_{00}(1, 0) = J_{00}(0, 1) = J_{00}(1, 1) = 1 \\
 J_{00,\xi}(0, 0) &= J_{00,\xi}(1, 0) = J_{00,\xi}(0, 1) = J_{00,\xi}(1, 1) = 0 \\
 J_{00,\eta}(0, 0) &= J_{00,\eta}(1, 0) = J_{00,\eta}(0, 1) = J_{00,\eta}(1, 1) = 0 \\
 J_{10}(0, 0) &= J_{10}(0, 1) = 0, \quad J_{10}(1, 0) = J_{10}(1, 1) = 1 \\
 J_{10,\xi}(0, 0) &= J_{10,\xi}(1, 0) = J_{10,\xi}(0, 1) = J_{10,\xi}(1, 1) = 1 \\
 &\dots\dots\dots
 \end{aligned} \right\}$$

The above equations decide the constants of equation (34) as  $d_{ij}^{(\xi)} = d_{ij}^{(\eta)} = d_{ij} = 0$ . Then we get equation (32). Accordingly statement (i) leads to statement (ii). On the other hand if equation (32) holds true, substitution of it in equation (30) gives directly equation (31).

**CONVERGENCE CRITERION OF GENERALIZED STRAINS**

[Lemma 2] *If  $J_{ij}(\xi, \eta)$  satisfies the following equation for a certain constant  $t (\geq 2)$*

$$J_{ij}(\xi, \eta) = \xi^i \eta^j \quad (i, j = 0, 1, 2, \dots, i + j \leq t) \tag{35}$$

then it holds that

$$\lim_{n \rightarrow \infty} |\bar{w}_n - w^*|_{\Omega}^2 = \lim_{a \rightarrow 0} \text{const.} \max\{a^{2(t-1)}, a^{2(1-\rho)}\} = 0 \tag{36}$$

The above lemma shows the condition that generalized strains of  $\bar{w}_n(x, y)$  converge to those of  $w^*(x, y)$  in the sense of square mean norm.

[Proof of lemma 2] Let us denote the assemblage of the elements which involve singular points by  $\Omega''$  and the assemblage of the other elements in  $\Omega$  by  $\Omega'$ . In this paper  $\Omega''$  is assumed as a single element at the most right and down element in figure 1. At first we treat the domain  $\Omega'$ . From equation (2') we obtain

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \sum_{p(\Omega')} \iint_{\Omega_p} [\partial^s w^* / \partial x^m \partial y^{s-m}]_{0_p}^2 dx dy &= \lim_{n \rightarrow \infty} \sum_{p(\Omega')} [\partial^s w^* / \partial x^m \partial y^{s-m}]_{0_p}^2 ab \\
 &= \lim_{a \rightarrow 0} \iint_{\Omega'} \{\partial^s w^*(x, y) / \partial x^m \partial y^{s-m}\}^2 dx dy \\
 &< \lim_{a \rightarrow 0} \int_0^{\frac{\pi}{2}} \int_{a\epsilon}^{\sqrt{A^2+B^2}} (Kr^{-\rho-s+2})^2 r dr d\theta = \lim_{a \rightarrow 0} F_{2(s-2)}(a) / a^{2(s-2)}
 \end{aligned} \tag{37}$$

where

$$F_{\sigma}(a) \equiv \frac{\pi}{2} K^2$$

$$\begin{cases} \left| \frac{(\sqrt{A^2+B^2})^{-2\rho+2-\sigma} a^{\sigma} - \kappa^{-2\rho+2-\sigma} a^{2(1-\rho)}}{(2-2\rho-\sigma)} \right|, & (\sigma \neq 2-2\rho) \\ |a^{\sigma} \log(\sqrt{A^2+B^2}/\kappa) - a^{\sigma} \log a|, & (\sigma = 2-2\rho) \end{cases} \quad (38)$$

Integrating the second derivatives of equation (29) on the domain  $\Omega'$  and using Schwartz's inequality we get

$$\iint_{\Omega'} (\bar{w}_{n,xx} - w_{,xx}^*)^2 dx dy \leq 2 \left[ \sum_{p(\Omega')} \iint_{\omega_p} \left( \sum_{s=0}^t \sum_{m=0}^s L_{sm, \xi\xi} \kappa^{s-m} a^{s-2} [\partial^s w^* / \partial x^m \partial y^{s-m}]_{0p} \right)^2 dx dy \right. \\ \left. + \sum_{p(\Omega')} \iint_{\omega_p} (\bar{R}_{ntp,xx} - R_{t,xx})^2 dx dy \right] \quad (39)$$

About the first term of the right hand side of the above equation the following inequality is obtained with use of equation (37).

$$\lim_{n \rightarrow 0} \sum_{p(\Omega')} \iint_{\omega_p} \left\{ \sum_{s=0}^t \sum_{m=0}^s L_{sm, \xi\xi} \kappa^{s-m} a^{s-2} [\partial^s w^* / \partial x^m \partial y^{s-m}]_{0p} \right\}^2 dx dy \\ < \frac{(t+1)(t+2)}{2} \sum_{m=0}^t \sum_{s=0}^m \max_{0 \leq \xi, \eta \leq 1} (L_{sm, \xi\xi}(\xi, \eta))^2 \kappa^{2(s-m)} \cdot \lim_{a \rightarrow 0} F_{2(s-2)}(a) \quad (40)$$

In the same way about the second term of the right hand side of equation (39) we have

$$\lim_{n \rightarrow \infty} \sum_{p(\Omega')} \iint_{\omega_p} (\bar{R}_{ntp,xx} - R_{t,xx})^2 dx dy < 2(t+2) \sum_{m=0}^{t+1} \left\{ 12 \sum_{q=1}^4 \sum_{l=0}^2 \max_{0 \leq \xi, \eta \leq 1} (I_{mq, \xi\xi}^{(l)}(\xi, \eta))^2 \right. \\ \left. + \max_{0 \leq \xi, \eta \leq 1} (I_{m, \xi\xi}^*(\xi, \eta))^2 \right\} \lim_{a \rightarrow 0} F_{2(t-1)}(a) \quad (41)$$

From equations (39) to (41) we can conclude that if  $L_{sm, \xi\xi}$  satisfies the following equation for a certain integer  $t (\geq 2)$ .

$$L_{sm, \xi\xi} = 0 \quad (0 \leq m \leq s \leq t) \quad (42)$$

then we get

$$\lim_{n \rightarrow \infty} \iint_{\Omega'} (\bar{w}_{n,xx} - w_{,xx}^*)^2 dx dy = \lim_{a \rightarrow 0} \text{const} \cdot \max \{ a^{2(t-1)}, a^{2(1-\rho)} \} = 0 \quad (43)$$

In the same way if  $L_{sm, \xi\eta}$  satisfies the following equation for a certain integer  $t (\geq 2)$

$$L_{sm,\xi\eta} = 0 \quad (0 \leq m \leq s \leq t) \tag{44}$$

then we get

$$\lim_{n \rightarrow \infty} \iint_{\Omega'} (\bar{w}_{n,xy} - w_{,xy}^*)^2 dx dy = \lim_{a \rightarrow 0} \text{const} \cdot \max\{a^{2(t-1)}, a^{2(1-\rho)}\} = 0 \tag{45}$$

In the same way, when  $L_{sm,\eta\eta}$  satisfies the following equation for a certain integer  $t (\geq 2)$

$$L_{sm,\eta\eta} = 0 \quad (0 \leq m \leq s \leq t)$$

then we get

$$\lim_{n \rightarrow \infty} \iint_{\Omega'} (\bar{w}_{n,y\eta} - w_{,y\eta}^*)^2 dx dy = \lim_{a \rightarrow 0} \text{const} \cdot \max\{a^{2(t-1)}, a^{2(1-\rho)}\} = 0 \tag{46}$$

The above relations of equations (43), (45) and (46) with equation (6) and **Lemma I** show that when equation (35) is satisfied for a certain integer  $t (\geq 2)$ , then we get

$$\lim_{n \rightarrow \infty} \|\bar{w}_n - w^*\|_{\Omega'}^2 = \lim_{a \rightarrow 0} \text{const} \cdot \max\{a^{2(t-1)}, a^{2(1-\rho)}\} = 0 \tag{47}$$

In the next place we treat the domain  $\Omega''$ . From equation (29),

$$\begin{aligned} \iint_{\Omega''} (\bar{w}_{n,xx} - w_{,xx}^*)^2 dx dy \leq & 2 \left[ \iint_{\Omega''} \left( \sum_{s=0}^1 \sum_{m=0}^s L_{sm,\xi\xi\eta\eta} a^{s-2} [\partial^s w^* / \partial x^m \partial y^{s-m}]_{0n} \right)^2 dx dy \right. \\ & \left. + \iint_{\Omega''} (R_{n,p,xx} - R_{1,xx})^2 dx dy \right] \end{aligned}$$

where  $p=1$  as the singular point is supposed to be at the origin of  $\Omega$  in figure 1. If  $J_{ij}(\xi, \eta)$  satisfies equation (35) the first term of the right hand side of the above equation obviously becomes zero. The second term of the right hand side of the above equation is transformed, substituting equation (22) and (27), using Schwartz's inequality and substituting equation (2) with the assumption that  $\xi_0=0, \eta_0=0$  with loss of generality and we obtain

$$\begin{aligned} \iint_{\Omega''} (\bar{w}_{n,xxx} - w_{,xxx}^*)^2 dx dy & < 3 \sum_{m=0}^2 \left\{ \sum_{q=1}^4 \sum_{l=0}^2 \max_{0 \leq \xi, \eta \leq 1} |I_{mq}^{(l)}(\xi, \eta)(\theta_{1q}^{(l)})^{-\rho}| \right. \\ & \left. + \max_{0 \leq \xi, \eta \leq 1} |I_{m,\xi\xi}^*(\xi, \eta)(\theta_{1xx})^{-\rho}| \right\}^2 K^2 a^{-2\rho} \iint_{\Omega''} dx dy \\ & = 3 \sum_{m=0}^2 \left\{ \sum_{q=1}^4 \sum_{l=0}^2 \max_{0 \leq \xi, \eta \leq 1} |I_{mq,\xi\xi}^{(l)}(\xi, \eta)(\theta_{1q}^{(l)})^{-\rho}| \right. \end{aligned}$$

$$+ \max_{0 \leq \xi, \eta \leq 1} |I_{m, \xi \xi}^*(\xi, \eta)(\theta_{1xx})^{-\rho}|^2 K^2 a^{2(1-\rho)\kappa} \quad (48)$$

where

$$0 < \theta_{pxx} < 1$$

In the similar manner when  $J_{ij}(\xi, \eta)$  satisfies equation (35) we have

$$\begin{aligned} \iint_{\mathcal{D}^n} (\bar{w}_{n,xy} - w_{,xy}^*)^2 dx dy &< 3 \sum_{m=0}^2 \left\{ \sum_{q=1}^4 \sum_{l=0}^2 \max_{0 \leq \xi, \eta \leq 1} |I_{mq, \xi \eta}^{(l)}(\xi, \eta)(\theta_{1q}^{(l)})^{-\rho}| \right. \\ &\left. + \max_{0 \leq \xi, \eta \leq 1} |I_{m, \xi \eta}^*(\xi, \eta)(\theta_{1xy})^{-\rho}| \right\}^2 K^2 \kappa^{-1} a^{2(1-\rho)} \end{aligned} \quad (49)$$

$$\begin{aligned} \iint_{\mathcal{D}^n} (\bar{w}_{n,yy} - w_{,yy}^*)^2 dx dy &< 3 \sum_{m=0}^2 \left\{ \sum_{q=1}^4 \sum_{l=0}^2 \max_{0 \leq \xi, \eta \leq 1} |I_{mq, \eta \eta}^{(l)}(\xi, \eta)(\theta_{1q}^{(l)})^{-\rho}| \right. \\ &\left. + \max_{0 \leq \xi, \eta \leq 1} |I_{m, \eta \eta}^*(\xi, \eta)(\theta_{1yy})^{-\rho}| \right\}^2 K^2 \kappa^{-3} a^{2(1-\rho)} \end{aligned} \quad (50)$$

where

$$0 < \theta_{pxy}, \theta_{pyy} < 1$$

In view of equations (48) to (50) we conclude that if  $J_{ij}(\xi, \eta)$  satisfies equation (35) the following equation is obtained.

$$\lim_{n \rightarrow \infty} |\bar{w}_n - w^*|_{\mathcal{D}^n}^2 = \lim_{a \rightarrow 0} \text{const} \cdot a^{2(1-\rho)} = 0 \quad (51)$$

As equation (47) and equation (51) together are equivalent to equation (36), we have **lemma 2**.

**[Lemma 3]** *If  $J_{ij}(\xi, \eta)$  satisfies the following equation for a certain integer constant  $t$  ( $\geq 2$ )*

$$J_{ij}(\xi, \eta) = \xi^i \eta^j \quad (i, j = 0, 1, 2, \dots, i + j \leq t) \quad (35)$$

then it holds that

$$\lim_{n \rightarrow \infty} \left| \sum_{p=1}^n Ar_p[w^*, \bar{w}_{np} - w^*] \right| = \lim_{a \rightarrow 0} \text{const} \cdot \max\{a^{t-1}, a^{2(1-\rho)}\} = 0 \quad (52)$$

**[Proof of lemma 3]**  $\sum_{p=1}^n Ar_p[w^*, \bar{w}_{np} - w^*]$  of equation (52) can be expanded as follows :



$$\begin{aligned}
 \sum_{p=1}^n \text{Ave}_p[w^*, \bar{w}_{np} - w^*] &= \sum_{p=1}^n [D \iint_{\omega_p} (w_{,xxx}^* + \nu w_{,xy}^*) (\bar{w}_{np,x} - w_{,x}^*) dx dy \\
 &+ D \iint_{\omega_p} (w_{,xx}^* + \nu w_{,y}^*) (\bar{w}_{np,xx} - w_{,xx}^*) dx dy \\
 &+ D \iint_{\omega_p} (w_{,yyy}^* + \nu w_{,xxy}^*) (\bar{w}_{np,y} - w_{,y}^*) dx dy \\
 &+ D \iint_{\omega_p} (w_{,yy}^* + \nu w_{,xx}^*) (\bar{w}_{np,yy} - w_{,yy}^*) dx dy \\
 &- D \iint_{\omega_p} (w_{,xxxx}^* + (2 - \nu) w_{,xxy}^*) (\bar{w}_{np} - w^*) dx dy \\
 &- D \iint_{\omega_p} (w_{,xxx}^* + (2 - \nu) w_{,xy}^*) (\bar{w}_{np,x} - w_{,x}^*) dx dy \\
 &- D \iint_{\omega_p} (w_{,yyy}^* + (2 - \nu) w_{,xxy}^*) (\bar{w}_{np} - w^*) dx dy \\
 &- D \iint_{\omega_p} (w_{,yy}^* + (2 - \nu) w_{,xy}^*) (\bar{w}_{np,y} - w_{,y}^*) dx dy] \tag{53}
 \end{aligned}$$

As the singularity of the partial derivatives of the deflection  $w^*$  is supposed to be independent of the directions of derivatives, it is sufficient to examine the

convergence of  $\iint_{\Omega} |w_{,xx}^*| |\bar{w}_{n,xx} - w_{,xx}^*| dx dy$ ,  $\iint_{\Omega} |w_{,xxx}^*| |\bar{w}_{n,x} - w_{,x}^*| dx dy$  and

$\iint_{\Omega} |w_{,xxxx}^*| |\bar{w}_{np} - w^*| dx dy$  for the examination of the convergence of the right hand side of the above equation. At the first place we examine the convergence of these equations in the domain  $\Omega'$ . From equation (29) we have

$$\begin{aligned}
 &\lim_{n \rightarrow \infty} \iint_{\Omega'} |w_{,xx}^*| |\bar{w}_{n,xx} - w_{,xx}^*| dx dy \\
 &\leq \lim_{a \rightarrow 0} \sum_{p(\Omega')} \iint_{\omega_p} |w_{,xx}^*| \left[ \sum_{s=0}^l \sum_{m=0}^s L_{sm, \xi \xi} \kappa^{s-m} a^{s-2} [\partial^s w^* / \partial x^m \partial y^{s-m}]_{0,p} \right] dx dy \\
 &+ \lim_{a \rightarrow 0} \sum_{p(\Omega')} \iint_{\omega_p} |w_{,xx}^*| (|\bar{R}_{ntp,xx}| + |R_{t,xx}|) dx dy
 \end{aligned}$$

Substituting equations (22), (27) and (2'), the above equation reduces as follows :

$$\lim_{n \rightarrow \infty} \iint_{\Omega'} |w_{,xx}^*| |\bar{w}_{n,xx} - w_{,xx}^*| dx dy$$

$$\begin{aligned}
&< \sum_{s=0}^t \sum_{m=0}^s \max_{0 \leq \xi, \eta \leq 1} |L_{sm, \xi \xi}(\xi, \eta)| \kappa^{s-m} \lim_{a \rightarrow 0} \left\{ a^{s-2} \int_0^{\frac{\pi}{2}} \int_{ax}^{\sqrt{A^2+B^2}} (Kr^{-\rho})(K^{-\rho-s+2}) r dr d\theta \right\} \\
&+ \sum_{m=0}^{t+1} \left\{ \sum_{q=1}^4 \sum_{l=0}^2 \max_{0 \leq \xi, \eta \leq 1} |I_{mq, \xi \xi}^{(l)}(\xi, \eta)| \right. \\
&+ \left. \max_{0 \leq \xi, \eta \leq 1} |I_{m, \xi \xi}^*(\xi, \eta)| \right\} \lim_{a \rightarrow 0} \left\{ a^{t-1} \int_0^{\frac{\pi}{2}} \int_{ax}^{\sqrt{A^2+B^2}} (Kr^{-\rho})(K^{-\rho-t+1}) r dr d\theta \right\} \\
&\leq \sum_{s=0}^t \sum_{m=0}^s \max_{0 \leq \xi, \eta \leq 1} |L_{sm, \xi \xi}(\xi, \eta)| \kappa^{s-m} \lim_{a \rightarrow 0} F_{s-2}(a) \\
&+ \sum_{m=0}^{t+1} \left\{ \sum_{q=1}^4 \sum_{l=0}^2 \max_{0 \leq \xi, \eta \leq 1} |I_{mq, \xi \xi}^{(l)}(\xi, \eta)| + \max_{0 \leq \xi, \eta \leq 1} |I_{m, \xi \xi}^*(\xi, \eta)| \right\} \lim_{a \rightarrow 0} F_{t-1}(a) \quad (54)
\end{aligned}$$

In the same way about  $\iint_{\mathcal{Q}'} |w_{,xxx}^*| |\bar{w}_n, x - w_{,x}^*| dx dy$  and  $\iint_{\mathcal{Q}'} |w_{,xxxx}^*| |\bar{w}_n - w^*| dx dy$  we obtain the following inequalities

$$\begin{aligned}
\lim_{n \rightarrow \infty} \iint_{\mathcal{Q}'} |w_{,xxx}^*| |\bar{w}_n, x - w_{,x}^*| dx dy &< \sum_{s=0}^t \sum_{m=0}^s \max_{0 \leq \xi, \eta \leq 1} |L_{sm, \xi \xi}(\xi, \eta)| \kappa^{s-m} \lim_{a \rightarrow 0} F_{s-1}(a) \\
&+ \sum_{m=0}^{t+1} \left\{ \sum_{q=1}^4 \sum_{l=0}^2 \max_{0 \leq \xi, \eta \leq 1} |I_{mq, \xi \xi}^{(l)}(\xi, \eta)| + \max_{0 \leq \xi, \eta \leq 1} |I_{m, \xi \xi}^*(\xi, \eta)| \right\} \lim_{a \rightarrow 0} F_t(a) \quad (55)
\end{aligned}$$

$$\begin{aligned}
\lim_{n \rightarrow \infty} \iint_{\mathcal{Q}'} |w_{,xxxx}^*| |\bar{w}_n - w^*| dx dy &< \sum_{s=0}^t \sum_{m=0}^s \max_{0 \leq \xi, \eta \leq 1} |L_{sm}(\xi, \eta)| \kappa^{s-m} \lim_{a \rightarrow 0} F_s(a) \\
&+ \sum_{m=0}^{t+1} \left\{ \sum_{q=1}^4 \sum_{l=0}^2 \max_{0 \leq \xi, \eta \leq 1} |I_{mq}^{(l)}(\xi, \eta)| + \max_{0 \leq \xi, \eta \leq 1} |I_m^*(\xi, \eta)| \right\} \lim_{a \rightarrow 0} F_{t+1}(a) \quad (56)
\end{aligned}$$

If  $J_{ij}(\xi, \eta)$  satisfies equation (35) for a certain integer  $t (\geq 2)$ , equations (54) to (56) converge to zero and the convergence orders are as follows :

$$\lim_{n \rightarrow \infty} \iint_{\mathcal{Q}'} |w_{,xx}^*| |\bar{w}_n, xx - w_{,xx}^*| dx dy = \lim_{a \rightarrow 0} \text{const} \cdot \max \{ a^{t-1}, a^{2(1-\rho)} \} \quad (57)$$

$$\lim_{n \rightarrow \infty} \iint_{\mathcal{Q}'} |w_{,xxx}^*| |\bar{w}_n, x - w_{,x}^*| dx dy = \lim_{a \rightarrow 0} \text{const} \cdot \max \{ a^t, a^{2(1-\rho)} \} \quad (58)$$

$$\lim_{n \rightarrow \infty} \iint_{\mathcal{Q}'} |w_{,xxxx}^*| |\bar{w}_n - w^*| dx dy = \lim_{a \rightarrow 0} \text{const} \cdot \max \{ a^{t+1}, a^{2(1-\rho)} \} \quad (59)$$

Equations (57) to (59) mean

$$\lim_{n \rightarrow \infty} \left| \sum_{\mathcal{P}(\mathcal{Q}')} A_{r,p} [w^*, \bar{w}_n - w^*] \right| = \lim_{a \rightarrow 0} \text{const} \cdot \max \{ a^{t-1}, a^{2(1-\rho)} \} \quad (60)$$

At the second place we examine the convergence in the domain  $\Omega''$ . If  $J_{ij}(\xi, \eta)$  satisfies the equation (35), from equation (29) with equations (22) and (27) we get

$$\begin{aligned} & \lim_{n \rightarrow \infty} \iint_{\Omega''} |w_{,xx}| |\bar{w}_n - w_{,xx}^*| dx dy \\ & \leq \lim_{a \rightarrow 0} \iint_{\Omega''} |w_{,xx}| \sum_{m=0}^2 \left\{ \sum_{q=1}^4 \sum_{l=0}^2 I_{mq, \xi\xi}^{(l)}(\xi, \eta) [\partial^2 w^* / \partial x^m \partial y^{2-m}]_{\theta_{pq}^{(l)}} \right. \\ & \quad \left. + |I_{m, \xi\xi}^*(\xi, \eta) [\partial^2 w^* / \partial x^m \partial y^{2-m}]_{\theta_{pxx}}| \right\} dx dy \end{aligned}$$

Substitution of equation (2') in the above equation yield

$$\begin{aligned} & \lim_{n \rightarrow \infty} \iint_{\Omega''} |w_{,xx}| |\bar{w}_n - w_{,xx}^*| dx dy \leq \sum_{m=0}^2 \left\{ \sum_{q=1}^4 \sum_{l=0}^2 \max_{0 \leq \xi, \eta \leq 1} |I_{mq, \xi\xi}^{(l)}(\xi, \eta)| (\sqrt{1+\kappa^2} \theta_{pq}^{(l)})^{-\rho} \right. \\ & \quad \left. + \max_{0 \leq \xi, \eta \leq 1} |I_{m, \xi\xi}^*(\xi, \eta)| (\sqrt{1+\kappa^2} \theta_{pxx})^{-\rho} \right\} \lim_{a \rightarrow 0} \int_0^{\frac{\pi}{2}} \int_0^{\alpha\sqrt{1+\kappa^2}} (Ka^{-\rho})(Kr^{-\rho}) r dr d\theta \\ & = \sum_{m=0}^2 \left\{ \sum_{q=1}^4 \sum_{l=0}^2 \max_{0 \leq \xi, \eta \leq 1} |I_{mq, \xi\xi}^{(l)}(\xi, \eta)| (\sqrt{1+\kappa^2} \theta_{pq}^{(l)})^{-\rho} \right. \\ & \quad \left. + \max_{0 \leq \xi, \eta \leq 1} |I_{m, \xi\xi}^*(\xi, \eta)| (\sqrt{1+\kappa^2} \theta_{pxx})^{-\rho} \right\} \cdot \lim_{a \rightarrow 0} \frac{\pi}{2} K^2 \frac{(\sqrt{1+\kappa^2})^{2-\rho} a^{2(1-\rho)}}{(2-\rho)} \quad (61) \end{aligned}$$

In the same way if  $J_{ij}(\xi, \eta)$  satisfies equation (35) we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \iint_{\Omega''} |w_{,xxx}| |\bar{w}_n - w_{,xxx}^*| dx dy \\ & \leq \sum_{m=0}^2 \left\{ \sum_{q=1}^4 \sum_{l=0}^2 \max_{0 \leq \xi, \eta \leq 1} |I_{mq, \xi\xi}^{(l)}(\xi, \eta)| (\sqrt{1+\kappa^2} \theta_{pq}^{(l)})^{-\rho} \right. \\ & \quad \left. + \max_{0 \leq \xi, \eta \leq 1} |I_{m, \xi\xi}^*(\xi, \eta)| (\sqrt{1+\kappa^2} \theta_{pxx})^{-\rho} \right\} \frac{\pi}{2} K^2 \lim_{a \rightarrow 0} \frac{(\sqrt{1+\kappa^2})^{1-\rho} a^{2(1-\rho)}}{1-\rho} \quad (62) \end{aligned}$$

$$\begin{aligned} & \lim_{n \rightarrow \infty} \iint_{\Omega''} |w_{,xxx}| |\bar{w}_n - w_{,xxx}^*| dx dy \\ & \leq \sum_{m=0}^2 \left\{ \sum_{q=1}^4 \sum_{l=0}^2 \max_{0 \leq \xi, \eta \leq 1} |I_{mq}^{(l)}(\xi, \eta)| (\sqrt{1+\kappa^2} \theta_{pq}^{(l)})^{-\rho} + \max_{0 \leq \xi, \eta \leq 1} |I_m^*(\xi, \eta)| \right. \\ & \quad \left. \cdot (\sqrt{1+\kappa^2} \theta_p)^{-\rho} \right\} \frac{\pi}{2} K^2 \lim_{a \rightarrow 0} \begin{cases} (\sqrt{1+\kappa^2})^{-\rho} a^{2(1-\rho)} / \rho & (\rho \neq 0) \\ a^2 \log(a\sqrt{1+\kappa^2}) & (\rho = 0) \end{cases} \quad (63) \end{aligned}$$

When  $J_{ij}(\xi, \eta)$  satisfies the equation (35), from equations (61) to (63) we have

$$\lim_{n \rightarrow \infty} \left| \sum_{p \in (Q^n)} A_{r_p} [w^*, \bar{w}_{np} - w^*] \right| = \lim_{a \rightarrow 0} \text{const} \cdot a^{2(1-\rho)} \quad (64)$$

Equations (60) and (64) prove **lemma 3**.

[**Lemma 4**] If  $J_{ij}(\xi, \eta)$  satisfies the following equation for a certain integer constant  $t$  ( $\geq 2$ )

$$J_{ij}(\xi, \eta) = \xi^i \eta^j \quad (i, j = 0, 1, 2, \dots, i + j \leq t) \quad (35)$$

then it holds that

$$\lim_{n \rightarrow \infty} \left| \sum_{p=1}^n A_{r_p} [w^*, \tilde{w}_{np} - w^*] \right| = \lim_{a \rightarrow 0} \text{const} \cdot \max\{a^{t-1}, a^{2(1-\rho)}\} = 0 \quad (65)$$

[*Proof of lemma 4*] As  $w^*$  satisfies **conditions** [I] and [II] in the previous section, the following relation exists :

$$\sum_{q=1}^n A_{r_q} [w^*, w^*] = 0 \quad (66)$$

Let us define the function  $\hat{w}_n(x, y)$ , which is assumed to belong to the family of the functions that satisfies **conditions** [I] and [II], and the nodal values of which  $\hat{w}_n(x_h, y_h)$ ,  $\hat{w}_{n,x}(x_h, y_h)$  and  $\hat{w}_{n,y}(x_h, y_h)$  ( $h=1, 2, \dots, k$ ) are settled equal to  $\tilde{w}_n(x_h, y_h)$ ,  $\tilde{w}_{n,x}(x_h, y_h)$  and  $\tilde{w}_{n,y}(x_h, y_h)$  respectively. Obviously  $\hat{w}_n$  satisfies the following equation :

$$\sum_{p=1}^n A_{r_p} [w^*, \hat{w}_n] = 0 \quad (67)$$

Equations (66) and (67) lead to

$$\sum_{p=1}^n A_{r_p} [w^*, \tilde{w}_{np} - w^*] = \sum_{p=1}^n A_{r_p} [w^*, \tilde{w}_{np} - \hat{w}_n] \quad (68)$$

Now let us examine the properties of the function  $\tilde{w}_n(x, y)$  to reveal the properties of  $\hat{w}_n(x, y)$ . **Condition** [i] which ordinary finite element methods demand, leads to

$$\begin{aligned} \tilde{w}_n(x, y), \tilde{w}_{n,x}(x, y), \tilde{w}_{n,y}(x, y) < \infty \\ \text{(bounded properties of rigid body displacement)} \end{aligned}$$

On the other hand equations (5) (positive definiteness of  $\tilde{w}_n$ ), (9) and (10) give the following inequality :

$$|\bar{w}_n|_Q \leq [II[\bar{w}_n] + AB\|f\|_Q^2/(4\theta^2)]^{1/2} + \sqrt{AB}\|f\|_Q/(2\theta)$$

From this inequality we have easily

$$\iint_Q \bar{w}_{n,xx}^2 dx dy, \quad \iint_Q \bar{w}_{n,xy}^2 dx dy, \quad \iint_Q \bar{w}_{n,yy}^2 dx dy < \infty$$

(bounded properties of strains of  $\bar{w}_n$ )

The above inequalities mean that the whole second derivatives of  $\bar{w}_n$  are bounded at almost everywhere in  $\Omega$ , and that the order of singularity of  $\bar{w}_n$  is, at the most, same as that allowed in  $w^*$ . The mean value theorem shows, for example,

$$\{\bar{w}_{n,x}(x_h+a, y_h) - \bar{w}_{n,x}(x_h, y_h)\}/h = \bar{w}_{n,xx}(x_h+a\xi, y_h) \quad (0 \leq \xi \leq a)$$

As  $\bar{w}_{n,xx}$  is bounded, the above equation shows that  $\bar{w}_n$  is smooth even for  $a \rightarrow 0$ . The above mentioned properties of  $\bar{w}_n$  means that  $\hat{w}_n(x, y)$  and its derivatives can be settled bounded for any value of  $a$ . These properties of  $\hat{w}_n$  imply that limiting value of the right hand side of equation (68) is obtainable by a procedure analogous to that of  $\sum_{p=1}^n Ar_p[w^*, \bar{w}_{np} - w^*]$ . In view of the derivation of equation (52), we obtain **lemma 4**. From **theorem 2** with **lemma 2, 3** and **4**, we conclude the following theorem.

**[Theorem 3]** *If  $J_{ij}(\xi, \eta)$  satisfies the following equation for a certain integer constant  $t$  ( $\geq 2$ )*

$$J_{ij}(\xi, \eta) = \xi^i \eta^j \quad (i, j = 0, 1, 2, \dots, i+j \leq t) \quad (35)$$

then it holds that

(i) *for the case when a shape function used is conforming*

$$\lim_{n \rightarrow \infty} [\bar{w}_n - w^*]_Q = \lim_{a \rightarrow 0} \text{const} \cdot a^{1-\rho} = 0 \quad (69-a)$$

(ii) *for the case when a shape function used is non-conforming*

$$\lim_{n \rightarrow \infty} [\bar{w}_n - w^*]_Q = \lim_{a \rightarrow 0} \text{const} \cdot \max\{a^{(t-1)/2}, a^{1-\rho}\} = 0 \quad (69-b)$$

The above theorem shows that if  $J_{ij}(\xi, \eta)$  satisfies equation (35) for  $i+j \leq 2$ , the energy of the generalized strain of the finite element solution converges to those of the exact solution in the sense of square mean. equation (35) for  $i+j \leq 2$  means the convergence criterion of the generalized strains.

### CONVERGENCE CRITERION OF FINITE ELEMENT SOLUTION

**Theorem 3** with **theorem 1** (i) gives the following theorem :

[**Theorem 4**] If  $J_{ij}(\xi, \eta)$  satisfies the following equation for a certain integer constant  $t (\geq 2)$

$$J_{ij}(\xi, \eta) = \xi^i \eta^j \quad (i, j = 0, 1, 2, \dots, i + j \leq t) \quad (35)$$

then it holds that

(i) for the case when a shape function used is conforming

$$\lim_{n \rightarrow \infty} \max_{x, y \in \mathcal{Q}} |\tilde{w}_n - w^*| = \lim_{a \rightarrow 0} \text{const} \cdot a^{1-\theta} = 0 \quad (70-a)$$

(ii) for the case when a shape function used is non-conforming

$$\lim_{n \rightarrow \infty} \max_{x, y \in \mathcal{Q}} |\tilde{w}_n - w^*| = \lim_{a \rightarrow 0} \text{const} \cdot \max\{a^{(t-1)/2}, a^{1-\theta}\} = 0 \quad (70-b)$$

**Theorem 1** to **4** are derived for the case when the shape function is expressed in the form of equation (14) and the element shape is a rectangle. However these theorems are also valid for some another types of shape functions and element shapes with a slight modification of equations (14) and (28).

For example, for the case when the element shape is a right-angled triangle, a pair of which construct a rectangle and nodes of the element is chosen at its corner  $q = 1, 2$  and  $3$  as shown in figure 4, the theorems also hold only if the shape function (14) and equation (28) are replaced by the following equations (14') and (28') respectively.

$$w_{np}(x, y) = \sum_{q=1}^3 [G_q^{(0)}(\xi, \eta) w_{pq}^{(0)} + a G_q^{(1)}(\xi, \eta) w_{pq}^{(1)} + b G_q^{(2)}(\xi, \eta) w_{pq}^{(2)}] \quad (14')$$

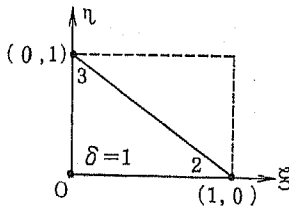


Fig. 4 Right-angled triangular element

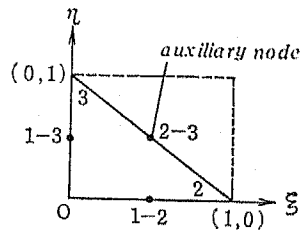


Fig. 5 Right-angled triangular element with auxiliary node

$$\begin{aligned}
 J_{00} &= G_1^{(0)} + G_2^{(0)} + G_3^{(0)} \\
 J_{10} &= G_2^{(0)} + G_1^{(1)} + G_2^{(1)} + G_3^{(1)} \\
 J_{01} &= G_3^{(0)} + G_1^{(2)} + G_2^{(2)} + G_3^{(2)} \\
 J_{11} &= G_3^{(1)} + G_2^{(2)} \\
 J_{i0} &= G_2^{(0)} + iG_2^{(1)} \quad (i \geq 2) \\
 J_{0j} &= G_3^{(0)} + jG_3^{(2)} \quad (j \geq 2) \\
 J_{i1} &= G_2^{(2)} \quad (i \geq 2) \\
 J_{1j} &= G_3^{(1)} \quad (j \geq 2) \\
 J_{ij} &= 0 \quad (i \geq 2, j \geq 2)
 \end{aligned} \tag{28'}$$

Next example is a case when the element shape is a right angled triangle and the auxiliary nodes are added at the middle points of the sides of element besides the ordinary nodes at the vertices as shown in figure 5. For this case the shape function has 12 degree of freedom and equations (14) and (28) are replaced by the following equations (14'') and (28'') respectively.

$$\begin{aligned}
 w_{np}(x, y) &= \sum_{q=1}^3 [G_q^{(0)} w_{pq}^{(0)} + aG_q^{(1)} w_{pq}^{(1)} + bG_q^{(2)} w_{pq}^{(2)}] \\
 &\quad + \{ b\mathcal{G}_{12}(\xi, \eta)\phi_{12} + \sqrt{a^2 + b^2}\mathcal{G}_{23}(\xi, \eta)\phi_{23} + a\mathcal{G}_{13}(\xi, \eta)\phi_{13} \}
 \end{aligned} \tag{14''}$$

$$\begin{aligned}
 J_{00} &= G_1^{(0)} + G_2^{(0)} + G_3^{(0)} \\
 J_{10} &= G_1^{(1)} + G_2^{(0)} + G_2^{(1)} + G_3^{(1)} + \mathcal{G}_{13} + \kappa\mathcal{G}_{23} \\
 J_{01} &= G_1^{(2)} + G_2^{(2)} + G_3^{(0)} + G_3^{(2)} + \mathcal{G}_{12} + \kappa^{-1}\mathcal{G}_{23} \\
 J_{11} &= G_2^{(2)} + G_3^{(1)} + \frac{1}{2}\mathcal{G}_{13} + \frac{1}{2}\mathcal{G}_{12} + \left(\frac{\kappa}{2} + \frac{\kappa^{-1}}{2}\right)\mathcal{G}_{23} \\
 J_{i0} &= G_2^{(0)} + iG_2^{(1)} + \kappa i \left(\frac{1}{2}\right)^{i-1} \mathcal{G}_{23} \quad (i \geq 2) \\
 J_{0j} &= G_3^{(0)} + jG_3^{(2)} + \kappa^{-1}j \left(\frac{1}{2}\right)^{j-1} \mathcal{G}_{23} \quad (j \geq 2) \\
 J_{i1} &= G_2^{(2)} + \left(\frac{1}{2}\right)^i \mathcal{G}_{12} + \left[\kappa i \left(\frac{1}{2}\right)^i + \kappa^{-1} \left(\frac{1}{2}\right)^j\right] \mathcal{G}_{23} \quad (i \geq 2) \\
 J_{1j} &= G_3^{(1)} + \left(\frac{1}{2}\right)^j \mathcal{G}_{13} + \left[\kappa^{-1}j \left(\frac{1}{2}\right)^j + \kappa \left(\frac{1}{2}\right)^i\right] \mathcal{G}_{23} \quad (j \geq 2)
 \end{aligned} \tag{28''}$$

$$J_{ij} = (\kappa i + \kappa^{-1} j) \left( \frac{1}{2} \right)^{i+j-1} g_{23} \quad (i, j \geq 2) \quad (28'' \text{ cont.})$$

where  $\phi_{12}$ ,  $\phi_{23}$  and  $\phi_{13}$  denote the nodal slopes at the middle points of the element sides 1-2, 2-3 and 1-3 respectively to the directions perpendicular to the sides as shown in figure 5.  $g_{12}(\xi, \eta)$ ,  $g_{23}(\xi, \eta)$  and  $g_{13}(\xi, \eta)$  denote weighting functions corresponding to  $\phi_{12}$ ,  $\phi_{23}$  and  $\phi_{13}$  respectively.

## RESULTS AND DISCUSSION

From **theorems 4** and **5** we can express the convergence criterion as follows : The condition of uniform convergence of the finite element solutions to the exact solutions, and at the same time the condition of convergence of the generalized strain of the finite element solutions to those of the exact solutions in the sense of square mean (but not uniform) is that  $J_{ij}(\xi, \eta)$  satisfies equation (35) for  $i + j \leq 2$ .

CLOUGH et al.<sup>2)</sup> presented convergence criterion obtained nearly by experience with a large number of numerical calculations and without rigorous mathematical proof as follows :

- (a) Every possible rigid body displacement must be included.<sup>22)</sup>
- (b) Every uniform strain state should be included.<sup>22)</sup>
- (c) Condition of compatibility should be satisfied along the boundaries between elements.<sup>6)</sup>

The discriminants  $J_{00} = 1$ ,  $J_{10} = \xi$ ,  $J_{01} = \eta$  and  $J_{20} = \xi^2$ ,  $J_{11} = \xi\eta$ ,  $J_{02} = \eta^2$  of equation (35) correspond to the items (a) and (b) of the above conditions respectively. Equation (35) for  $i + j \leq 2$  shows a concrete way of realizing the above conditions (a) and (b). Corresponding to the above condition (c), **lemma 4** shows that the discontinuities along the element boundaries between adjacent elements converge to zero with increase of the element number in the sense of the energy. Therefore the above condition (c) is not necessary for the correct convergence, and can be relaxed to a more weak condition as stated above. On the influences of singularity of  $w^*$  the theory proves that the problems, the exact solutions of which have singularities can be solved by the finite element method in the sense of equations (69) and (70), only if **condition [II]** (equation (2)) is satisfied.

**Remark :** In recent years, STRANG<sup>13)</sup> et al. , CIARLET,<sup>14)</sup> LASCAUX et al.<sup>15)</sup> and KIKUCHI<sup>20)21)</sup> derived the fundamental inequality to determine the convergence properties of the finite element solutions, using Pythagorean theorem on strains, as follows :



$$|\bar{w}_n - w^*|_{\Omega}^2 \leq |\bar{w}_n - w^*|_{\Omega}^2 + \frac{1}{2} \left\{ \sup_{v_n} \sum_{p=1}^n Ar_p[w^*, v_n p] / |v_n|_{\Omega} \right\}^2 \quad (71)$$

They also showed that to pass the Patch Test means to satisfy the following equation :

$$\sup_{v_n} \sum_{p=1}^n Ar_p[P(2), v_n] / |v_n|_{\Omega} = 0 \quad (72)$$

where  $P(2)$  is arbitrary polynomials of degree  $\leq 2$  on  $\Omega$ .

The author believes, it is very significant for the development of the convergence theories of finite element method to make clear the relations of equation (18) (the fundamental inequality on strains of this report) against equations (71) and (72). To carry into execution this problem **lemma 4** and its derivation should be examined more precisely and carefully.

These problems will be treated elsewhere by the author.

### CONCLUSIONS

The convergence criterion of the finite element method applied to thin plate bending is derived theoretically in the form of the discriminants for shape functions as shown in **theorem 4**. In this paper the shape functions are assumed to be expressed in the form of equations (14), (14') and (14'').

The derived condition of the convergence does not contradict with the empirically obtained condition so called "constant strains".

It is also proved that the finite element solution can converge to the exact solution even for the case where exact solution has singularity in such a extent as is shown in **condition [II]**.

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