(II) Radius of Curvature of Bent Yarn

by

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INTRODUCTION

The deformation curve which a yarn centeral line shows in large deformation is changed by an equivalent thin rod. Then the pitch of yarn is varied by the direct force varying at each point of the centeral line, and so the cross section is variable. T. Suhara¹) has studied this problem as an elastic rod with variable cross section and M. Mizuno²) has studied the deformation of spring coil. It is an interesting problem in applied mechanics from the point of view of the large deformation theory, but in bending fabrics the woven materials and conditions are large factors to solve the problem mechanically. The relations between the crease of fabrics and the viscoelastic theory were studied by K. Murakami³) but in a few experiments of the woven materials the crease of yarn is too complicated to discuss it viscoelastically. Many questions about it have not been solved yet and so it is not unworthy to study bent yarn on the elastic theory.

Now let us consider in this paper the small radius of curvature of a formed yarn in bending fabrics. In this case the mutual frictions between the fibers are negligible in order to simplify this problem. The radius of curvature of yarn centeral line on the elastic theory and the local curvature on the differential geometry are discussed mechanically in this paper.

THEOREM

The straight filament having the same pitch in a state of unloading is deformed under the compresion force w to the direction of the initial central line at the both ends, rotational ends. It is assumed that the deformation curve of the central line is APOB in Fig 1. When the coordinate axes are determined as shown in Fig.1, the tangent of the deformation curve at the middle point of the yarn length is parallel to the loading direction and the deformation curve is OPA. θ is the angle between the tangent at an arbitrary point P on the deformation curve of the centeral line of the filament and the load-direction, where $\pi > \theta > 0$, and the angle at the point A is α . Therefore the orthogonal axes (x, y) are determined as the normal and





tangential directions at the point O. If the position of an arbitary point Pis determined in the case of Fig 1., the deformation of centeral line of the filament as well as the large deformation of the rod can be determined when there is the deformation curve on a plane, whatever the conditions of loading and holding at both ends may be.

Now, let l_0 be total legth of the filament in unloading, l total length of the filament in loading, h_0 pitch of the filament in unloading, S the

length of OP along the deformation curve of the centeral line of the filament, S_0 the equivalent length of S in unloading. E, and G young's modulus and modulus of rigidity of the fiber, and I moment of inertia of the fiber. J represents the polar moment of inertia of the cross section in the simple case of circular cross section and it is the same with the cross section of other type, then,

$$2I = J$$

And also let r be herix radius of the fiber and A, B, C coefficients of rigidity for the direct, bending and shearing forces introduced respectively by S. Timoshenko⁴)

$$A_0 = h_0 \cos\beta/2\pi r^3 \left(\sin^2\beta/EI + \cos^2\beta/2GI\right)$$

$$B_0 = h_0 \cos\beta/\pi r \left(1 + \sin^2\beta/EI + \cos^2\beta/2GI\right)$$

$$C_0 = h_0 \cos\beta/\pi r^3 \left(\cos^2\beta/EI + \sin^2\beta/2GI\right)$$

(a)

where β is the herix angle of the fiber and is assumed to be constant in bending, h_0 one pitch length of the filament, A, B, C coefficient of rigidity for A_0, B_0, C_0 at point P under loading respectively, h one pitch at point P under loading, and T, Q, M the direct force, shearing force and bending moment at point P respectively, then,

$$T = -w\cos\theta, \ Q = w\sin\theta, \ M = w \int_{s}^{t} \sin\theta \ ds$$
(b)
$$T/A_{0} = (ds - ds_{0})/ds_{0} = (h - h_{0})/h_{0}$$

Therefore,

$$ds/ds_0 = h/h_0 = 1 + T/A_0 = 1 - w/A_0 \cdot \cos\theta \qquad (c)$$

$$A = A_0 h/h_0 = A_0 (1 - w/A_0 \cdot \cos\theta)$$

$$B = B_0 h/h_0 = B_0 (1 - w/A_0 \cdot \cos\theta) \qquad (d)$$

$$C = C_0 h/h_0 = C_0 (1 - w/A_0 \cdot \cos\theta)$$

From the curvature of the deformation curve of centeral line of the filament,

$$d\theta/ds = M/B + 1/C \cdot d\theta/ds \tag{1}$$

This equation is the fundamental equation of the deformation curve and the first term represents the bending moment and the second term the curvature by the shearing force. When the equations (b) and (d) are introduced into the equation (1),

$$d\theta/ds = w \int_{s}^{l} \sin\theta ds/B_{0} \left(1 - w/A_{0} \cdot \cos\theta\right) + w \cos\theta/C_{0} \left(1 - w/A_{0} \cdot \cos\theta\right) \cdot d\theta/ds$$

To solve the above equation easily, the following are put in it.

$$s = l\alpha, \ w/A_0 = \lambda, \ l^2 w/B_0 = \mu$$
$$w/C_0 = \nu, \ \nu + \lambda = \tau = w (1/A_0 + 1/C_0)$$
(e)

Then,

$$(1 - \tau \cos\theta) \, d\theta / d\alpha = \mu \int_{\alpha}^{1} \sin\theta d\alpha \tag{2}$$

$$\left\{ (1 - \tau \cos\theta) \, d\theta / d\alpha \right\} d/d\alpha \left\{ (1 - \tau \cos\theta) \, d\theta / d\alpha \right\} d\alpha = -\mu \left(1 - \tau \cos\theta \right) \sin\theta \cdot d\theta$$

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$$(1 - \tau \cos\theta)^2 \left(\frac{d\theta}{d\alpha} \right)^2 = \mu \left(2\cos\theta - \tau \cos^2\theta + c \right)$$
(3)

Where c is the integral constant and at point A, $\theta = \alpha$

$$d\theta/d\alpha = (ld\theta/ds) = 0 \tag{f}$$

Therefore,

$$c = -\left(2\cos\alpha - \tau\cos^2\alpha\right) = 1/\tau \cdot \left\{(1 - \tau\cos\alpha)^2 - 1\right\}$$
(4)

Introducing the equation (4) into the equation (3),

$$(1 - \tau \cos\theta)^2 \cdot (d\theta/d\alpha)^2 = \mu/\tau \cdot \left\{ (1 - \tau \cos\alpha)^2 - (1 - \tau \cos\theta)^2 \right\}$$

Refering the equation (e), put into as following:-

$$\sqrt{\tau/\mu} = \sqrt{(1/A_0 + 1/C_0)/(l^2/B_0)} = r/l \cdot \sqrt{f}$$
 (g)

This problem is discussed in the case of $\tau \leq 1$ and may be treated by the same method in the case of $\tau > 1$, from $d\theta/d\alpha \leq 0$

$$d\alpha = \sqrt{f \cdot r/l} \cdot (1 - \tau \cos\theta) / \sqrt{(1 - \tau \cos\alpha)^2 - (1 - \tau \cos\theta)^2} \cdot d\theta$$
(5)

From the equation (e),

$$ds \Rightarrow ld\alpha$$

If $ds/d\theta$ is put into R in the equation (5),

$$R = \sqrt{f \cdot r \left(1 - \tau \cos\theta\right)} / \sqrt{(1 - \tau \cos\alpha)^2 - (1 - \tau \cos\theta)^2} \tag{6}$$

Therefore the radius of curvature at point O, $\theta = 0$,

$$R = \sqrt{f} \cdot r \left(1 - \tau\right) / \sqrt{(1 - \tau \cos \alpha)^2 - (1 - \tau)^2}$$

Then the loading direction is parallel to the x-axis, $\alpha = \pi/2$, and the theoretical radius of curvature at point O in this case,

$$R = \sqrt{f \cdot r (1 - \tau)} / \sqrt{2\tau - \tau^2}$$
(7)

If the radius of curvature equals to the diameter of the warp or fill, the limitation of loading must be known, then

R = 2r

Therefore,

$$2\sqrt{2\tau - \tau^2} = \sqrt{f(1 - \tau)} \tag{8}$$

Solving the equation (8),

$$\tau = 4 + f \pm 2\sqrt{4 + f} / (4 + f) \tag{9}$$

Only the plus in the above equation can be taken by the condition $\tau \leq 1$, then

$$\tau = 4 + f + 2\sqrt{4+f}/(4+f) \tag{9'}$$

Next, the length of OP along the deformation curve of the centeral line of the filament, S, the equivalent length of S under unloading, S_0 , and x, y are shown:—

$$S = \int_{0}^{s} ds = \sqrt{f} \cdot r \int_{0}^{\pi/2} (1 - \tau \cos\theta) / \sqrt{F(\theta)} \cdot d\theta$$
(10)
$$S_{0} = \int_{0}^{s} 1 / (1 - \lambda \cos\theta) \, ds = \sqrt{f} \, r \Big\{ (1 + \nu/\lambda) \int_{0}^{\pi/2} 1 / \sqrt{F(\theta)} \cdot d\theta - \nu/\lambda \int_{0}^{\pi/2} 1 / (1 - \lambda \cos\theta) \sqrt{F(\theta)} \cdot d\theta \Big\}$$
(11)

$$x = \int_{0}^{s} \sin\theta ds = \sqrt{fr} \left[\frac{1}{\tau} - \sqrt{F(\theta)} \right]_{0}^{\pi/2} = \sqrt{fr} \cdot \frac{1}{\tau} \sqrt{2\tau - \tau^{2}}$$
(12)
$$y = \int_{0}^{s} \cos\theta ds = \sqrt{fr} \left[\frac{\pi/2}{\tau} \cos\theta (1 - \tau \cos\theta) / \sqrt{F(\theta)} \cdot \frac{d\theta}{\tau} \right]$$
(13)

$$y = \int_0^s \cos\theta ds = \sqrt{f} r \int_0^{\pi/2} \cos\theta (1 - \tau \cos\theta) / \sqrt{F(\theta)} \cdot d\theta$$
(13)

where $F(\theta)$ is $1 - (1 - \tau \cos \theta)^2$

NUMERICAL EXAMPLES

The radius of curvature of the small bent part on the warp or fill in the plain fabrics with acetate yarn 100 deniers is calculated in this paper. The warp is an axis in bending the fill or the fill is an axis in bending the warp. Then, D is diameter of the yarn, d diameter of the fiber and N yarn count.

The yarn is about 53' by the relations between the yarn count and the deniers. T is the twist in turns per inch and F is the twist multiplier. Then,

 $T = F\sqrt{N} \quad (\text{in}^{-1})$

Therefore,

 $h_0 = 25.4/T$ (mm)

where suffix w represents the warp and f the fill. Acording to W. Tsuji's⁵⁾ data and their calculations,

> $E = 400 \text{ kg/mm}^2$, $G = 80 \text{ kg/mm}^2$ d = 0.01 mm, D = 0.13 mm $h_{ow} = 0.735 \text{ mm}$, $h_{of} = 1,000 \text{ mm}$ $I = \pi d^4/64 = 4.91 \times 10^{-10} \text{ mm}$

and

$$f = \sqrt{\frac{(2\sin^2\beta + \cos^2\beta)/E + (2\cos^2\beta + \sin^2\beta)/2G}{(1 + \sin^2\beta)/E + \cos^2\beta/2G}}$$
(g')

$$\tau = w\pi r^3/h_0 \cos\beta \left\{ (2\sin^2\beta + \cos^2\beta)/EI + (2\cos^2\beta + \sin^2\beta)/2GI \right\} \quad (e')$$

Introducing the above data into (g') and the equation (9),

$$f_{w} = 1.436, \qquad f_{f} = 1.459$$

 $\tau_{w} = 0.142, \qquad \tau_{f} = 0.144$

and from the relation (e')

$$\tau_w = 5.979 \times 10^4 \times w_w$$
$$\tau_f = 5.471 \times 10^4 \times w_f$$

Since τ from the equation (9) equals to τ from the equation (e') eidenticaly,

$$w_f = 2.378 \times 10^{-3}$$
 (g)
 $w_f = 2.632 \times 10^{-3}$ (g)

If the yarn has 100 fibers and W is the bending force on the yarn,

$$W_{w} = 2.378 \times 10^{-1} (g)$$

 $W_{f} = 2.632 \times 10^{-1} (g)$

Generally in the case of R = cr, where c is constant,

$$\tau = c^2 + f + c\sqrt{c^2 + f}/(c^2 + f) \tag{9"}$$

The relation between the bending force W and the radius of curvature of the yarn R is represented in Fig. 2 by the same mothod. Next,

$$x = 0.065\sqrt{f} \cdot 1/\tau \cdot \sqrt{2\tau - \tau^2} \tag{12'}$$

The relation between the radius of curvature of the yarn and the x-coordinate

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of the loaing point is shown in Fig. 2 from the equation (12').





DISCUSSION

(1) In this paper the relations between the radius of curvature for the small bent part of the crease and the loads are determined elastically. In this case the herix angle β is assumed to be constant and in case that it is variable J. Prescott⁶) studied with the tension of spring. However this generalized bending theory is very complicated and so the case that it is variable will be discussed in the later paper.

It is very interesting that the bending theory which I. Shibuya⁷) used for the

wave-like plate may be applied to the larger part of the bent fabrics. In fact E. Alexander and others⁸) assumed fibers to be spiral curve and sine curve and measured the crimp of fibers by the tension test. Considering the spiral model for the small part of the bent yarn and the sinsoidal model for the bent fabrics, the relations between the bending loads and the radius of curvature of the bent part may be discussed enough.

The materials of fabrics do not show pure elastical behaviors, but they should behave themselves viscoelastically acording to K. Murakami's experiments. But in his paper the tensile strain is very small, a few percent at maximum, which W.J. Hamburger and others⁹ have also analized likewise, and therefore it may be safely regarded as the elastic region.

Generally, as yarn consists of many filaments, not a filament of the complete circular cross section and the cross section of fiber is complicated, it is difficult to solve the question. According to Meredith's method, ¹⁰) which regards the cross section of fiber as an approximate circle, it is easy to solve the question——in determining the cross section area, m is the mass per unit length which is measured by the cantilever type micro-balance, ρ the apparent density of the fiber, then,

the cross section area of fiber $=\frac{m}{\rho}$

The diameter of fiber is determined by the above relation.

The bending stiffness of yarn equals to the product of the bending stiffness of the fiber and the number of the fibers according to W. Tsuji's experimental results. As he has proved that the theoretical values of the bending stiffness of yarns equal to their experimental values, the theoretical values are taken in this paper. The moment of inertia of the yarn cross section differs by the spaces between fibers and its error seems to come from these spaces.

The calculations of S, S_0 and y are elliptic integrals, and are not important in this paper. The differences between S and S_0 are very small, and it is expected that y is near the value of the radius of curvature of bent yarn.

(2) A single yarn is made of continuous filaments which are considered to lie on right cylindrical surfaces along circular-herical path. The cylindrical surfaces are concentric and are spaced one fiber diameter apart. All fibers on all surface are subjected to the same number of turns about the yarn axis per unit length along that same yarn axis. According to S. Backer's thesis¹¹) from the point of view of the differential geometry, it follows that herix angles formed by the fibers at each concentric surface will vary with the maximum herix angle occuring at the outermost fiber, and zero herix angle at the yarn axis. It is assumed that differences in path length of filaments at different concentric surfaces occur in the spinning or twisting, without the presence of fiber tension or compression. The local herix angle for a given surface is constant for all fibers on that surface, and

(14)

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where, β_r' is the fiber herix angle on the cylindrical surface of radius r and T is the twist per inch of the yarn. In particular, the outer fibers of the yarn form the herix angle β' with the yarn axis, where $\beta + \beta' = \pi/2$, and if D is the yarn diameter,

$$\tan \beta' = \pi DT \tag{15}$$

No. 7

The single yarn is assumed to form a torus when bent around a fill, as illustrated in Fig 3. The radius of the bent warp is designated as a, and the radius of the torus as r. The torus radius, r, equals the sum of the radii of the warp (crown) and the fill (inner).



Fig 3. Schematic diagram of bent yarn.

It has been indicated in equations (14) and (15) that the helix angle $\beta_{r'}$ of a straight singles yarn is independent upon the local radius, r, of the yarn and upon its twist T. This angle, $\beta_{r'}$, is therefore constant for a given r and T at every point along the yarn. However, when the yarn is assumed to be the bent form of Fig 3. the local helix angle is no longer a function of r and T alone in a case where complete freedom of fiber or strand movement exists.

Then O and P are the torus center and a point on a fiber as it twists around the warp. X_1 , X_2 and X_3 are mutually orthogonal axes forming the Cartesian coordinate system. In specular, X_3 is the axis of the torus—i.e., the axis of the filling cross yarn. The circular path of the warp lies in the X_3 plane. The intersection of a plane through X_3 , making an angle θ with X_1 , is indicated in Fig 3. as passing through point P. The circular intersection of the torus and the indicated plane is enlarged on the right side of Fig. 3. J is the yarn axis at the circular section. OJ intersects the circle at H. JQ in the plane of the section is perpendicular to JH. Angle θ lies between the X_1 coordinate and the vector OJ. Angle ϕ lies between vector OJ and vector JP. The ratio between θ and ϕ is assumed to be constant and is designated as λ —i.e.,

 $\psi = \lambda \theta$

(16)

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This simply means that in proceeding along the warp axis through an angle θ about the torus axis, the fiber twists around its yarn axis through a corresponding, ϕ . For each revolution around the torus axis ($\theta=2\pi$), there are λ revolution or turns of the fiber about the warp axis.

The parameter r/a (=g) varies from 1.5 to 2.5, the normal range of diameter variation encountered in practical fabrics. Parameter λ is varied from 0.5 to 1.50. If F is defined as the twist multiplier,

$$T = F \sqrt{N}$$

where T is the twist in turns per inch, and N is the yarn count. If the specific volume of the yarn is taken to be 1.1, so that

$$D = 2a = 1/K\sqrt{N}$$

where K depends upon the yarn system used (K = 28 for cotton, 15.4 for woolen cut, 22.8 for worsted), then

$$\lambda = (r/a) \pi F/K \tag{17}$$

For r/a equal to 2—i.e., where the warp and fill are of equal diameter—the range " λ equals 0.5 to 1.25" represents a range in twist multiplier from "F equals 2.23 to 5.58" in cotton system "1.23 to 3.07" for woolen cut, and "1.38 to 4.54" in the worsted system.

Then k, the local curvature of each point on bent yarn, is determined as following:

$$k = \sqrt{\frac{(\cos\lambda\theta - g)^2 + \lambda^4 + 2\lambda^2\cos\theta(\cos\lambda\theta - g) + 4\lambda^2\sin^2\theta - \lambda^2\sin^2\lambda\theta(\cos\theta - g)^2 / \{(\cos\lambda\theta - g)^2 + \lambda^2\}}{a\left\{(cs\lambda\theta - g)^2 + \lambda^2\right\}}}$$
(18)

 Table 1. Note on the warp and fill

 of acetate varn.

	warp 4.75 0.74		fill 3.50 1.00	
F				
ho (mm)				
β	60°55′		67°45′	
g	2.0	2.5	1.5	2.0
2	1.062	1.338	0.584	0.784

The results of calculations in the examples of table 1 are represented in Fig. 4 and 5, where suffix w and f show the warp and fill. When the curvature of the warp is zero at $\phi=0^{\circ}$, the warp is flat. When the curvature of the fill is minus at $\phi=0^{\circ}$, the fill is bent to the opposite side. The warp and fill have the maximum local curvature at $\phi=45^{\circ}$, and at many degrees the curvatures decrease. There is little difference in the curvature

between the warp and fill from $\psi = 90^{\circ}$ to 180°. The curvature of each point along the warp (crown) is also shown as a function of ϕ in Fig 4 and 5. If the bending loads as the warp (crown) covers the half side of the fill (inner), as illustrated in Fig 3, are known by the relation between the loads and the radius of curvature of the central line of bent yarn, the relations



Fig 4. The curvatures of bent warp yarn of g=2.0 and 2.5.



between the loads and the curvature of each point in bent yarn may be determined. There is little difference in the curvature of g=2.0 of the warp and fill, of g=2.0 and 2.5 of the warp and of g=1.5 and 2.0 of the fill from $\phi = 90^{\circ}$ to 180° but there is a little difference in the curvatures at $\phi = 45^{\circ}$ and yarn is bent with the smallest radius of curvature at this degree. It is shown that both the warp and fill have larger curvatures in the case of smaller g.

CONCLUSION

In this paper the relations between the radius of curvature of the small bent part and the bending loads are determined elastically in the example of acetate yarn (100 den.) Next, the local curvature of each point on bent yarn is determined by the differential geometry.

The conclusion which have been reached can be generalized as follows.

(1) The bending loads of fill is larger than that of warp, since the warp twist multiplier is larger than the fill. The coordinate of loading point

x is about constant in any radius of curvature and there is little difference in the radius of curvature in the warp and fill.

(2) The calculations of S, S_0 and y are elliptic integrals and are not important in this paper. The difference between S and S_0 is every small and it is expected that y is near the value of the radius of curvature of bent yarn.

(3) When the local curvature of the warp is zero at $\phi=0^{\circ}$, the warp is flat. When the local curvature of the fill is minus at $\phi=0^{\circ}$, the fill is bent to the opposite side. The warp and fill have the maximum local curvature at $\phi=45^{\circ}$ and the local curvatures at many degrees decrease. There is little difference in the local curvature between the warp and fill from $\phi=90^{\circ}$ to 180° .

(4) There is a little difference in the local curvature at $\phi=45^{\circ}$ and the yarn is bent with the smallest radius of curvature at this degree, but there is little difference at the other degrees. It is shown that both the warp and fill have larger curvature in the case of smaller g.

Therefore, if the bending loads as the warp (crown) covers the half side of the fill (inner) are known by the relation between the loads and radius of curvature of the central line of bent yarn, the relations between the loads and the local curvature of each point in bent yarn may be determined.

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