Inventory Model with Stochastic Demand for the Lost Sales Case

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An important characteristic of the inventory process generating demands is what happens when the system is out of stock. In this paper, it is evaluated that when the system is out of stock, demands are lost sales.

Lost Sales

Demand is one of the most difficult element to consider in making stock control decisions, so that it has been customary to make the very simple assumptions about demand distributions. For example, the simple problem of the lost sales case was presented by T. C. Fry.[1]

"A retail chain store, with limited storage facilities, sells on the average 10 boxes of dog-biscuit per week. The usual practice is to stock up Monday morning. How many packages should be adopted as the standard Monday morning stock, in order not to lose more than one sale out of a hundred ?"

If each week is begun with r packages in stock, no sales will be lost unless the demand exceeds r during the week. suppose the chance of a demand for x packages obeys a Poisson probability law with parameter λ.

The expected number of the lost sales per week is

$$E=\sum_{x=r}^{\infty}(x-r)p(x) \tag{1}$$

where $p(x)=\dfrac{\lambda^x e^{-\lambda}}{x!}$. If, then, we were to keep records for a large number of weeks, the number of purchasers would not differ much from 10m, nor the number of lost sales from Em. Therefore the proportion of lost sales would be very close to

$$\frac{E}{10}=\frac{1}{10}\sum_{x=r}^{\infty}(x-r)p(x). \tag{2}$$

The problem is to find the smallest value of r for which this expression (2) is less than 0.01, that is,

(3) $$\frac{E}{10}=\sum_{x=r-1}^{\infty} p(x)-\frac{r}{10}\sum_{x=r}^{\infty} p(x)\leq 0.01$$

By actual computation, Fry has shown that the least safety stock, under the condition of the problem, was 16.

If λ is so large that the normal approximation to the Poisson law may be used, then (3) may be solved explicitly for r. Since the sum in (3) is approximately equal to $1-\Phi\left(\frac{r-\lambda}{\sqrt{\lambda}}\right)$, r should be chosen so that $(r-\lambda)/\sqrt{\lambda}=2.326$ or $r=2.326\sqrt{\lambda}+\lambda$.

Let c_1 be carrying cost and c_2 shortage cost. If the total demand x is less than or equal to the inventory $(r\geq x)$, this situation is represented by

$$c_1\sum_{x=0}^{r}(r-x)p(x)\cdots\cdots\cdots\cdots(r\geq x)$$

Where $p(x)$ is the demand distribution. If the demand is greater than inventory $(r<x)$, this situation is represented by

$$c_2\sum_{x=r+1}^{\infty}(x-r)p(x)\cdots\cdots\cdots\cdots(r<x)$$

For this reason, the total expected cost is

(4) $$\Gamma(r)=c_1\sum_{x=0}^{r}(r-x)p(x)+c_2\sum_{x=r+1}^{\infty}(x-r)p(x).$$

It is evident that the minimum of the equation (4) occurs for a value of r^* such that

(5) $$p(x\leq r^*-1)<\frac{c_2}{c_1+c_2}<p(x\leq r^*)$$

Similarly if the random variable x is continuous,

(6) $$\Gamma(r)=c_1\int_0^r (r-x)f(x)\,dx+c_2\int_r^{\infty}(x-r)f(x)dx$$

where $f(x)$ is the demand distribution. we can solve,

$$\int_0^{\infty}(r-x)f(x)dx$$

but it remains to evaluate the expected number of lost sales, that is,

$$\int_r^\infty (x-r)\,f(x)\,dx=\int_r^\infty xf(x)dx-rF(r)$$

where $F(x)$ is the complementary cumulative of $f(x)$.

For the case of $f(x)$ being the normal distribution, Hadley and Whitin[2] have shown that

(7)
$$\begin{aligned}\int_r^\infty x\,f(x)dx &=\int_r^\infty xn(x;\mu,\sigma)dx\\ &=\int_r^\infty \frac{x}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)dx\\ &=\sigma\phi\left(\frac{r-\mu}{\sigma}\right)+\mu\Phi\left(\frac{r-\mu}{\sigma}\right)\end{aligned}$$

where $n(x;\mu,\sigma)$ is the normal density function with mean μ and variance σ^2, and

$$\phi(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}$$

$$\Phi(z)=\int_z^\infty \phi(x)dx$$

Similarly for the Poisson distribution they have shown that

(8)
$$\begin{aligned}\sum_{x=r+1}^{\infty}(x-r)\,p(x;\lambda)&=\sum_{x=0}^{\infty}xp(r+x;\lambda)\\ &=\lambda P(r-1;\lambda)-rP(r;\lambda)\end{aligned}$$

where
$$p(x;\lambda)=\frac{\lambda^x}{x!}e^{-\lambda}$$

and
$$P(r)=\sum_{x=r}^{\infty}p(x)$$

We often encount these two distribution in inventory analysis, therefore these identities are particularly useful for many models since most of them have terms such type that

$$\int_r^\infty xf(x)dx$$

or

$$\sum_{x=r}^{\infty} xp(x)$$

Safety Factor

One of the basic inventory analysis problem can be answered with an estimate of the maximum reasonable demand during a lead time. By Brown one way of estimating the maximum reasonable demand is to estimate the expected demand and then to add an allowance for protection against the uncertainty inherent in any forecast. The allowance is the product of the safety factor and the standard deviation of the errors in forecasting over a lead time.

The safety factor is the control element that reflects what we consider reasonable in protecting service to our customers. In other words, the safety factor is a kind of barometer whether our customers are satistied with service or not.

By the above statement, the service function is defined by

(9) $$f(\alpha)=\frac{1}{\sigma}(1-p)$$

where σ is the standard deviation of the forecast errors and p is the desired service rate. Furthermore, if $p(x)$ may be supposed to be the Poisson distribution function, we can derive $f(\alpha)$ from (8), that is,

(10) $$f(\alpha)=\frac{1}{\lambda}\sum_{x=\alpha+1}^{\infty}(x-\alpha)\ p(x)$$

$$=P(\alpha-1\ ;\ \lambda)-\frac{\alpha}{\lambda}P(\alpha;\lambda)$$

where $f(\alpha)$ is considered in the long run and we can define α as a safety factor.

Suppose that the standard deviation of the forecast errors is equivalent to $\lambda=10$ and also that desired service rate is $p=0.95$. The service function in this case is

$$f(\alpha)=\frac{1}{10}(1-0.95)=0.005$$

We should choose the value of α such that $f(\alpha)=0.005$. We can find the value of the service function is 0.005, which corresponds to a safety factor of $\alpha \doteqdot 16$.

We considered the service function in the long run, but if t is the number of periods between from Table successive reviews of the inventory, the service function will be modified as follows,

$$f(\alpha)=\frac{t}{\sigma}(1-p) \tag{11}$$

If the probability density function is prescribed, we can determine the service function $f(\alpha)$, and we can go a step further to know the rate of lost sales.

Table (The Service Function for a Poisson Distribution of Forecast Errors)

Safety factor α	Service function $f(\alpha)$
8	0.24604
9	0.17932
10	0.12511
11	0.08415
12	0.05310
13	0.03225
14	0.01870
15	0.01035
16	0.00545
17	0.00277
18	0.00134
19	0.00062
20	0.00028

References

1. Fry, T. C., Probability and its Engineering Uses, Van Nostrand, 1928, pp. 229–232
2. Hadley, G. and T. M. Whitin, Analysis of Inventory Systems, prentice-Hall, 1963, p. 167
3. Brown, R. G., Statistical Forecasting for Inventory Control, McGraw-Hill, 1959, pp. 105–127