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ON A CONFORMAL KILLING VECTOR FIELD IN A COMPACT ALMOST KÄHLERIAN MANIFOLD

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Abstract. In this paper, we will prove that in a compact almost Kählerian manifold M^n, any conformal Killing vector field is Killing if $n \geq 4$.

1. Introduction

Let M be an n-dimensional Riemannian manifold. We denote respectively by g_{ij} and ∇_j the metric and the covariant derivative in terms of local coordinates $\{x^i\}$, where Latin indices run over the range $\{1, 2, \cdots, n\}$. A conformal Killing vector field u^i in M is given by

$$\nabla_k u_j + \nabla_j u_k = 2\rho g_{kj}, \tag{1.1}$$

where $u_i = g_{ir}u^r$ and ρ is a scalar function, called the associated scalar of u^i. If ρ vanishes identically, then the vector field is called Killing.

Also, a conformal Killing vector field is Killing in a compact Kählerian manifold [3]. In a Sasakian manifold, any conformal Killing vector field is uniquely decomposed into the summation of Killing and closed conformal Killing [2].

In [1], Y. Ogawa has studied differential operators in a almost Kählerian manifold. Using of the operators of the almost Kählerian manifold, we prove the following theorem:

Theorem. *In a compact almost Kählerian manifold M^n, any conformal Killing vector field $(n \geq 4)$ is Killing.*

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2. Preliminaries

We represent tensors by their components with respect to the natural basis and use the summation convention. For a differential p-form

$$u = \frac{1}{p!}\, u_{i_1\cdots i_p}\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

with skew symmetric coefficients $u_{i_1\cdots i_p}$, the coefficients of its exterior differential du and the exterior codifferential δu are given by

$$(du)_{i_1\cdots i_{p+1}} = \sum_{a=1}^{p+1}(-1)^{a+1}\nabla_{i_a} u_{i_1\cdots \widehat{i_a}\cdots i_{p+1}} \text{ and } (\delta u)_{i_2\cdots i_p} = -\nabla^h u_{hi_2\cdots i_p},$$

where $\nabla^h = g^{hj}\nabla_j$ and $\widehat{i_a}$ means i_a to be deleted.

We consider an almost Hermitian manifold M^n $(n = 2m)$ with positive definite metric g_{ji} and almost complex structure $\phi_j{}^i$. We put $\phi_{ji} = \phi_j{}^r g_{ri}$. An almost Hermitian manifold is called almost Kählerian if the 2-form ϕ_{ji} is closed. We want to recall some operators for differential forms in the almost Kählerian manifold. Denote by $\mathcal{F}^p$ the set of all p-forms. The operators $\Gamma, \gamma : \mathcal{F}^p \to \mathcal{F}^{p+1}$, $C, c, \vartheta : \mathcal{F}^p \to \mathcal{F}^{p-1}$ and $\Phi : \mathcal{F}^p \to \mathcal{F}^p$ are defined respectively by

$$(\Gamma u)_{i_0\cdots i_p} = \sum_{a=0}^{p}(-1)^a \phi_{i_a}{}^r \nabla_r u_{i_0\cdots \widehat{i_a}\cdots i_p},$$

$$(\gamma u)_{i_0\cdots i_p} = \sum_{a\neq b}(-1)^a \nabla_{i_a}\phi_{i_b}{}^r \cdot u_{i_0\cdots \widehat{i_a}\cdots r\cdots i_p},$$

$$(Cu)_{i_2\cdots i_p} = \phi^{rs}\nabla_r u_{si_2\cdots i_p}, (cu)_{i_2\cdots i_p} = \sum_{a=2}^{p} \nabla^r \phi^s{}_{i_a} \cdot u_{ri_2\cdots s\cdots i_p},$$

$$(\vartheta u)_{i_2\cdots i_p} = \sum_{a=2}^{p} \phi_{i_a}{}^r \nabla^s \phi_r{}^t \cdot u_{ti_2\cdots s\cdots i_p},$$

$$(\Phi u)_{i_1\cdots i_p} = \sum_{a=1}^{p} \phi_{i_a}{}^r u_{i_1\cdots r\cdots i_p}$$

for any p-form u, where we put $\phi^{ji} = g^{rj}\phi_r{}^i$. For any 0-form u_0 and 1-form u_1, we define $\gamma u_0 = Cu_0 = cu_0 = \vartheta u_0 = \Phi u_0 = 0$ and $cu_1 = \vartheta u_1 = 0$. In the almost Kählerian manifold, we know $*\Gamma* = -C, *\gamma* = -c$ and $*\Phi* = (-1)^p\Phi$ for any p-form, where $*$ means the dual mapping [1].

We denote by L (resp. Λ) the exterior (resp. interior) product with the associated 2-form ϕ, then the operators $L : \mathcal{F}^p \to \mathcal{F}^{p+2}$ and Λ :

$\mathcal{F}^p \to \mathcal{F}^{p-2}$ are written by $Lu = \phi \wedge u$ and $\Lambda u = (-1)^p * L * u$ for any p-form u. Λ is trivial on 0 and 1-forms. These local expressions are defined by

$$(Lu)_{kji_1\cdots i_p} = \phi_{kj}u_{i_1\cdots i_p} - \sum_{a=1}^{p} \phi_{i_a j}u_{i_1\cdots k\cdots i_p} - \sum_{b=1}^{p} \phi_{ki_b}u_{i_1\cdots j\cdots i_p} + \sum_{a<b} \phi_{i_a i_b}u_{i_1\cdots k\cdots j\cdots i_p},$$

$$(\Lambda u)_{i_3\cdots i_p} = \frac{1}{2}\phi^{rs}u_{rsi_3\cdots i_p}.$$

For the operators above, we find from [1]:

(2.1) $(d\Lambda - \Lambda d)u = -(C + c)u,$ (2.2) $(dL - Ld)u = 0,$
(2.3) $(\Gamma\Lambda - \Lambda\Gamma)u = (\delta - \vartheta)u,$ (2.4) $(\gamma\Lambda - \Lambda\gamma)u = \vartheta u,$
(2.5) $(\Lambda L - L\Lambda)u = (m - p)u,$ (2.6) $(\delta L - L\delta)u = (\Gamma + \gamma)u.$

Moreover in a compact almost Kählerian manifold, it follows from [1] that for a p-form u and a $(p+1)$-form v

(2.7) $(\Gamma u, v) = (u, Cv),$ (2.8) $(\gamma u, v) = (u, cv),$

where $(\ ,\ )$ denotes the global inner product.

3. Proof of theorem

From (1.1), we find $\delta u = -n\,\rho$. Operating $\phi_h^{\ k}$ to (1.1), we obtain $(\Gamma u)_{hj} - (d\Phi u)_{hj} + (\gamma u)_{hj} = 4(L\rho)_{hj}$. We will use the similar arrangements of indices without any special notice. This equation may be written as follows:

$$n(\Gamma u - d\Phi u + \gamma u) + 4L\delta u = 0. \tag{3.1}$$

If we operate Λd to (3.1) and regard to (2.1)$\sim$(2.5), then we have

$$(n-4)d\delta u + n(C\Gamma u + c\Gamma u + C\gamma u + c\gamma u) = 0,$$

which denotes that $(n-4)(\delta u, \delta u) + n(\Gamma u + \gamma u, \Gamma u + \gamma u) = 0$ from (2.7) and (2.8). Thus we find $\delta u = 0$ $(n \geq 6)$ and

$$\Gamma u + \gamma u = 0 \ (n \geq 4). \tag{3.2}$$

Substituting (3.2) into (3.1) and owing to (2.6) and (3.2), we get $nd\Phi u - 4\delta Lu = 0$. Applying δ to this, we find $\delta d\Phi u = 0$, namely $d\Phi u = 0$ $(n \geq 4)$. From (3.1) we obtain $L\delta u = 0$, which means that $\delta u = 0$ $(n \geq 4)$, that is $\rho = 0$ $(n \geq 4)$. Consequently, we complete the proof of Theorem.

References

[1] Y. Ogawa, *Operators on almost Kählerian spaces*, Natur. Sci. Rep. Ochanomizu Univ. **21** (1970), 1–17.

[2] M. Okumura, *On infinitesimal conformal and projective transformations of normal contact spaces*, Tohoku Math. J. **14** (1962), 398–412.

[3] Y. Tashiro, *On conformal and projective transformations in Kählerian manifolds*, Tohoku Math. J. **14** (1962), 317–320.

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