THE TOPOLOGICAL TYPES OF HYPERSURFACE SIMPLE $K3$ SINGULARITIES

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Abstract

We give a result that relates the diffeomorphism type of the link of a non-degenerate semi-quasi-homogeneous hypersurface simple $K3$ singularity with the singularities of the normal $K3$ surface that appears as the exceptional divisor of the resolution of the singularity. As a result, we show that the links are diffeomorphic to the connected sum of copies of $S^2 \times S^3$. Moreover, we also show that the topological types of hypersurface simple $K3$ singularities defined by non-degenerate semi-quasi-homogeneous polynomials are all different.

1 Introduction

Let $f = f(z_1, \dots, z_n)$ be a polynomial defining an isolated singularity at the origin of $\mathbb{C}^n$. The intersection

$$L := f^{-1}(0) \cap S_\epsilon^{2n-1}$$

of the hypersurface $f^{-1}(0)$ and a small $(2n-1)$-sphere S_ϵ^{2n-1} with the center at the origin is a closed spin $(2n-3)$-manifold, which is called the *link* of the singularity. The homeomorphism type of the embedding $L \hookrightarrow S_\epsilon^{2n-1}$ determines the topological type of the isolated hypersurface singularity (see Theorem 2.3).

The *simple* $K3$ *singularity* was defined in Ishii-Watanabe [8] as a Gorenstein purely elliptic singularity of type $(0, 2)$, which is a three-dimensional analogue of the simple elliptic singularity in dimension 2. Its geometric characterization was also given in [8] as follows:

Definition 1.1. A three-dimensional normal isolated singularity (X, x) is called a *simple* $K3$ *singularity* if the exceptional divisor of a $\mathbb{Q}$-factorial terminal modification is an irreducible normal $K3$ surface, where a *normal* $K3$ *surface* means a normal surface whose resolution is a $K3$ surface.

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A normal $K3$ surface has only rational double points as its singularities from Artin [1, 2]. Moreover, Shimada [21] determined all possible configurations of rational double points on normal $K3$ surfaces.

Boyer, Galicki and Matzeu showed in [3] that the links of hypersurface simple $K3$ singularities defined by non-degenerate quasi-homogeneous polynomials are all diffeomorphic to some connected sum of $S^2 \times S^3$ by using Sasakian structures. In this paper, more generally, we investigate the topological types of hypersurface simple $K3$ singularities defined by non-degenerate semi-quasi-homogeneous polynomials.

First we focus on the links of hypersurface simple $K3$ singularities defined by non-degenerate semi-quasi-homogeneous polynomials. It is known that the link of a three-dimensional hypersurface isolated singularity is a simply connected closed spin C^∞-manifold of dimension 5. Due to Smale's result, its diffeomorphism type is determined by the second homology group $H_2(M)$, where every (co)homology group is a (co)homology group with integer coefficients unless otherwise stated (see Theorem 3.1).

Let $f(z) = \sum_k a_k z^k$ be a polynomial in $\mathbb{C}[z_1, \ldots, z_n]$, where $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n_{\geq 0}$. Then the Newton diagram $\Gamma_+(f)$ of f is the convex hull of $\bigcup_{a_k \neq 0}(k + \mathbb{R}^n_{\geq 0})$ in $\mathbb{R}^n_{\geq 0}$ and the Newton boundary $\Gamma(f)$ of f is the union of the compact faces of $\Gamma_+(f)$. For a face Δ of $\Gamma(f)$, we put

$$f_\Delta(z) := \sum_{k \in \Delta} a_k z^k.$$

We say that the polynomial f is *non-degenerate* if

$$\partial f_\Delta / \partial z_1 = \cdots = \partial f_\Delta / \partial z_n = 0$$

has no solutions in $(\mathbb{C} \backslash \{0\})^n$ for any face Δ of $\Gamma(f)$. We say that a hypersurface singularity defined by f at the origin is *non-degenerate* if f is a non-degenerate polynomial.

The non-degenerate hypersurface simple $K3$ singularities are classified as follows:

Theorem 1.2 (Watanabe [25]). *Let $f = \sum a_k z^k \in \mathbb{C}[z_1, \ldots, z_4]$ be a non-degenerate polynomial defining an isolated singularity at the origin of $\mathbb{C}^4$. Then the singularity is a simple $K3$ singularity if and only if $\Gamma(f)$ contains $(1,1,1,1)$ and the face $\Delta_0(f)$ of $\Gamma(f)$ containing $(1,1,1,1)$ in its relative interior is of dimension 3.*

Definition 1.3. Let $f \in \mathbb{C}[z_1, \ldots, z_4]$ be a non-degenerate polynomial defining a simple $K3$ singularity at the origin, and let $\Delta_0(f)$ be the face of $\Gamma(f)$ containing $(1,1,1,1)$ in its relative interior. Then the *weight-vector* $\alpha(f)$ of f is the vector $\alpha(f) = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{Q}^4_{>0}$ with $\sum \alpha_i = 1$ such that the 3-dimensional polygon $\Delta_0(f)$ is perpendicular to $\alpha(f)$ in $\mathbb{R}^4$.

Yonemura [26], and independently Fletcher [5], classified all vectors $\alpha \in \mathbb{Q}^4_{>0}$ that appear as the weight-vector $\alpha(f)$ of a non-degenerate polynomial f defining a hypersurface simple $K3$ singularity, and made the famous list of ninety-five weight-vectors, which is also called Reid's 95 examples. They also provide a non-degenerate quasi-homogeneous polynomial defining a hypersurface simple $K3$ singularity for each weight-vector in the list. *Throughout this paper, we use the numbering of the weight-vectors given in Yonemura's list* [26, Table 2.2].

DEFINITION 1.4. Let (X, x) be a hypersurface simple $K3$ singularity defined by a polynomial $f = \sum a_k z^k$. We say that f is *semi-quasi-homogeneous* if $f_{\Delta_0} = 0$ defines an isolated singularity at the origin, where

$$f_{\Delta_0} := \sum_{k \in \Delta_0(f)} a_k z^k$$

is the *principal part* of f.

Moreover, we have the following result, proved by Tomari [23] (see also [26, Theorem 3.1]) and Yonemura [26, Theorem 4.2]:

THEOREM 1.5. *Let (X, x) be a hypersurface simple $K3$ singularity defined by a non-degenerate semi-quasi-homogeneous polynomial of weight-vector $\alpha = (p_1/p, \ldots, p_4/p)$, where $p, p_1, \ldots, p_4$ are positive integers such that* $\gcd(p_1, \ldots, p_4) = 1$. *Then (X, x) has a unique minimal resolution $\pi' : (X', K') \to (X, x)$, which is given by the weighted blow-up of $\mathbb{C}^4$ with weight $(p_1, \ldots, p_4)$. The exceptional divisor K' is a normal $K3$ surface with only rational double points of type A_l, and the ADE-type $R_{K'}$ of* $\mathrm{Sing}(K')$ *is determined by α, where the ADE*-type *is a finite formal sum of symbols A_l $(l \geq 1)$, D_m $(m \geq 4)$ and E_n $(n = 6, 7, 8)$ with non-negative integer coefficients.*

There is a list of $\mathrm{Sing}(K')$ for 95 weight-vectors in [26]. The following are well-defined:

$$R(\alpha) := R_{K'} := \sum a_l A_l + \sum d_m D_m + \sum e_n E_n.$$

$$r(\alpha) := r(K') := \sum a_l l + \sum d_m m + \sum e_n n,$$

where $r(\alpha)$ is called the *total Milnor number*.

We show that the second homology groups of the links are free when the hypersurface simple $K3$ singularities are defined by non-degenerate and semi-quasi-homogeneous polynomials. In order to calculate the second homology groups of the links in higher dimensions, the monodromy of the Milnor fibration is often used (see also [12] and [9]). However a different method is described in this paper, which uses the information of the normal $K3$ surfaces that appears as the exceptional divisor of the resolution of the singularity. The main result is as follows:

THEOREM 1.6. *The link L of a hypersurface simple $K3$ singularity (X, x) defined by a non-degenerate semi-quasi-homogeneous polynomial of weight-vector α is diffeomorphic to the connected sum of $21 - r(\alpha)$ copies of $S^2 \times S^3$.*

The plan of this paper is as follows. In §2 and in §3, we recall known results of topological types of the hypersurface singularities and Smale's result in [22], respectively. In §4, we prove Theorem 1.6. From Smale's result, the key point of the proof of Theorem 1.6 is to calculate the second homology group $H_2(L, \mathbb{Z})$ of the link L of a hypersurface simple $K3$ singularity (X, x). In §5, we give a partial affirmative answer for Orlik's Conjecture 3.2 stated in [16]. In §6, we show that the topological types of hypersurface simple $K3$

singularities defined by non-degenerate semi-quasi-homogeneous polynomials are different when the weight-vectors are different (see Theorem 6.1). In order to show this, we use Lê Dũng Tráng's result in [10]: the characteristic polynomial of the monodromy of the Milnor fibration is a topological invariant. As a corollary, we give a partial affirmative answer for Saeki's problem stated in [18] for four variables, which is related to Zariski's multiplicity problem [27]: the weight-vectors of non-degenerate semi-quasi-homogeneous polynomials defining simple $K3$ singularities are *topological* invariants (see Corollary 6.3).

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2 Topological types of hypersurface singularities

Let n be an integer ≥ 2, and let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ and $g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be germs of holomorphic functions with isolated critical points at the origin. We put $V_f := f^{-1}(0)$ and $V_g := g^{-1}(0)$.

Definition 2.1. We say that f and g are *topologically equivalent* if there exists a homeomorphism germ $\varphi : (\mathbb{C}^n, 0) \xrightarrow{\sim} (\mathbb{C}^n, 0)$ satisfying $\varphi(V_f) = V_g$.

Let ϵ be a sufficiently small positive real number. We put $D^{2n}_\epsilon := \{z \in \mathbb{C}^n \mid \|z\| \leq \epsilon\}$ and $S^{2n-1}_\epsilon := \partial D^{2n}_\epsilon$. The pair $(S^{2n-1}_\epsilon, S^{2n-1}_\epsilon \cap V_f)$ (or simply $S^{2n-1}_\epsilon \cap V_f$) is called the *link* of the singularity.

Definition 2.2. We say that f and g are *link equivalent* if $(S^{2n-1}_\epsilon, S^{2n-1}_\epsilon \cap V_f)$ is homeomorphic to $(S^{2n-1}_{\epsilon'}, S^{2n-1}_{\epsilon'} \cap V_g)$ for all sufficiently small ϵ and ϵ'.

The link equivalence implies the topological equivalence because $(D^{2n}_\epsilon, D^{2n}_\epsilon \cap V_f)$ is homeomorphic to the cone over the link $(S^{2n-1}_\epsilon, S^{2n-1}_\epsilon \cap V_f)$ (see [12, Theorem 2.10]). The converse was proved by Saeki in [19]. Therefore we have the following:

Theorem 2.3 ([12, 19]). *Two germs f and g of holomorphic functions with isolated critical points are topologically equivalent if and only if f and g are link equivalent.*

Let $h : F \to F$ be the characteristic homeomorphism of the Milnor fiber F of the fibration

$$\phi : S^{2n-1}_\epsilon \setminus (S^{2n-1}_\epsilon \cap V_f) \to S^1$$

associated to f. We denote by

$$\Delta_f(t) := \det(tI - h_*)$$

the characteristic polynomial of the monodromy $h_* : H_{n-1}(F, \mathbb{C}) \to H_{n-1}(F, \mathbb{C})$ on $H_{n-1}(F, \mathbb{C})$. The Milnor number $\mu(f)$ of f is defined by

$$\mu(f) := \dim_{\mathbb{C}} \mathbb{C}[[z_1, \ldots, z_n]]/(\partial f/\partial z_1, \ldots, \partial f/\partial z_n).$$

Then we have the following:

Theorem 2.4 ([10] Theorem 3.3). *If two germs f and g are topologically equivalent, then $\Delta_f(t) = \Delta_g(t)$ and $\mu(f) = \mu(g)$ hold.*

3 Smale's theorem

Let $f \in \mathbb{C}[z_1, \ldots, z_4]$ be a polynomial defining an isolated singularity at the origin of $\mathbb{C}^4$. Then the link

$$K := f^{-1}(0) \cap S^7_\epsilon$$

is a simply connected closed spin C^∞-manifold of dimension 5. By the following result of Smale, the diffeomorphism type of the link K is determined by $H_2(K)$.

THEOREM 3.1 (Smale [22]). *There exists a one-to-one correspondence φ from the set of isomorphism classes of simply connected closed spin C^∞-manifolds of dimension 5 to the set of isomorphism classes of finitely generated abelian groups.*

Let M be a simply connected closed spin C^∞-manifold of dimension 5, and let $H_2(M)$ be $F \oplus T$, where F is the free part and T is the torsion part. Then the correspondence φ is given by $\varphi(M) := F \oplus (1/2)T$, where $T = (1/2)T \oplus (1/2)T$.

As a corollary, we have the following:

COROLLARY 3.2. *Let M be a simply connected closed spin 5-dimensional C^∞-manifold. If $H_2(M)$ is free of rank r, then M is diffeomorphic to the connected sum of r copies of $S^2 \times S^3$.*

4 Proof of Theorem 1.6

4.1 Preparations

Let L be the link of a hypersurface simple $K3$ singularity (X, x) defined by a non-degenerate semi-quasi-homogeneous polynomial f of weight-vector α. In order to show Theorem 1.6, it is enough to show the following proposition according to Smale's classification.

PROPOSITION 4.1. *The second homology group $H_2(L)$ of the link L is a free group of the rank $21 - r(\alpha)$.*

Further the following proposition also holds:

PROPOSITION 4.2. *Let (X_1, x_1) and (X_2, x_2) be hypersurface simple $K3$ singularities defined by non-degenerate semi-quasi-homogeneous polynomials f_1 and f_2, respectively, of the same weight-vector $\alpha := \alpha(f_1) = \alpha(f_2)$. If Proposition 4.1 holds for (X_1, x_1), then Proposition 4.1 holds for (X_2, x_2).*

PROOF. We put $\alpha = (\ p_1/p,\ \ p_2/p,\ \ p_3/p,\ \ p_4/p\ )$, where $p, p_1, \ldots, p_4$ are positive integers such that $\gcd(p_1, \ldots, p_4) = 1$. We choose a sufficiently large N, and consider the space $\mathbb{C}[z_1, \ldots, z_4]_N$ of all polynomials of degree $\leq N$. Then there exists a Zariski open dense subset $\mathcal{U}_{\alpha,N}$ of the linear subspace

$$\overline{\mathcal{U}}_{\alpha,N} := \{\ \sum a_k z^k \ \mid\ a_k = 0 \text{ for any } k = (k_1, \ldots, k_4) \text{ with } k \cdot \alpha \leq 1\ \}$$

of $\mathbb{C}[z_1, \ldots, z_4]_N$ such that $\mathcal{U}_{\alpha,N}$ contains both f_1 and f_2, and that, if $g \in \mathcal{U}_{\alpha,N}$, then g is a non-degenerate semi-quasi-homogeneous polynomial defining a hypersurface simple $K3$ singularity of weight-vector α. Consider the universal family

$$\mathcal{X}_{\alpha,N} := \{ \ (x, g) \in \mathbb{C}^4 \times \mathcal{U}_{\alpha,N} \ \mid \ g(x) = 0 \ \} \quad \subset \quad \mathbb{C}^4 \times \mathcal{U}_{\alpha,N}$$

of hypersurface simple $K3$ singularities of weight-vector α defined by polynomials in $\mathcal{U}_{\alpha,N}$. Then we have a simultaneous minimal resolution of these singularities, because, by [26, Theorem 3.1], the weighted blow-up of $\mathbb{C}^4$ with weight $(p_1, \ldots, p_4)$ yields the minimal resolution for each member $X_g := \{g = 0\}$ of the family $\mathcal{X}_{\alpha,N} \to \mathcal{U}_{\alpha,N}$. In particular, the exceptional divisors $K_g^{'}$ of the minimal resolution of X_g form a family over $\mathcal{U}_{\alpha,N}$, and all members are normal $K3$ surfaces with the same type of rational double points by [26, Theorem 4.2]. Since $\mathcal{U}_{\alpha,N}$ is connected, we have the required result. □

Therefore, in proving Proposition 4.1, *we can assume that* (X, x) *is the hypersurface simple* $K3$ *singularity defined by the non-degenerate quasi-homogeneous polynomial of weight-vector* α *given in Yonemura* [26, Table 2.2].

Note the condition that a polynomial contains a term of the form z_i^n or $z_i^n z_j$, in Yonemura's paper [26], which is equal to the necessary conditions for a polynomial having an isolated singularity (see [20, Corollary 1.6]).

Summarizing the above results, it is enough to show the following theorem in order to show Theorem 1.6,

THEOREM 4.3. *Let* L *be the link of a hypersurface simple* $K3$ *singularity* (X, x) *defined by a non-degenerate quasi-homogeneous polynomial of weight-vector* α*. Then the second homology group* $H_2(L)$ *is a free group of the rank* $21 - r(\alpha)$*.*

4.2 Resolutions of (X, x)

The link of a hypersurface simple $K3$ singularity (X, x) defined by a non-degenerate quasi-homogeneous polynomial of weight-vector α is considered as the boundary of the neighborhood of the smooth exceptional divisor in the ambient space of the singularity (X, x). In order to show Theorem 4.3, first we consider the weighted blow-up of $\mathbb{C}^4$ at the origin $0 \in \mathbb{C}^4$ by using the method of toric varieties (see [7], [14], [15]).

Let $p := (p_1, p_2, p_3, p_4)$ be the quadruple of positive integers with $\gcd(p_i, p_j, p_k) = 1$ for all distinct i, j, k. Then the weighted blow-up

$$\Pi : (\ V, \ \mathbb{P}(p_1, p_2, p_3, p_4)\ ) \to (\ \mathbb{C}^4, \ 0\ )$$

with weight p, where $\mathbb{P}(p_1, p_2, p_3, p_4)$ is the weighted projective space, is constructed as follows: Let $N := \mathbb{Z}^4$ and let $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be the dual $\mathbb{Z}$-module of N. A subset σ of $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ is called a *cone* if there exists $n_1, \ldots, n_s \in N$ such that σ is written as

$$\sigma := \{ \sum_{i=1}^{s} t_i n_i \mid t_i \in \mathbb{R}_{\geq 0} \},$$

which we simply denote $\sigma := \langle n_1, \dots, n_s \rangle$ and call $n_1, \dots, n_s$ the generators of σ. For a cone σ in $N_{\mathbb{R}}$, we define the dual cone of σ by

$$\check{\sigma} := \{m \in M_{\mathbb{R}} \mid m(u) \geq 0 \text{ for any } u \in \sigma\},$$

and associate a normal variety

$$X_\sigma := \operatorname{Spec} \mathbb{C}[\check{\sigma} \cap M]$$

with the cone σ, where $\mathbb{C}[\check{\sigma} \cap M]$ is a $\mathbb{C}$-algebra generated by z^m for $m \in \check{\sigma} \cap M$. We assume that the generators $n_1, \dots, n_s$ of a cone σ consist of primitive elements of N, i.e. each n_i satisfied $n_i\mathbb{R} \cap N = n_i\mathbb{Z}$. We define the determinant $\det \sigma$ of a cone σ as the greatest common divisor of all (s, s) minors of the matrix (n_{ij}), where $n_i := (n_{i1}, \dots, n_{i4})$.

Let $\sigma \subset N_{\mathbb{R}} = \mathbb{R}^4$ be the first quadrant of $\mathbb{R}^4$, i.e. $\sigma := \langle e_1, e_2, e_3, e_4 \rangle$ where $e_1 := (1, 0, 0, 0), \dots, e_4 := (0, 0, 0, 1)$. We divide the cone σ into four cones by adding the vector $p = (p_1, p_2, p_3, p_4)$ in σ.

$$\sigma := \bigcup_{i=1}^{4} \sigma_i, \quad \text{where } \sigma_i := (p, e_j, e_k, e_l).$$

From the inclusion $\sigma_i \subset \sigma$, we obtain natural morphisms

$$\Pi_i : V_i \to \operatorname{Spec} \mathbb{C}[\check{\sigma} \cap M] = \mathbb{C}^4,$$

where $V_i := \operatorname{Spec} \mathbb{C}[\check{\sigma}_i \cap M]$. Let

$$V := \bigcup_{i=1}^{4} V_i$$

be the union of V_i which is glued along the images of Π_i. Then we have a morphism

$$\Pi : V \to \mathbb{C}^4,$$

where

$$V - \Pi^{-1}(0) \simeq \mathbb{C}^4 - \{0\} \quad \text{and} \quad \Pi^{-1}(0) = \mathbb{P}(p_1, p_2, p_3, p_4).$$

Let X' be the proper transform of X by Π. Let

$$\pi := \Pi \mid_{X'}$$

and the exceptional set

$$K' := \pi^{-1}(0).$$

Then

$$\pi : (X', K') \to (X, x)$$

is the weighted blow-up with weight p, and K' is a normal $K3$ surface. The minimality of the weighted blow-up and the singularities on normal $K3$ surfaces were stated in Theorem 1.5.

4.3 Resolutions of (X', K')

In this section, we study the resolution of (X', K') because we need *a smooth exceptional divisor $\widetilde{K'}$ in the ambient space of the singularity* to show Theorem 4.3 (see §4.5). Due to Theorem 1.5, it is enough to consider isolated cyclic quotient singularities of dimension 3.

Let $\mathcal{C}_n = \{g\}$ be a cyclic group of order n. The generator g acts on $\mathbb{C}^3$ by

$$g : (z_1, z_2, z_3) \to (\xi z_1, \xi^{q_1} z_2, \xi^{q_2} z_3),$$

where ξ is a primitive nth root of unity and q_1, q_2 are integers satisfying $0 < q_1,\ q_2 < n$ and $\gcd(n, q_1) = \gcd(n, q_2) = 1$. Note that, in this case, we can take the canonical generator $g \in \mathcal{C}_n$ and define the canonical way of the resolution which is minimal (see [6]). Here we express the singularities in terms of toric varities as follows:

$$\mathbb{C}^3/\mathcal{C}_n = \operatorname{Spec} \mathbb{C}[\check{\sigma} \cap \mathbb{Z}^3], \quad \text{where } \sigma = \langle (n, -q_1, -q_2), (0,1,0), (0,0,1) \rangle.$$

We denote this cyclic quotient singularity by $N^n_{q_1,q_2}$.

Consider the case of a nondegenerate hypersurface simple $K3$ singularity with weight-vector $\alpha = (p_1/p, p_2/p, p_3/p, p_4/p)$. From the results of Yonemura [26, Proposition 3.4], the cone is expressed as follows:

$$\sigma = \langle (a, p_k, p_l), (0,1,0), (0,0,1) \rangle,$$

where $a = a_{ij} := \gcd(p_i, p_j) \geq 2$ if $p_i \mid p$ or $a = p_i \geq 2$ if $p_i \nmid p$ for $\{i,j,k,l\} = \{1,2,3,4\}$. Moreover, the condition $a \mid (p_k + p_l)$ for $a = a_{ij}$ or p_i is always satisfied from $p = \sum_{i=1}^4 p_i$ and [26, Proposition 2.3]. Therefore by changing the generator of $\mathcal{C}_a$, we have $(p_k, p_l) \equiv (a-1, 1) \mod a$. The following lemma holds.

Lemma 4.4. *The cone is expressed as*

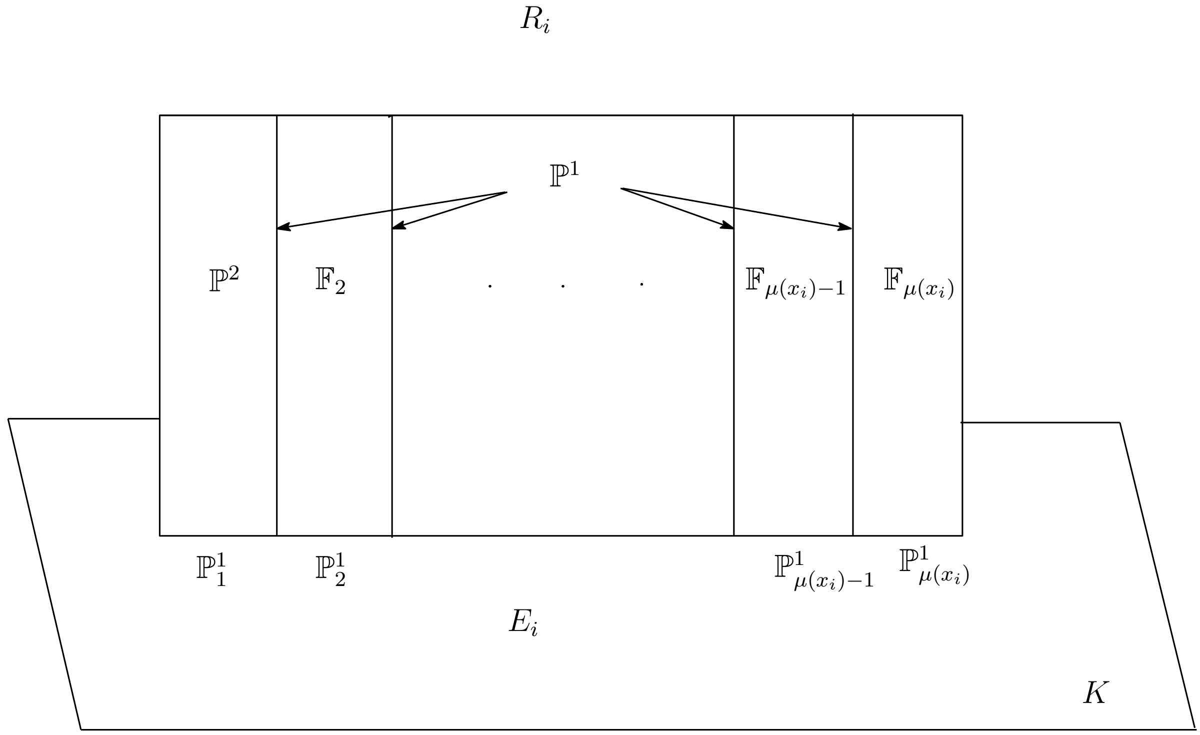
$$\sigma = \langle (a, a-1, 1), (0,1,0), (0,0,1) \rangle,$$

where $a = a_{ij}$ or p_i for $\{i,j\} \subset \{1,2,3,4\}$.

Note that the above cone σ defines the singularity $N^a_{1-a,-1}$. By changing the generator of $\mathcal{C}_a$ again, we have $N^a_{1-a,-1} \cong N^a_{1,a-1}$. It follows from the result of Fujiki [6, Lemma 6] that $N^a_{1,a-1} \cong N^a_{a-1,a-1}$.

Set $\operatorname{Sing}(K') := \{x_1, \ldots, x_n\}$, and the Milnor number of x_i is denoted by $\mu(x_i)$. Then $r(\alpha) = r(K') = \sum_{i=1}^n \mu(x_i)$.

Proposition 4.5. *Let (X', K') be the weighted blow-up of (X, x) with weight p. Let $\widetilde{\pi}' : (\widetilde{X'}, \widetilde{K'}) \to (X', K')$ be a toric resolution of (X', K'). Then $\widetilde{X'}$ is a smooth six-dimensional manifold and $\widetilde{K'} = K \cup R$, where K is a smooth $K3$ surface, $R = \bigcup_{i=1}^n R_i$ is the disjoint union of the union R_i of rational surfaces, and $K \cap R_i$ is a tree of $\mu(x_i)$ rational curves $\mathbb{P}^1$. More precisely, $R_i = \bigcup_{j=1}^{\mu(x_i)} \mathbb{F}_j$, where $\mathbb{F}_j (2 \leq j \leq \mu(x_i))$ is isomorphic to the Hirzebruch surface of degree $j \geq 2$ and $\mathbb{F}_1$ is isomorphic to the projective plane $\mathbb{P}^2$ such that $\mathbb{F}_j$ and $\mathbb{F}_{j+1}$ intersect transversally with $\mathbb{F}_j \cap \mathbb{F}_{j+1} = \mathbb{P}^1$, and no three of the Hirzebruch surfaces intersect.*



PROOF. For each singularity x_i on K', it follows from Lemma 4.4 that it is enough to consider the cone $\sigma = \langle (a, a-1, 1), (0, 1, 0), (0, 0, 1) \rangle$, where $a = \mu(x_i)+1$. By using Oka's method in [15], a simplicial subdivision of σ is obtained by adding new 1-dimensional cones R_λ for $1 \leq \lambda \leq a-1$,

$$R_\lambda := \frac{1}{a-(\lambda-1)}(a, a-1, 1) + \frac{1}{a-(\lambda-1)}(0,1,0) + \frac{a-\lambda}{a-(\lambda-1)}(\lambda-1, \lambda-1, 1) = (\lambda, \lambda, 1),$$

and subdividing by induction. The orbit O_{R_λ} has dimension 2 and the orbit closure $V(R_\lambda)$ is constructed from the cones of σ containing R_λ. From the configuration of the cones of σ, due to the known results on two-dimensional compact non-singular toric varieties in Fulton [7] and Fujiki [6, Corollary to Lemma 6](see also [14] and [24]), we have required results: the orbit closure $V(R_\lambda)$ for $1 \leq \lambda \leq a-2$ is isomorphic to the Hirzebruch surface $\mathbb{F}_j$ of degree $j \geq 2$, and the orbit closure $V(R_{a-1})$ is isomorphic to the projective plane $\mathbb{P}^2$. □

4.4 Homology group $H_*(\widetilde{K'})$

Here we study the homology group $H_*(\widetilde{K'})$, which will be used to show Theorem 4.3.

LEMMA 4.6. *Let $R = \bigcup_{i=1}^n R_i$, where $R_i = \bigcup_{j=1}^{\mu(x_i)} \mathbb{F}_j$ in Proposition 4.5. Then*

$$H_*(R) \cong \begin{cases} \mathbb{Z}^{r(K')} & \text{if } * = 2, 4, \\ \mathbb{Z}^n & \text{if } * = 0, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. First we calculate the homology group $H_*(R_i)$ of R_i. For $j = 1$, we have

$$H_*(\mathbb{F}_1) \cong H_*(\mathbb{P}^2) \cong \begin{cases} \mathbb{Z} & \text{if } * = 0, 2, 4, \\ 0 & \text{otherwise.} \end{cases}$$

For $2 \leq j \leq \mu(x_i)$, we have

$$H_*(\mathbb{F}_j) \cong \begin{cases} \mathbb{Z} & \text{if } * = 0, 4, \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } * = 2, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from the Mayer-Vietoris sequence by induction on the number $\mu(x_i) \geq 2$ that

$$H_*(R_i) \cong H_*(\bigcup_{j=1}^{\mu(x_i)} \mathbb{F}_j) \cong \begin{cases} \mathbb{Z}^{\mu(x_i)} & \text{if } * = 2, 4, \\ \mathbb{Z} & \text{if } * = 0, \\ 0 & \text{otherwise,} \end{cases}$$

which also holds for $\mu(x_i) = 1$. Since $R = \bigcup_{i=1}^n R_i$ and $R_i \cap R_j = \emptyset$ for $i \neq j$, $H_*(R) = H_*(\bigcup_{i=1}^n R_i) \cong \bigoplus_{i=1}^n H_*(R_i)$, which completes the proof. □

LEMMA 4.7.

$$H_*(K \cap R) \cong \begin{cases} \mathbb{Z}^{r(K')} & \textit{if } * = 2, \\ \mathbb{Z}^n & \textit{if } * = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

PROOF. It follows from Theorem 1.5 that $K \cap R = \bigcup_{i=1}^n E_i$, where $E_i = \bigcup_{j=1}^{\mu(x_i)} \mathbb{P}^1_j$ is a tree of $\mu(x_i)$ rational curves $\mathbb{P}^1_j$ and $E_i \cap E_j = \emptyset$ for $i \neq j$. Consider the Mayer-Vietoris sequence by induction on the number $\mu(x_i)$, we have

$$H_*(E_i) \cong \begin{cases} \mathbb{Z}^{\mu(x_i)} & \text{if } * = 2, \\ \mathbb{Z} & \text{if } * = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore we obtain $H_*(K \cap R) = H_*(\bigcup_{i=1}^n E_i) = \bigoplus_{i=1}^n H_*(E_i)$, which completes the proof. □

Then we obtain the homology group $H_*(\widetilde{K'})$ of $\widetilde{K'} = K \cup R$.

PROPOSITION 4.8.

$$H_*(\widetilde{K'}) \cong \begin{cases} \mathbb{Z} & \textit{if } * = 0, \\ \mathbb{Z}^{22} & \textit{if } * = 2, \\ \mathbb{Z}^{r(K')+1} & \textit{if } * = 4, \\ 0 & \textit{otherwise.} \end{cases}$$

PROOF. Since K is a smooth $K3$ surface, we have

$$H_*(K) \cong \begin{cases} \mathbb{Z} & \text{if } * = 0, 4, \\ \mathbb{Z}^{22} & \text{if } * = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the Mayer-Vietoris sequence,

$$\to H_*(K \cap R) \to H_*(K) \oplus H_*(R) \to H_*(\widetilde{K'}) \to .$$

It follows from the above lemmas and the exactness that we have the homology groups $H_*(\widetilde{K'})$. $\square$

4.5 Proof of Theorem 4.3

PROOF. Let $N := N(\widetilde{K'})$ be a smooth neighborhood of $\widetilde{K'}$ in $\widetilde{X'}$. Then $L := \partial N$ is the link of the singularity. From Smale's result, it is enough to show $H_2(L)$ is a free group of the rank $21 - r(K')$. Consider the exact sequence of a pair (N, L)

$$\to H_3(N,L) \to H_2(L) \to H_2(N) \to H_2(N,L) \to H_1(L) \to .$$

Note that L is simply connected. By the Poincaré-Lefschetz duality, the universal coefficient theorem and Proposition 4.8 we have

$$\begin{aligned}
&H_3(N,L) \cong H^3(N) \cong H^3(\widetilde{K'}) \cong \mathrm{Hom}(H_3(\widetilde{K'}),\mathbb{Z}) \oplus \mathrm{Ext}(H_2(\widetilde{K'}),\mathbb{Z}) = 0,\\
&H_2(N) \cong H_2(\widetilde{K'}) \cong \mathbb{Z}^{22},\\
&H_2(N,L) \cong H^4(N) \cong H^4(\widetilde{K'}) \cong \mathrm{Hom}(H_4(\widetilde{K'}),\mathbb{Z}) \oplus \mathrm{Ext}(H_3(\widetilde{K'}),\mathbb{Z}) = \mathbb{Z}^{r(K')+1},\\
&H_1(L) = 0.
\end{aligned}$$

Hence we obtain the required result $H_2(L) \cong \mathbb{Z}^{21-r(K')} = \mathbb{Z}^{21-r(\alpha)}$. $\square$

REMARK 4.9. There exist 3-dimensional hypersurface singularities which are not hypersurface simple $K3$ singularities, but whose links are diffeomorphic to the connected sum of some copies of $S^2 \times S^3$. For example, the link of the hypersurface singularity defined by

$$f = x^2 + y^2 + z^c + w^d, \text{ where } 2 \leq c \leq d,$$

is diffeomorphic to the connected sum of n copies of $S^2 \times S^3$, where $n = \gcd(c,d) - 1$ (see [9]).

5 Orlik's Conjecture

Consider the Milnor fibration as stated in §2. Then the matrix $(tI - h_*)$ is equivalent to a diagonal matrix since the polynomial ring $\mathbb{C}[t]$ is a principal ideal domain. Therefore there exist unimodilar matrices $U(t)$ and $V(t)$ with entries in $\mathbb{C}[t]$ so that

$$U(t)(tI - h_*)V(t) = \mathrm{diag}(m_1(t), \ldots, m_{\mu(f)}(t)),$$

where $m_i(t)$ divides $m_{i+1}(t)$ for $1 \leq i \leq \mu(f)$. The minimal polynomial $m_{\mu(f)}(t)$ contains each irreducible factor of the characteristic polynomial $\Delta_f(t)$. Suppose f is quasi-homogenous. Then Orlik's Conjecture 3.2 for the homology group of the link L in [16] is as follows:

$$H_{n-2}(L;\mathbb{Z}) \cong \mathbb{Z}_{m_1(1)} \oplus \mathbb{Z}_{m_2(1)} \oplus \cdots \oplus \mathbb{Z}_{m_{\mu(f)}(1)},$$

where $\mathbb{Z}_1$ is the trivial group and $\mathbb{Z}_0$ is the infinite cyclic group.

For three variables, the conjecture folds from the result of Orlik and Wagreich in [17]. In general, the characteristic polynomial $\Delta_f(t)$ of the monodromy and the Milnor number $\mu(f)$ are calculated from the weight-vector $\alpha(f)$ by means of the formula of Milnor and Orlik [13]. The results are given in Table 5.1, where Φ_k denotes the kth cyclotomic polynomial.

By using this table, we can calculate $m_i(1)$ for $1 \leq i \leq \mu(f)$ for each quasi-homogeneous defining polynomial in Yonemura's list: for example, in the case No.1 of Table 5.1, we have $\Delta_f(t) = \Phi_1^{21}\Phi_2^{20}\Phi_4^{20}$ and $\mu(f) = 81$. Then $m_i(t) = 1$ for $1 \leq i \leq 60, m_{61}(t) = \Phi_1, m_i(t) = \Phi_1\Phi_2\Phi_4$ for $62 \leq i \leq 81$. Hence we have $m_i(1) = 0$ for $61 \leq i \leq 81$. Together with Theorem 1.6, we have the following corollary.

Corollary 5.1. *Let f be a non-degenerate semi-quasi-homogeneous polynomial defining hypersurface simple $K3$ singularity at the origin. Let L be the associated link of the singularity defined by f. Then $H_2(L;\mathbb{Z}) \cong \mathbb{Z}_{m_1(1)} \oplus \mathbb{Z}_{m_2(1)} \oplus \cdots \oplus \mathbb{Z}_{m_{\mu(f)}(1)}$.*

6 Topological types

From the result of Theorem 1.6, the links of hypersurface simple $K3$ singularities defined by non-degenerate semi-quasi-homogeneous polnomials are diffeomorphic when the total Milnor numbers are the same: for example, the links associated with the polynomials No.1: $x^4+y^4+z^4+w^4$ and No.5: $x^2+y^6+z^6+w^6$ in Yonemura's list are diffeomorphic to the connected sum of 21 copies of $S^2 \times S^3$ because both exceptional divisors are smooth. In this section, we consider topological types of hypersurface simple $K3$ singularities defined by non-degenerate semi-quasi-homogeneous polynomials.

Theorem 6.1. *Let f and g be non-degenerate semi-quasi-homogeneous polynomials defining hypersurface simple $K3$ singularities (X_f, x) and (X_g, x') at the origin of weight-vector $\alpha(f)$ and $\alpha(g)$, respectively. If $\alpha(f) \neq \alpha(g)$, then $\Delta_f(t) \neq \Delta_g(t)$. Moreover, (X_f, x) and (X_g, x') are not topological equivalent.*

For the proof, we use the following lemma, which follows from the result of Lê and Ramanujam [11, Theorem 2.1] via the argument in [4, page 74].

Lemma 6.2. *Let $f = f(z_1, \ldots, z_4)$ be a semi-quasi-homogeneous polynomial defining a simple $K3$ singularity at the origin, and let f_{Δ_0} be the principal part of f. Then f and f_{Δ_0} are topologically equivalent.*

Proof of Theorem 6.1. By Lemma 6.2, we can assume that f and g are non-degenerate and quasi-homogeneous. It follows from Table 5.1 that we have $\Delta_f(t) \neq \Delta_g(t)$ for all 95 weight-vectors in [26]. Together with Theorem 2.4, we have required results. □

As a corollary, we give a partial affirmative answer for Saeki's problem stated in [18]: whether weight-vectors are topological invariants or not.

No.	$\Delta_f(t)$	$\mu(f)$	No.	$\Delta_f(t)$	$\mu(f)$
1	$\Phi_1^{21}\,\Phi_2^{20}\,\Phi_4^{20}$	81	49	$\Phi_1^{8}\,\Phi_2^{6}\,\Phi_3^{8}\,\Phi_6^{7}\,\Phi_7^{8}\,\Phi_{14}^{6}\,\Phi_{21}^{8}\,\Phi_{42}^{6}$	296
2	$\Phi_1^{10}\,\Phi_2^{10}\,\Phi_3^{7}\,\Phi_4^{8}\,\Phi_6^{8}\,\Phi_{12}^{6}$	90	50	$\Phi_1^{13}\,\Phi_2^{12}\,\Phi_3^{13}\,\Phi_5^{13}\,\Phi_6^{13}\,\Phi_{10}^{12}\,\Phi_{15}^{13}\,\Phi_{30}^{12}$	377
3	$\Phi_1^{18}\,\Phi_2^{16}\,\Phi_3^{17}\,\Phi_6^{16}$	100	51	$\Phi_1^{12}\,\Phi_2^{12}\,\Phi_3^{12}\,\Phi_4^{12}\,\Phi_6^{13}\,\Phi_9^{12}\,\Phi_{12}^{12}\,\Phi_{18}^{12}\,\Phi_{36}^{12}$	434
4	$\Phi_1^{12}\,\Phi_2^{10}\,\Phi_3^{12}\,\Phi_4^{10}\,\Phi_6^{11}\,\Phi_{12}^{11}$	132	52	$\Phi_1^{3}\,\Phi_2^{2}\,\Phi_3^{3}\,\Phi_4^{2}\,\Phi_6^{3}\,\Phi_9^{3}\,\Phi_{12}^{3}\,\Phi_{18}^{2}\,\Phi_{36}^{2}$	87
5	$\Phi_1^{21}\,\Phi_2^{20}\,\Phi_3^{21}\,\Phi_6^{21}$	125	53	$\Phi_1^{7}\,\Phi_2^{6}\,\Phi_3^{6}\,\Phi_6^{6}\,\Phi_9^{5}\,\Phi_{18}^{4}$	91
6	$\Phi_1^{16}\,\Phi_2^{12}\,\Phi_5^{16}\,\Phi_{10}^{13}$	144	54	$\Phi_1^{6}\,\Phi_3^{3}\,\Phi_7^{6}\,\Phi_{21}^{4}$	96
7	$\Phi_1^{19}\,\Phi_2^{18}\,\Phi_4^{19}\,\Phi_8^{18}$	147	55	$\Phi_1^{7}\,\Phi_2^{6}\,\Phi_4^{4}\,\Phi_5^{7}\,\Phi_{10}^{7}\,\Phi_{20}^{5}$	117
8	$\Phi_1^{15}\,\Phi_2^{14}\,\Phi_3^{14}\,\Phi_4^{14}\,\Phi_6^{14}\,\Phi_{12}^{13}$	165	56	$\Phi_1^{3}\,\Phi_2^{2}\,\Phi_3^{3}\,\Phi_5^{3}\,\Phi_6^{2}\,\Phi_{10}^{3}\,\Phi_{15}^{4}\,\Phi_{30}^{3}$	95
9	$\Phi_1^{12}\,\Phi_2^{12}\,\Phi_4^{12}\,\Phi_5^{11}\,\Phi_{10}^{12}\,\Phi_{20}^{11}$	228	57	$\Phi_1^{5}\,\Phi_2^{4}\,\Phi_3^{4}\,\Phi_4^{5}\,\Phi_6^{4}\,\Phi_8^{4}\,\Phi_{12}^{5}\,\Phi_{24}^{3}$	95
10	$\Phi_1^{20}\,\Phi_2^{20}\,\Phi_3^{20}\,\Phi_4^{20}\,\Phi_6^{21}\,\Phi_{12}^{20}$	242	58	$\Phi_1^{11}\,\Phi_2^{10}\,\Phi_4^{10}\,\Phi_8^{11}\,\Phi_{16}^{10}$	165
11	$\Phi_1^{10}\,\Phi_2^{8}\,\Phi_3^{9}\,\Phi_5^{10}\,\Phi_6^{8}\,\Phi_{10}^{8}\,\Phi_{15}^{9}\,\Phi_{30}^{7}$	252	59	$\Phi_1^{10}\,\Phi_3^{9}\,\Phi_7^{10}\,\Phi_{21}^{10}$	208
12	$\Phi_1^{16}\,\Phi_2^{14}\,\Phi_3^{16}\,\Phi_6^{15}\,\Phi_9^{16}\,\Phi_{18}^{14}$	272	60	$\Phi_1^{11}\,\Phi_2^{10}\,\Phi_3^{10}\,\Phi_6^{10}\,\Phi_9^{11}\,\Phi_{18}^{10}$	187
13	$\Phi_1^{14}\,\Phi_2^{14}\,\Phi_3^{13}\,\Phi_4^{14}\,\Phi_6^{14}\,\Phi_8^{14}\,\Phi_{12}^{13}\,\Phi_{24}^{13}$	322	61	$\Phi_1^{4}\,\Phi_2^{4}\,\Phi_4^{2}\,\Phi_7^{4}\,\Phi_{14}^{5}\,\Phi_{28}^{3}$	102
14	$\Phi_1^{12}\,\Phi_2^{12}\,\Phi_3^{12}\,\Phi_6^{12}\,\Phi_7^{12}\,\Phi_{14}^{12}\,\Phi_{21}^{12}\,\Phi_{42}^{11}$	492	62	$\Phi_1^{6}\,\Phi_2^{4}\,\Phi_4^{4}\,\Phi_5^{6}\,\Phi_{10}^{5}\,\Phi_{20}^{5}$	102
15	$\Phi_1^{8}\,\Phi_3^{4}\,\Phi_5^{8}\,\Phi_{15}^{5}$	88	63	$\Phi_1^{14}\,\Phi_2^{12}\,\Phi_5^{13}\,\Phi_{10}^{12}$	126
16	$\Phi_1^{6}\,\Phi_2^{6}\,\Phi_3^{3}\,\Phi_4^{6}\,\Phi_6^{4}\,\Phi_8^{6}\,\Phi_{12}^{4}\,\Phi_{24}^{3}$	102	64	$\Phi_1^{5}\,\Phi_2^{4}\,\Phi_3^{5}\,\Phi_4^{5}\,\Phi_6^{5}\,\Phi_8^{4}\,\Phi_{12}^{6}\,\Phi_{24}^{5}$	119
17	$\Phi_1^{8}\,\Phi_3^{8}\,\Phi_5^{6}\,\Phi_{15}^{7}$	104	65	$\Phi_1^{4}\,\Phi_3^{4}\,\Phi_{11}^{4}\,\Phi_{33}^{5}$	152
18	$\Phi_1^{14}\,\Phi_3^{13}\,\Phi_9^{12}$	112	66	$\Phi_1^{18}\,\Phi_7^{17}$	120
19	$\Phi_1^{15}\,\Phi_2^{14}\,\Phi_4^{14}\,\Phi_8^{12}$	105	67	$\Phi_1^{8}\,\Phi_3^{6}\,\Phi_7^{8}\,\Phi_{21}^{7}$	152
20	$\Phi_1^{10}\,\Phi_2^{10}\,\Phi_3^{9}\,\Phi_4^{10}\,\Phi_6^{10}\,\Phi_8^{10}\,\Phi_{12}^{10}\,\Phi_{24}^{9}$	230	68	$\Phi_1^{5}\,\Phi_2^{4}\,\Phi_3^{5}\,\Phi_5^{5}\,\Phi_6^{5}\,\Phi_{10}^{4}\,\Phi_{15}^{6}\,\Phi_{30}^{5}$	153
21	$\Phi_1^{20}\,\Phi_5^{19}$	96	69	$\Phi_1^{9}\,\Phi_2^{8}\,\Phi_4^{8}\,\Phi_8^{9}\,\Phi_{16}^{6}$	117
22	$\Phi_1^{12}\,\Phi_3^{10}\,\Phi_5^{12}\,\Phi_{15}^{11}$	168	70	$\Phi_1^{8}\,\Phi_2^{6}\,\Phi_3^{7}\,\Phi_6^{6}\,\Phi_9^{8}\,\Phi_{18}^{7}$	130
23	$\Phi_1^{11}\,\Phi_2^{10}\,\Phi_3^{11}\,\Phi_4^{6}\,\Phi_6^{11}\,\Phi_{12}^{7}$	105	71	$\Phi_1^{12}\,\Phi_3^{12}\,\Phi_5^{11}\,\Phi_{15}^{12}$	176
24	$\Phi_1^{14}\,\Phi_2^{14}\,\Phi_3^{13}\,\Phi_4^{12}\,\Phi_6^{14}\,\Phi_{12}^{12}$	154	72	$\Phi_1^{14}\,\Phi_3^{13}\,\Phi_5^{14}\,\Phi_{15}^{14}$	208
25	$\Phi_1^{18}\,\Phi_3^{17}\,\Phi_9^{18}$	160	73	$\Phi_1^{3}\,\Phi_2^{2}\,\Phi_5^{3}\,\Phi_{10}^{3}\,\Phi_{25}^{3}\,\Phi_{50}^{2}$	129
26	$\Phi_1^{8}\,\Phi_2^{8}\,\Phi_4^{4}\,\Phi_5^{8}\,\Phi_{10}^{9}\,\Phi_{20}^{5}$	132	74	$\Phi_1^{5}\,\Phi_2^{4}\,\Phi_4^{5}\,\Phi_8^{5}\,\Phi_{16}^{4}\,\Phi_{32}^{4}$	135
27	$\Phi_1^{8}\,\Phi_2^{8}\,\Phi_3^{8}\,\Phi_4^{8}\,\Phi_6^{9}\,\Phi_8^{6}\,\Phi_{12}^{9}\,\Phi_{24}^{7}$	182	75	$\Phi_1^{9}\,\Phi_2^{4}\,\Phi_{11}^{9}\,\Phi_{22}^{5}$	153
28	$\Phi_1^{12}\,\Phi_3^{12}\,\Phi_7^{12}\,\Phi_{21}^{13}$	264	76	$\Phi_1^{8}\,\Phi_2^{4}\,\Phi_{13}^{8}\,\Phi_{26}^{5}$	168
29	$\Phi_1^{6}\,\Phi_2^{4}\,\Phi_3^{6}\,\Phi_5^{5}\,\Phi_6^{4}\,\Phi_{10}^{4}\,\Phi_{15}^{5}\,\Phi_{30}^{3}$	130	77	$\Phi_1^{11}\,\Phi_2^{10}\,\Phi_{13}^{11}\,\Phi_{26}^{11}$	285
30	$\Phi_1^{4}\,\Phi_2^{4}\,\Phi_4^{4}\,\Phi_5^{3}\,\Phi_8^{4}\,\Phi_{10}^{4}\,\Phi_{20}^{3}\,\Phi_{40}^{3}$	132	78	$\Phi_1^{12}\,\Phi_2^{10}\,\Phi_{11}^{12}\,\Phi_{22}^{11}$	252
31	$\Phi_1^{7}\,\Phi_2^{6}\,\Phi_3^{6}\,\Phi_4^{6}\,\Phi_6^{6}\,\Phi_8^{6}\,\Phi_{12}^{5}\,\Phi_{24}^{5}$	133	79	$\Phi_1^{7}\,\Phi_2^{6}\,\Phi_4^{7}\,\Phi_8^{7}\,\Phi_{16}^{7}\,\Phi_{32}^{6}$	207
32	$\Phi_1^{12}\,\Phi_2^{6}\,\Phi_7^{12}\,\Phi_{14}^{7}$	132	80	$\Phi_1^{4}\,\Phi_2^{4}\,\Phi_4^{4}\,\Phi_{11}^{4}\,\Phi_{22}^{5}\,\Phi_{44}^{4}$	186
33	$\Phi_1^{10}\,\Phi_2^{6}\,\Phi_3^{10}\,\Phi_6^{7}\,\Phi_9^{9}\,\Phi_{18}^{6}$	140	81	$\Phi_1^{9}\,\Phi_2^{6}\,\Phi_{13}^{9}\,\Phi_{26}^{7}$	207
34	$\Phi_1^{8}\,\Phi_2^{4}\,\Phi_3^{8}\,\Phi_5^{8}\,\Phi_6^{4}\,\Phi_{10}^{5}\,\Phi_{15}^{8}\,\Phi_{30}^{4}$	184	82	$\Phi_1^{13}\,\Phi_2^{12}\,\Phi_{11}^{13}\,\Phi_{22}^{13}$	285
35	$\Phi_1^{6}\,\Phi_2^{6}\,\Phi_4^{6}\,\Phi_7^{5}\,\Phi_{14}^{6}\,\Phi_{28}^{5}$	150	83	$\Phi_1^{5}\,\Phi_2^{4}\,\Phi_3^{5}\,\Phi_6^{5}\,\Phi_9^{5}\,\Phi_{18}^{4}\,\Phi_{27}^{5}\,\Phi_{54}^{4}$	245
36	$\Phi_1^{9}\,\Phi_2^{8}\,\Phi_4^{8}\,\Phi_5^{8}\,\Phi_{10}^{8}\,\Phi_{20}^{7}$	153	84	$\Phi_1^{4}\,\Phi_3^{3}\,\Phi_9^{4}\,\Phi_{27}^{3}$	88
37	$\Phi_1^{13}\,\Phi_2^{12}\,\Phi_4^{13}\,\Phi_8^{12}\,\Phi_{16}^{12}$	195	85	$\Phi_1^{9}\,\Phi_2^{6}\,\Phi_7^{8}\,\Phi_{14}^{6}$	99
38	$\Phi_1^{11}\,\Phi_2^{10}\,\Phi_3^{11}\,\Phi_5^{11}\,\Phi_6^{10}\,\Phi_{10}^{11}\,\Phi_{15}^{11}\,\Phi_{30}^{10}$	319	86	$\Phi_1^{4}\,\Phi_5^{3}\,\Phi_{25}^{4}$	96
39	$\Phi_1^{13}\,\Phi_2^{12}\,\Phi_3^{13}\,\Phi_6^{13}\,\Phi_9^{12}\,\Phi_{18}^{12}$	221	87	$\Phi_1^{12}\,\Phi_{13}^{11}$	144
40	$\Phi_1^{15}\,\Phi_2^{12}\,\Phi_7^{15}\,\Phi_{14}^{13}$	195	88	$\Phi_1^{6}\,\Phi_3^{5}\,\Phi_9^{6}\,\Phi_{27}^{6}$	160
41	$\Phi_1^{9}\,\Phi_2^{8}\,\Phi_3^{8}\,\Phi_4^{9}\,\Phi_6^{8}\,\Phi_8^{8}\,\Phi_{12}^{8}\,\Phi_{24}^{7}$	187	89	$\Phi_1^{14}\,\Phi_{11}^{13}$	144
42	$\Phi_1^{19}\,\Phi_2^{18}\,\Phi_5^{19}\,\Phi_{10}^{19}$	189	90	$\Phi_1^{5}\,\Phi_2^{2}\,\Phi_{17}^{5}\,\Phi_{34}^{3}$	135
43	$\Phi_1^{6}\,\Phi_2^{6}\,\Phi_3^{6}\,\Phi_4^{6}\,\Phi_6^{7}\,\Phi_9^{5}\,\Phi_{12}^{6}\,\Phi_{18}^{6}\,\Phi_{36}^{5}$	200	91	$\Phi_1^{4}\,\Phi_2^{2}\,\Phi_{19}^{4}\,\Phi_{38}^{3}$	132
44	$\Phi_1^{15}\,\Phi_2^{14}\,\Phi_4^{15}\,\Phi_8^{15}\,\Phi_{16}^{14}$	231	92	$\Phi_1^{5}\,\Phi_2^{4}\,\Phi_{19}^{5}\,\Phi_{38}^{5}$	189
45	$\Phi_1^{12}\,\Phi_2^{12}\,\Phi_4^{12}\,\Phi_7^{12}\,\Phi_{14}^{13}\,\Phi_{28}^{12}$	342	93	$\Phi_1^{6}\,\Phi_2^{4}\,\Phi_{17}^{6}\,\Phi_{34}^{5}$	186
46	$\Phi_1^{4}\,\Phi_2^{4}\,\Phi_3^{4}\,\Phi_6^{4}\,\Phi_{11}^{4}\,\Phi_{22}^{4}\,\Phi_{33}^{4}\,\Phi_{66}^{3}$	244	94	$\Phi_1^{6}\,\Phi_{19}^{5}$	96
47	$\Phi_1^{7}\,\Phi_2^{6}\,\Phi_3^{6}\,\Phi_6^{6}\,\Phi_7^{7}\,\Phi_{14}^{6}\,\Phi_{21}^{6}\,\Phi_{42}^{5}$	247	95	$\Phi_1^{8}\,\Phi_{17}^{7}$	120
48	$\Phi_1^{6}\,\Phi_2^{6}\,\Phi_3^{5}\,\Phi_4^{6}\,\Phi_6^{6}\,\Phi_8^{6}\,\Phi_{12}^{5}\,\Phi_{16}^{6}\,\Phi_{24}^{5}\,\Phi_{48}^{5}$	258			

Table 5.1: Topological invariants

Corollary 6.3. *Let f and g be non-degenerate semi-quasi-homogenous polynomials defining hypersurface simple $K3$ singularities at the origin of weight-vector $\alpha(f)$ and $\alpha(g)$, respectively. If f and g are topologically equivalent, then $\alpha(f) = \alpha(g)$.*

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