Cohomology Algebras of Blocks of Finite Groups and Brauer Correspondence II *

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Abstract. Let k be an algebraically closed field of characteristic p. We shall discuss the cohomology algebras of a block ideal B of the group algebra kG of a finite group G and a block ideal C of the block ideal of kH of a subgroup H of G which are in Brauer correspondence and have a common defect group, continuing [4]. We shall define a (B,C)-bimodule L. The k-dual L^* induces the transfer map between the Hochschild cohomology algebras of B and C, which restricts to the inclusion map of the cohomology algebras of B into that of C under some condition. Moreover the module L induces a kind of refinement of Green correspondence between indecomposable modules lying in the blocks B and C; the block varieties of modules lying in B and C which are in Green correspondence will also be discussed.

Keywords: finite group, block, source modules, Brauer correspondence, Green correspondence, Hochschild cohomology, block cohomology, block variety

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1. Introduction

Throughout this paper we let k be an algebraically closed field of prime characteristic p.

Let G be a finite group of order divisible by p. Proposition 2.3 of Kessar, Linckelmann and Robinson [5] says that the cohomology algebra $H^*(G,B)$ of a block ideal B of the group algebra kG is contained in the cohomology algebra $H^*(H,C)$ of a suitably taken block ideal C, which satisfies $C^G = B$, of a suitably chosen subgroup H of G.

To understand such an inclusion via transfer map between the Hochschild cohomology algebras of the block ideals B and C we discussed in [4] under the following situation. Namely a block ideal B of kG has D as a defect group; H is a subgroup of G and C is a block ideal of kH such that the Brauer correspondent C^G is defined and $C^G = B$ and D is also a defect group of C. We considered the (C,B)-bimodule $M = CB$ and gave a necessary and sufficient condition for M to induce the transfer map from $HH^*(B)$ to $HH^*(C)$ which restricts to the inclusion map of $H^*(G,B)$ into $H^*(H,C)$.

In this paper we shall discuss under the following situation:

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Situation (BC). Let B be a block ideal of kG with a defect group D; let H be a subgroup of G containing $DC_G(D)$ and C a block ideal of kH such that $C^G = B$. Assume that C has D as a defect group.

We shall denote by G^{op} the opposite group of the group G and consider the group algebra kG as a $k[G \times G^{\mathrm{op}}]$-module through $(x, y)\alpha = x\alpha y$ for $x, y \in G$ and $\alpha \in kG$. The stabilizer of $1 \in G$ under the action of $G \times G^{\mathrm{op}}$ on G is the diagonal set $\Delta G = \{(g, g^{-1}) \mid g \in G\}$; hence we have a $k[G \times G^{\mathrm{op}}]$-isomorphism $kG \simeq k[G \times G^{\mathrm{op}}] \otimes_{\Delta G} k$.

Definition 1 Under Situation (BC), because the block ideal C has, as an indecomposable $k[H \times H^{\mathrm{op}}]$-module, ΔD as a vertex and $N_{G\times H^{\mathrm{op}}}(\Delta D) = \Delta N_H(D)(C_G(D)\times 1) \leqslant H\times H^{\mathrm{op}}$, the Green correspondent of C to $G\times H^{\mathrm{op}}$ is defined, which turns out to be a (B, C)-bimodule; we denote it by $L(B, C)$.

The module $L(B, C)$ will play crucial role in this paper, depending on the following fact, which will be proved in Section 2.

Theorem 1.1 *Under Situation* (BC) *let $L = L(B, C)$. The relatively L-projective element $\pi_L \in Z(B)$ and the relatively L^*-projective element $\pi_{L^*} \in Z(C)$ are both invertible. In particular, A being an arbitrary symmetric algebra over k of finite dimension, every finitely generated (B, A)-module is relatively L-projective and every finitely generated (C, A)-module is relatively L^*-projective.*

For the theory of *projectivity relative to bimodules over symmetric algebras*, see Section 5 Appendix, where we shall quote some definitions and results from Broué's lecture notes [3] for the convenience of the readers. We shall also state some facts which we shall use frequently.

Here we fix a symbol. For a subgroup S of a finite group T and a kT-module W we shall write ${}_S W$ for the restriction of W to kS.

Following Alperin, Linckelmann and Rouquier [1], we recall the definition of source modules of block ideals.

Definition 2 ([1, Definition 2]) Since, as an indecomposable $k[G \times G^{\mathrm{op}}]$-module, B has ΔD as a vertex and $G\times D^{\mathrm{op}} \geqslant \Delta D$, there exists an indecomposable direct summand X of the $k[G \times D^{\mathrm{op}}]$-module ${}_{G\times D^{\mathrm{op}}}B$ having ΔD as a vertex. The $k[G \times D^{\mathrm{op}}]$-module X is called a *source module* of the block B. The source module X is written as kGi, where i is a source idempotent. If X and X' are source modules of B, then they are conjugate under $N_G(D)$, namely $X \simeq X' \otimes t$ as $k[G \times D^{\mathrm{op}}]$-module for an element $t \in N_G(D)$.

We shall write $H^*(G, B; X)$ for the block cohomology of B with respect to the defect group D and the source idempotent i such that $X = kGi$.

Under Situation (BC) we can take a source module X of the block B and a source module Y of the block C in order that X and Y are in the Green correspondence with respect to $(G \times D^{\mathrm{op}}, \Delta D, H \times D^{\mathrm{op}})$. Then the (B, C)-bimodule $L = L(B, C)$ links the source modules X and Y in a similar way to induction and restriction of modules (Theorem 2.9). We should mention that the (B, C)-module $L(B, C)$ has already appeared in some works. In particular, in Alperin, Linckelmann and Rouquier [1] the case of $H = N_G(D, b_D)$, where (D, b_D) is a Sylow B-subpair, was treated. Theorem 5 in [1] corresponds to our Theorem 2.9; our proof of Theorem 2.9 is partly due to the argument of the proof of [1, Theorem 5].

One of our ingredients to prove Theorem 1.2 below is that the bimodule L is splendid with respect to X and Y, namely L is a direct summand of the tensor product $X \otimes_{kD} Y^*$ (Theorem 2.10). Another important property is that the relatively projective elements associated with tensor products of the bimodules L, X and Y and their duals, including such as $X^* \otimes_B L \otimes_C Y$, are all invertible (Theorem 2.11).

The following is one of our main theorems.

Theorem 1.2 *Let B be a block ideal of kG and $D \leqslant G$ a defect group of B. Assume that a subgroup H of G containing $DC_G(D)$ normalizes a subgroup Q of D and contains $QC_G(Q)$. Let (D, b_D) be a Sylow B-subpair and let $(Q, b_Q) \leqslant (D, b_D)$. Let C be a unique block ideal of kH covering the block ideal b_Q of $kQC_G(Q)$. Then $C^G = B$ and D is a defect group of C; hence (D, b_D) is also a Sylow C-subpair.*

Let j be a source idempotent of C such that $\mathrm{Br}_D(j)e_D = \mathrm{Br}_D(j)$, where $e_D \in kC_G(D)$ is the block idempotent of the block b_D; let $Y = kHj$. Let X be a source module of B which is the Green correspondent of Y with respect to $(G \times D^{\mathrm{op}}, \Delta D, H \times D^{\mathrm{op}})$. We let $L = L(B, C)$.

Then we have the following commutative diagram:

$$\begin{array}{ccccc} H^*(G,B;X) & \xrightarrow{\ \delta_D\ } & HH^*_{X^*\otimes_B L\otimes_C Y}(kD) & \underset{R_{X^*}}{\overset{R_X}{\rightleftarrows}} & HH^*_{L\otimes_C Y}(B) \\ \big\downarrow & & \big\downarrow & & \big\downarrow {\scriptstyle R_{L^*}} \\ H^*(H,C;Y) & \xrightarrow[\ \delta_D\ ]{} & HH^*_{Y^*}(kD) & \underset{R_{Y^*}}{\overset{R_Y}{\rightleftarrows}} & HH^*_Y(C) \end{array} .$$

The theorem above will be proved in Section 3.

In Section 4 we shall discuss block varieties of modules which are in Green correspondence.

If the cohomology algebra $H^*(G, B; X)$ is contained in the cohomology algebra $H^*(H, C; Y)$, then Kawai and Sasaki [4, Theorem 1.3 (i)] says that the inclusion map $\iota : H^*(G, B; X) \hookrightarrow H^*(H, C; Y)$ induces a surjective map $\iota^* : V_{H,C} \to V_{G,B}$ of varieties; this generalizes to varieties of modules as follows.

Theorem 1.3 *Under Situation* (BC) *let $L = L(B, C)$. Let X and Y be source modules of the blocks B and C, respectively. Assume that X and Y are in the Green correspondence with respect to* $(G \times D^{\mathrm{op}}, \Delta D, H \times D^{\mathrm{op}})$. *We let moreover $P \leqslant D$ and assume that the subgroup H contains $N_G(P)$.*

Assume that $H^*(G, B; X) \subset H^*(H, C; Y)$.

(i) *Assume that an indecomposable B-module U has an X-vertex belonging to $\mathscr{A}(G, P, H)$. Then the Green correspondent V of U to H with respect to (G, P, H) lies in the block C and the following holds:*

$$V_{G,B}(U) = \iota^* V_{H,C}(V).$$

(ii) *Assume that an indecomposable C-module V has a Y-vertex belonging to $\mathscr{A}(G, P, H)$. Then the Green correspondent U of V to G with respect to (G, P, H) lies in the block B and the following holds:*

$$V_{G,B}(U) = \iota^* V_{H,C}(V).$$

Benson and Linckelmann [2] showed that the block variety of an indecomposable kG-module lying in a block is determined by a particular vertex and a source that are *compatible* with a source modules of the block ideal. We have to state more precisely.

Definition 3 ([2, Proposition 2.5]) Let X be a source module of a block ideal B. Let U be an indecomposable kG-module lying in B. There exists a vertex Q of U such that

$$Q \leqslant D, \quad U \,\Big|\, X \otimes_{kQ} X^* \otimes_B U.$$

We would like to call such a vertex Q of U an *X-vertex*. For an X-vertex Q of U we can take a Q-source S such that

$$U \,\Big|\, X \otimes_{kQ} S, \quad S \,\Big|\, {}_Q X^* \otimes_B U.$$

We would like to call such a source a *(Q, X)-source*.

[2, Theorem 1.1] says that the block variety $V_{G,B}(U)$ in the block cohomology $H^*(G, B; X)$ is the pull back of the variety $V_Q(S)$ of S, where Q is an X-vertex and S is a (Q, X)-source of U. Depending on this fact, [2, Corollary 1.4] says that the Green correspondents have the same block varieties for a particular case. We would like to understand this phenomenon from more general context.

Under Situation (BC) and the assumption that the source modules X and Y of the block ideals B and C are in Green correspondence, our bimodule $L = L(B, C)$ gives rise to a kind of refinement of Green correspondence (Proposition 4.1). The theorem above depends on the fact that Green correspondents have a common vertex and a common source compatible with source modules of the block ideals; see Proposition 4.4, which will be proved by using the bimodule L.

2. Brauer Correspondence

First of all we state a couple of lemmas concerning source modules of block ideals. Let $X = kGi$ be a source module of the block B. Let $e_D \in kC_G(D)$ be the block idempotent such that $\mathrm{Br}_D(i)e_D \neq 0$. The following are well known.

Lemma 2.1 *The $k[DC_G(D)\times D^{\mathrm{op}}]$-module $k[DC_G(D)]\,\mathrm{Br}_D(i)$ is a source module of the block ideal $k[DC_G(D)]e_D$.*

Since kGi is a direct summand of $kG\,\mathrm{Br}_D(i)$ as $k[G \times D^{\mathrm{op}}]$-modules, we have

Lemma 2.2 *The source module $X = kGi$ is the Green correspondent of the source module $k[DC_G(D)]\,\mathrm{Br}_D(i)$ of the block $k[DC_G(D)]e_D$ with respect to $(G \times D^{\mathrm{op}}, \Delta D, DC_G(D) \times D^{\mathrm{op}})$.*

In the rest of this section we argue under Situation (BC) and we let $L = L(B,C)$.

Since $N_{G\times D^{\mathrm{op}}}(\Delta D) = \Delta D(C_G(D)\times 1) \leqslant H\times D^{\mathrm{op}}$, the Green correspondence between the indecomposable $k[G \times D^{\mathrm{op}}]$-modules with vertices in $\mathscr{A}(G \times D^{\mathrm{op}}, \Delta D, H\times D^{\mathrm{op}})$ and the indecomposable $k[H\times D^{\mathrm{op}}]$-modules with vertices in $\mathscr{A}(G \times D^{\mathrm{op}}, \Delta D, H \times D^{\mathrm{op}})$ makes sense.

Proposition 2.3 *Let Y be a source module of the block ideal C and let X be the Green correspondent of Y with respect to $(G \times D^{\mathrm{op}}, \Delta D, H \times D^{\mathrm{op}})$. Then the following hold.*

(i) *The $k[G \times D^{\mathrm{op}}]$-module X is a source module of the block $B = C^G$.*

(ii) *The Brauer constructions $X(D)$ and $Y(D)$ are isomorphic as $kC_G(D)$-modules.*

Hence if (D, b_D) is a Sylow C-subpair such that $b_DY(D) \neq 0$, then (D, b_D) is a Sylow B-subpair and $b_DX(D) \neq 0$.

Proof. (i) Because

$$ {}_{H\times D^{\mathrm{op}}}B = {}_{H\times D^{\mathrm{op}}}({}_{H\times H^{\mathrm{op}}}B) = {}_{H\times D^{\mathrm{op}}}(C \oplus C') = Y \oplus \cdots $$

and ${}_{H\times D^{\mathrm{op}}}B = {}_{H\times D^{\mathrm{op}}}({}_{G\times D^{\mathrm{op}}}B)$, there exists an indecomposable direct summand X' of ${}_{G\times D^{\mathrm{op}}}B$ such that $Y \mid {}_{H\times D^{\mathrm{op}}}X'$. Theorem of Burry, Carlson and Puig (Nagao and Tsushima [8, Theorem 4.4.6]) implies that X' is the Green correspondent of Y; therefore it follows that $X \simeq X' \mid_{G\times D^{\mathrm{op}}} B$. Namely X is a source module of B.

(ii) Since $B = C^G = \left(b_D{}^H\right)^G = b_D{}^G$, we see that (D, b_D) is a Sylow B-subpair. Because ${}_{H\times D^{\mathrm{op}}}X \simeq Y \oplus Y_0$, where Y_0 is projective relative to $\mathscr{Y}(G \times D^{\mathrm{op}}, \Delta D, H \times D^{\mathrm{op}})$, the $k[\Delta D]$-module ${}_{\Delta D}Y_0$ is projective relative to $\{\, \Delta Q \mid Q < D \,\}$. Therefore we see that $Y_0(D) = 0$, hence $X(D) \simeq Y(D)$ as $kC_G(D)$-modules.

Proposition 2.4 *Let X be a source module of B and Y a source module of C. If the Brauer constructions $X(D)$ and $Y(D)$ are isomorphic as $kC_G(D)$-modules, then the modules X and Y are in the Green correspondence with respect to $(G \times D^{\mathrm{op}}, \Delta D, H \times D^{\mathrm{op}})$.*

Proof. We write $X = kGi$ and $Y = kHj$, where i and j are source idempotents of B and C, respectively. Let $\widehat{X}(D) = k[DC_G(D)]\,\mathrm{Br}_D(i)$ and $\widehat{Y}(D) = k[DC_H(D)]\,\mathrm{Br}_D(j)$. Then a $kC_G(D)$-isomorphism from $X(D)$ to $Y(D)$ lifts to a $k[DC_G(D)]$-isomorphism from $\widehat{X}(D)$ to $\widehat{Y}(D)$, which is also an isomorphism of $k[DC_G(D) \times D^{\mathrm{op}}]$-modules. We see from Lemma 2.2 that

- the source module X of B is the Green correspondent of $\widehat{X}(D)$ with respect to $(G \times D^{\mathrm{op}}, \Delta D, DC_G(D) \times D^{\mathrm{op}})$,
- the source module Y of C is the Green correspondent of $\widehat{Y}(D)$ with respect to $(H \times D^{\mathrm{op}}, \Delta D, DC_G(D) \times D^{\mathrm{op}})$.

Thus the modules X and Y correspond under the Green correspondence with respect to $(G \times D^{\mathrm{op}}, \Delta D, H \times D^{\mathrm{op}})$.

Lemma 2.5 *The following holds.*

$$B \otimes_{kH} C \equiv L \oplus O(\mathscr{X}(G \times H^{\mathrm{op}}, \Delta D, H \times H^{\mathrm{op}})).$$

In particular, the module L is a (B, C)-bimodule.

Proof. Put $M = B \otimes_{kH} C$. We first show that L is a direct summand of M as $k[G \times H^{\mathrm{op}}]$-modules. Notice that the block ideal C is a direct summand of ${}_{H\times H^{\mathrm{op}}}M$. Let $M = L_1 \oplus \cdots \oplus L_n$ be a direct sum decomposition into indecomposable $k[G \times H^{\mathrm{op}}]$-modules. Then the block ideal C is a direct summand of ${}_{H\times H^{\mathrm{op}}}L_i$ for some L_i. Theorem of Burry, Carlson and Puig says that L_i is the Green correspondent of C. Thus we obtain that $L \simeq L_i$ and that $L \mid M$. This together with the fact that $M \mid k[G \times H^{\mathrm{op}}] \otimes_{k[H\times H^{\mathrm{op}}]} C = L \oplus O(\mathscr{X}(G \times H^{\mathrm{op}}, \Delta D, H \times H^{\mathrm{op}}))$ gives rise to our assertion.

Proposition 2.6 *Since L is isomorphic with a direct summand of ${}_{G\times H^{\mathrm{op}}}B$, we can take a source module X of the block ideal B as a direct summand of ${}_{G\times D^{\mathrm{op}}}L$. Then the Green correspondent Y of X to $H \times D^{\mathrm{op}}$ with respect to $(G \times D^{\mathrm{op}}, \Delta D, H \times D^{\mathrm{op}})$ is a source module of the block ideal C.*

Proof. Let us write ${}_{H\times H^{\mathrm{op}}}L = C \oplus C_0$, where $C_0 \equiv O(\mathscr{Y}(G \times H^{\mathrm{op}}, \Delta D, H \times H^{\mathrm{op}}))$. Then the $k[H \times H^{\mathrm{op}}]$-module C_0 has no indecomposable direct summands with vertex ΔD; so does the $k[H \times D^{\mathrm{op}}]$-module ${}_{H\times D^{\mathrm{op}}}C_0$.

The direct sum decomposition ${}_{H\times D^{\mathrm{op}}}X \mid {}_{H\times D^{\mathrm{op}}}L \simeq {}_{H\times D^{\mathrm{op}}}C \oplus {}_{H\times D^{\mathrm{op}}}C_0$ implies that the $k[H \times D^{\mathrm{op}}]$-module Y is a direct summand of ${}_{H\times D^{\mathrm{op}}}C$; thus the Green correspondent Y is a source module of the block C.

Before proving Theorem 1.1, we prepare the following lemma, which depends on the theory of projectivity relative to bimodules over symmetric algebras. A brief account of the theory is in Appendix.

Lemma 2.7 *Let B be a block ideal of kG and D a defect group of B. Let $H \leqslant G$ and C a block ideal of kH. Assume that D is a defect group of C. Suppose that an indecomposable (B, C)-bimodule S has a trivial source and a vertex which is a proper subgroup of ΔD. Then the relatively projective elements $\pi_S \in Z(B)$ and $\pi_{S^*} \in Z(C)$ are both nilpotent.*

Proof. Let ΔQ, $Q < D$, be a vertex of S. Because S has a trivial source, it follows that

$$S \mid k[G \times H^{\mathrm{op}}] \otimes_{k[\Delta Q]} k = kG \otimes_{kQ} kH.$$

Namely, being $X = {}_BkG_{kQ}$ and $Y = {}_{kQ}kH_C$, we have that $S \mid X \otimes_{kQ} Y$. Then Theorem 5.4 implies that

$$\widehat{\pi}_S \in \mathrm{Im}\,[{}^X\mathrm{Tr} : {}_{kQ}(X^*, X^*) \to {}_B(B, B)],$$
$$\widehat{\pi}_{S^*} \in \mathrm{Im}\,[{}^{Y^*}\mathrm{Tr} : {}_{kQ}(Y, Y) \to {}_C(C, C)].$$

Thus we see that the relatively S-projective element $\pi_S = \widehat{\pi}_S(1) \in Z(B)$ belongs to the image of the relative trace map $\mathrm{tr}_Q^G : B^Q \to Z(B)$ and that the relatively S^*-projective element $\pi_{S^*} = \widehat{\pi}_{S^*}(1) \in Z(C)$ belongs to the image of the relative trace map $\mathrm{tr}_Q^H : C^Q \to Z(C)$. Because the blocks B and C have D as a defect group and $Q < D$, we have that

$$\mathrm{tr}_Q^G(B^Q) \subset J(Z(B)), \quad \mathrm{tr}_Q^H(C^Q) \subset J(Z(C))$$

so that the elements π_S and π_{S^*} are nilpotent.

Proof of Theorem 1.1. Let $M = B \otimes_{kH} C$. Then, by Lemma 2.5, the (B, C)-bimodule M decomposes as follows:

$$M = L \oplus L_0, \quad L_0 \equiv O(\mathscr{X}(G \times H^{\mathrm{op}}, \Delta D, H \times H^{\mathrm{op}})).$$

We know from Kawai and Sasaki [4, Proposition 3.3 (b)] that the relatively projective elements $\pi_M \in Z(B)$ and $\pi_{M^*} \in Z(C)$ are invertible. Every indecomposable direct summand of L_0 has a proper subgroup of ΔD as a vertex and a trivial source; so does every indecomposable direct summand of L_0^*. We see from [6, Proposition 2.12 (iv)] that

$$\pi_M = \pi_L + \pi_{L_0}, \quad \pi_{M^*} = \pi_{L^*} + \pi_{L_0^*}.$$

Since the elements π_{L_0} and $\pi_{L_0^*}$ are nilpotent by Lemma 2.7, we have

$$\pi_L \equiv \pi_M \pmod{J(Z(B))}, \quad \pi_{L^*} \equiv \pi_{M^*} \pmod{J(Z(C))}$$

so that the elements $\pi_L \in Z(B)$ and $\pi_{L^*} \in Z(C)$ are invertible.

The last assertion follows from Theorem 5.6.

Proposition 2.8 (i) *The (B, B)-bimodule B is isomorphic to a direct summand of $L \otimes_C L^*$.*

(ii) *As a $k[H \times H^{\mathrm{op}}]$-module*

$$L^* \otimes_B L \equiv C \oplus O(\mathscr{Y}(G \times H^{\mathrm{op}}, \Delta D, H \times H^{\mathrm{op}})).$$

Proof. Theorems 1.1 and 5.3 imply that the block ideal B is a direct summand of $L \otimes_C L^*$ as a (B, B)-bimodule; the block ideal C is a direct summand of $L^* \otimes_B L$ as a (C, C)-bimodule. The assertion (ii) follows from the fact that

$$L^* \otimes_B L \Big|_{H \times H^{\mathrm{op}}} L \equiv C \oplus O(\mathscr{Y}(G \times H^{\mathrm{op}}, \Delta D, H \times H^{\mathrm{op}})).$$

Our (B, C)-bimodule $L(B, C)$ links source modules of the blocks B and C as follows.

Theorem 2.9 *Assume that a source module X of the block B and a source module Y of the block C are in the Green correspondence with respect to $(G \times D^{\mathrm{op}}, \Delta D, H \times D^{\mathrm{op}})$. Then the following hold.*

(i) $L^* \otimes_B X \equiv Y \oplus O(\mathscr{Y}(G \times D^{\mathrm{op}}, \Delta D, H \times D^{\mathrm{op}}))$.

(ii) $L \otimes_C Y \equiv X \oplus O(\mathscr{X}(G \times D^{\mathrm{op}}, \Delta D, H \times D^{\mathrm{op}}))$.

(iii) *If $D \lhd H$, then $L \otimes_C Y \simeq X$.*

Proof. Let $\mathscr{X} = \mathscr{X}(G \times D^{\mathrm{op}}, \Delta D, H \times D^{\mathrm{op}})$ and $\mathscr{Y} = \mathscr{Y}(G \times D^{\mathrm{op}}, \Delta D, H \times D^{\mathrm{op}})$. Notice that the source module X is isomorphic to a direct summand of the tensor product $L \otimes_C L^* \otimes_B X$ and the source module Y is isomorphic to a direct summand of the tensor product $L^* \otimes_B L \otimes_C Y$ because of Theorems 1.1 and 5.3.

(i) The tensor product $L^* \otimes_B X$ is a direct summand of the restriction ${}_{H\times D^{\mathrm{op}}}X \equiv Y \oplus O(\mathscr{Y})$. If $L^* \otimes_B X$ is relatively $\mathscr{Y}$-projective, then $L \otimes_C L^* \otimes_B X$ is, as a direct summand of $kG \otimes_{kH} L^* \otimes_B X$, relatively $\mathscr{Y}$-projective; hence so is $X \mid L \otimes_C L^* \otimes_B X$, a contradiction. Therefore $L^* \otimes_B X$ is not relatively $\mathscr{Y}$-projective so that the assertion holds.

(ii) The tensor product $L \otimes_C Y$ is a direct summand of the induced module $k[G \times D^{\mathrm{op}}] \otimes_C Y \equiv X \oplus O(\mathscr{X})$. If $L \otimes_C Y$ is relatively $\mathscr{X}$-projective, then $L^* \otimes_B L \otimes_C Y$ is, as a direct summand of the restriction ${}_{H\times D^{\mathrm{op}}}L \otimes_C Y$, relatively $\mathscr{Y}$-projective; hence so is $Y \mid L^* \otimes_B L \otimes_C Y$, a contradiction. Therefore $L \otimes_C Y$ is not relatively $\mathscr{X}$-projective so that the assertion holds.

(iii) From the fact (ii) we can write

$$L \otimes_C Y \simeq X \oplus X_0, \quad X_0 \equiv O(\mathscr{X}).$$

Since L has ΔD as a vertex and a trivial source, there exists an indecomposable direct summand W of the induced module $k[G \times D^{\mathrm{op}}] \otimes_{k\Delta D} k$ with vertex ΔD such that L is a direct summand of the induced module $k[G \times H^{\mathrm{op}}] \otimes_{k[G \times D^{\mathrm{op}}]} W$. Then we obtain

$$\begin{aligned} X \oplus X_0 \simeq L \otimes_{kH} Y &\mid k[G \times H^{\mathrm{op}}] \otimes_{k[G\times D^{\mathrm{op}}]} W \otimes_{kH} Y \\ &\mid W \otimes_{kD} kH_{kD} = \bigoplus_{t \in D\backslash H} {}^{(1,t)}W \end{aligned}$$

This implies that $X_0 = 0$ because the module ${}^{(1,t)}W$ has a vertex conjugate to ΔD.

The (B, C)-bimodule L is splendid with respect to source modules of B and C which are in Green correspondence. Namely

Theorem 2.10 *Let X and Y be as in Theorem 2.9. Then the (B, C)-bimodule $L(B, C)$ is isomorphic with a direct summand of the tensor product $X \otimes_{kD} Y^*$.*

Proof. We argue with the same notation as in the proof of Theorem 2.9. Tensoring Y^* to the both sides of $L \otimes_C Y \simeq X \oplus X_0$ on the right, we obtain

$$L \mid L \otimes_C Y \otimes_{kD} Y^* \simeq X \otimes_{kD} Y^* \oplus X_0 \otimes_{kD} Y^*$$

since $C \mid Y \otimes_{kD} Y^*$. Note that $X_0 \otimes_{kD} Y^*$ is a direct summand of $X_0 \otimes_{kD} kH = k[G \times H^{\mathrm{op}}] \otimes_{k[G\times D^{\mathrm{op}}]} X_0$. If $L \mid X_0 \otimes_{kD} Y^*$, then L is also projective relative to $\mathscr{X}$. However, this contradicts to the fact the module L has ΔD as a vertex. Therefore we conclude that $L \mid X \otimes_{kD} Y^*$.

Relatively projective elements associated with the tensor products of the bimodules L, X and Y and their duals are all invertible if X and Y are in Green correspondence.

Theorem 2.11 *Let X and Y be as in Theorem 2.9.*

(i) *Relatively projective elements* $\pi_{L\otimes_C Y} \in Z(B)$ *and* $\pi_{Y^*\otimes_C L^*} \in Z(kD)$ *are invertible.*

(ii) *Relatively projective elements* $\pi_{X^*\otimes_B L\otimes_C Y} \in Z(kD)$ *and* $\pi_{X^*\otimes_B L} \in Z(kD)$ *are invertible.*

(iii) *Relatively projective elements* $\pi_{Y^*\otimes_C L^*\otimes_B X} \in Z(kD)$ *and* $\pi_{L^*\otimes_B X} \in Z(C)$ *are invertible.*

Proof. Again we argue with the same notation as in the proof of Theorem 2.9.

(i) We see from Lemma 2.7 that the relatively projective elements $\pi_{X_0} \in Z(B)$ and $\pi_{X_0^*} \in Z(kD)$ are nilpotent. Therefore, a similar argument in the proof of Theorem 1.1 shows that t $\pi_{X\oplus X_0}$ and $\pi_{X^*\oplus X_0^*}$ are invertible; hence so are the elements $\pi_{L\otimes_C Y}$ and $\pi_{Y^*\otimes_C L^*}$.

(ii) Let $t_{X^*} : HH^*(B) \to HH^*(kD)$ be the transfer map associated with the (kD, B)-bimodule X^*. Then, since $\pi_{X^*\otimes_B L\otimes_C Y} = t_{X^*}(\pi_{L\otimes_C Y})$ and the element $\pi_{L\otimes_C Y}$ is invertible, Kawai and Sasaki [4, Proposition 3.5] implies that $\pi_{X^*\otimes_B L\otimes_C Y}$ is invertible. We also have that $\pi_{X^*\otimes_B L} = t_{X^*}(\pi_L)$ and the element π_L is invertible; hence again [4, Proposition 3.5] implies that the element $\pi_{X^*\otimes_B L}$ is invertible.

(iii) In the direct sum decomposition $Y^* \otimes_C L^* \otimes_B X \simeq X^* \otimes_B X \oplus X_0^* \otimes_B X$, every indecomposable direct summand of $X_0^* \otimes_B X$ has a proper subgroup of ΔD as a vertex and a trivial source. Therefore the element $\pi_{X_0^*\otimes_B X} \in Z(kD)$ is nilpotent; hence we see that $\pi_{Y^*\otimes_C L^*\otimes_B X} \equiv \pi_{X^*\otimes_B X} \pmod{J(Z(kD))}$ and that the element $\pi_{Y^*\otimes_C L^*\otimes_B X}$ is invertible. Furthermore, since $\pi_{Y^*\otimes_C L^*\otimes_B X} = t_{Y^*}(\pi_{L^*\otimes_B X})$, where $t_{Y^*} : HH^*(kD) \to HH^*(C)$ is the transfer map associated with Y^*, again [4, Proposition 3.5] implies that the element $\pi_{L^*\otimes_B X}$ is invertible.

3. Cohomology Algebras of Block Ideals

We continue our argument under Situation (BC); let $L = L(B, C)$. We assume that a source module X of B and a source module Y of C are in the Green correspondence with respect to $(G \times D^{\mathrm{op}}, \Delta D, H \times D^{\mathrm{op}})$. Let us take a Sylow C-subpair (D, b_D) satisfying $b_D Y(D) \neq 0$. Then Proposition 2.3 says that (D, b_D) is also a Sylow B-subpair and that $b_D X(D) \neq 0$. The cohomology algebras of blocks B and C with respect to the Sylow subpair (D, b_D) satisfy

$$H^*(G, B; X) \overset{\delta_D}{\rightarrowtail} HH^*_{X^*}(kD), \quad H^*(H, C; Y) \overset{\delta_D}{\rightarrowtail} HH^*_{Y^*}(kD).$$

We know from Theorems 1.1 and 2.11 that the relatively projective elements are invertible; hence we obtain from Kawai and Sasaki [4, Proposition 2.2] the following commutative diagram:

$$\begin{array}{ccc} HH^*_{Y^*\otimes_C L^*}(kD) & \underset{R_{Y^*}}{\overset{R_Y}{\rightleftarrows}} & HH^*_{L^*}(C)\cap HH^*_{Y}(C) \\ R_{Y^*\otimes_C L^*}\uparrow\downarrow R_{L\otimes_C Y} & & R_{L^*}\uparrow\downarrow R_L \\ HH^*_{L\otimes_C Y}(B) & = & HH^*_{L\otimes_C Y}(B) \end{array}.$$

The diagram above together with Theorem 2.11, [4, Lemma 4.4] and Linckelmann [6, Corollary 3.8] yields the following.

Lemma 3.1 *The following diagram commutes:*

$$\begin{array}{ccccccc} H^*(G,B;X) & \xrightarrow{\delta_D} & HH^*_{X^*}(kD) & \underset{R_{X^*}}{\overset{R_X}{\rightleftarrows}} & HH^*_X(B) & & \\ & & \uparrow & & \uparrow & & \\ & & HH^*_{Y^*\otimes_C L^*}(kD) & \underset{R_{Y^*\otimes_C L^*}}{\overset{R_{L\otimes_C Y}}{\rightleftarrows}} & HH^*_{L\otimes_C Y}(B) & \hookrightarrow & HH^*_L(B) \\ & & \| & & R_L\downarrow\uparrow R_{L^*} & & R_L\downarrow\uparrow R_{L^*} \\ & & HH^*_{Y^*\otimes_C L^*}(kD) & \underset{R_{Y^*}}{\overset{R_Y}{\rightleftarrows}} & HH^*_{L^*}(C)\cap HH^*_Y(C) & \hookrightarrow & HH^*_{L^*}(C) \\ & & \downarrow & & \downarrow & & \\ H^*(H,C;Y) & \xrightarrow[\delta_D]{} & HH^*_{Y^*}(kD) & \underset{R_{Y^*}}{\overset{R_Y}{\rightleftarrows}} & HH^*_Y(C) & & \end{array}.$$

In particular

Corollary 3.2 *If D is normal in H, then we have the following commutative diagram*

$$\begin{array}{ccccc} H^*(G,B;X) & \xrightarrow{\delta_D} & HH^*_{X^*}(kD) & \xrightarrow{R_X} & HH^*_X(B) \\ \downarrow & & \downarrow & & \downarrow R_{L^*} \\ H^*(H,C;Y) & \xrightarrow[\delta_D]{} & HH^*_{Y^*}(kD) & \xrightarrow[R_Y]{} & HH^*_Y(C) \end{array}.$$

Namely, the inclusion map $H^*(G,B;X) \hookrightarrow H^*(H,C;Y)$ is the restriction of the normalized transfer map R_{L^*}.

Proof. Theorem 2.9 and [5, Proposition 2.3] and Lemma 3.1 gives rise to the diagram.

We generalize this to the case that D is not necessarily normal in H. In Kawai and Sasaki [4] we discussed under the following situation:

> Let $b \in Z(kG)$ and $c \in Z(kH)$ be block idempotents and assume that the Brauer correspondent c^G is defined and $c^G = b$ and moreover that the blocks b and c have a common defect group D. Let $X = ckGb$. Let $Y = kGi$ be a source module of b and let $Z = kHj$ be a source module of c such that
>
> $$Y = kGi \,\Big|\, X^* \otimes_{kGb} Z = bkGj.$$

We considered bimodules

$$\begin{aligned} ckGi &= X \otimes_{kGb} Y, & ikGc &= Y^* \otimes_{kGb} X^*, \\ bkGj &= X^* \otimes_{kHc} Z, & jkGb &= Z^* \otimes_{kHc} X, \\ ikGj &= Y^* \otimes_{kGb} X^* \otimes_{kHc} Z, & jkGi &= Z^* \otimes_{kHc} X \otimes_{kGb} Y. \end{aligned}$$

We showed in [4, Theorem 1.1] that the relatively projective elements associated with the bimodules above are all invertible.

In this paper we take L^* instead of $X = ckGb$ above; we have gotten the corresponding facts to the above in Theorem 2.11; hence the conclusions in [4, Section 4] and [4, Theorem 1.2] hold by replacing $ckGb$ there with our (C, B)-bimodule L^*.

Thus the argument in [4, Proof of Theorem 1.2] yields

Lemma 3.3 *We have the following commutative diagram:*

$$\begin{array}{ccc} HH^*_{X^* \otimes_B L \otimes_C Y}(kD) & \underset{R_{X^*}}{\overset{R_X}{\rightleftarrows}} & HH^*_{L \otimes_C Y}(B) \\ \text{Id}\,\Big\| & & R_L \Big\uparrow\Big\downarrow R_{L^*} \\ HH^*_{Y^* \otimes_C L^* \otimes_B X}(kD) & \underset{R_{Y^*}}{\overset{R_Y}{\rightleftarrows}} & HH^*_Y(C) \cap HH^*_{L^*}(C) \end{array} .$$

We obtain from Lemmas 3.1 and 3.3 that

$$\begin{aligned} HH^*_{Y^* \otimes_C L^* \otimes_B X}(kD) &= HH^*_{X^* \otimes_B L \otimes_C Y}(kD) \\ &= R_{X^*}(HH^*_{L \otimes_C Y}(B)) = HH^*_{Y^* \otimes_C L^*}(kD). \end{aligned}$$

Consequently we have

Proposition 3.4 *Under Situation* (BC) *assume that a source module X of B and a source module Y of C are in the Green correspondence with respect to*

$(G \times D^{\rm op}, \Delta D, H \times D^{\rm op})$. *Then we have the following commutative diagram*

$$
\begin{array}{ccccccc}
H^*(G,B;X) & \xrightarrow{\ \delta_D\ } & HH^*_{X^*}(kD) & \underset{R_{X^*}}{\overset{R_X}{\rightleftarrows}} & HH^*_X(B) & & \\
 & & \uparrow & & \uparrow & & \\
 & & HH^*_{X^*\otimes_B L\otimes_C Y}(kD) & \underset{R_{X^*}}{\overset{R_X}{\rightleftarrows}} & HH^*_{L\otimes_C Y}(B) & \hookrightarrow & HH^*_L(B) \\
 & & \| & & R_L \Updownarrow R_{L^*} & & R_L \Updownarrow R_{L^*} \\
 & & HH^*_{Y^*\otimes_C L^*\otimes_B X}(kD) & \underset{R_{Y^*}}{\overset{R_Y}{\rightleftarrows}} & HH^*_{L^*}(C) \cap HH^*_Y(C) & \hookrightarrow & HH^*_{L^*}(C) \\
 & & \downarrow & & \downarrow & & \\
H^*(H,C;Y) & \xrightarrow[\ \delta_D\ ]{} & HH^*_{Y^*}(kD) & \underset{R_{Y^*}}{\overset{R_Y}{\rightleftarrows}} & HH^*_Y(C) & &
\end{array}
.
$$

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. We know from Theorem 2.10 that $L \mid X \otimes_{kD} Y^* = kGi \otimes_{kD} jkH$. For a subgroup $R \leqslant D$ we let $(R, b_R) \leqslant (D, b_D)$ be the B-subpair and let $(R, c_R) \leqslant (D, b_D)$ be the C-subpair. Then Kessar, Linckelmann and Robinson [5, Proposition 2.3] implies for $R \leqslant D$ that

$$E_H((R, c_R), (D, b_D)) \subseteq E_G((R, b_R), (D, b_D)).$$

Therefore we have from Linckelmann [6, Theorem 5.7] that $H^*(G, B; X) \subseteq H^*(H, C; Y)$ and that $\delta_D H^*(G, B; X) \subseteq HH^*_{X^*\otimes_B L\otimes_C Y}(kD)$. Then [4, Proof of Theorem 1.2] gives the left rectangle of the diagram; we get from the diagram just before the theorem the right rectangle of the diagram.

4. Green Correspondence and Block Varieties of Modules

In this section we investigate relationship between the Green correspondence of indecomposable modules and the Brauer correspondence of blocks.

We argue under Situation (BC); let $L = L(B, C)$. We let moreover $P \leqslant D$ and assume that the subgroup H contains $N_G(P)$.

Proposition 4.1 *Assume that an indecomposable B-module U and an indecomposable C-module V have vertices in $\mathscr{A}(G, P, H)$ and are in the Green correspondence with respect to (G, P, H). Then the followings hold:*

$$L \otimes_C V \equiv U \oplus O(\mathscr{X}(G, P, H)), \quad L^* \otimes_B U \equiv V \oplus O(\mathscr{Y}(G, P, H)).$$

Proof. Since the module V lies in the block C, we have

$$L \otimes_C V \,\Big|\, kG \otimes_{kH} V \equiv U \oplus O(\mathscr{X}(G, P, H)).$$

If $L \otimes_C V$ is relatively $\mathscr{X}(G, P, H)$-projective, then we would have

$$V \,\Big|\, L^* \otimes_B L \otimes_C V \,\Big|\, {}_H L \otimes_C V \equiv O(\mathscr{Y}(G, P, H)),$$

a contradiction. Thus $L \otimes_C V$ is not relatively $\mathscr{X}(G, P, H)$-projective; hence we have that $L \otimes_C V \equiv U \oplus O(\mathscr{X}(G, P, H))$. Let us write

$$L \otimes_C V \simeq U \oplus U_0, \quad U_0 \equiv O(\mathscr{X}(G, P, H)).$$

Tensor with $L^* \otimes_B -$ to obtain

$$V \,\Big|\, L^* \otimes_B L \otimes_C V \simeq L^* \otimes_B U \oplus L^* \otimes_B U_0.$$

Because $L^* \otimes_B U_0 \,\big|\, {}_H U_0 \equiv O(\mathscr{Y}(G, P, H))$, we see that $V \,\big|\, L^* \otimes_B U$. This together with the fact that $L^* \otimes_B U \,\big|\, {}_H U \equiv V \oplus O(\mathscr{Y}(G, P, H))$ implies that $L^* \otimes_B U \equiv V \oplus O(\mathscr{Y}(G, P, H))$, as required.

Here we recall a general fact. For a family $\mathscr{F}$ of subgroups of a finite group G and a subgroup H of G we set

$$H \wedge_G \mathscr{F} = \{\, H \cap {}^g F \mid g \in G, F \in \mathscr{F} \,\}.$$

Mackey double coset formula yields

Lemma 4.2 *Let $\mathscr{F}$ be a family of subgroups of a finite group G. If a kG-module U is relatively $\mathscr{F}$-projective, then for a subgroup H of G the kH-module ${}_H U$ is relatively $H \wedge_G \mathscr{F}$-projective.*

Lemma 4.3 (i) *Assume that a direct summand W of the $k[H \times D^{\mathrm{op}}]$-module kH is relatively $\mathscr{Y}(G \times D^{\mathrm{op}}, \Delta D, H \times D^{\mathrm{op}})$-projective. Then for a subgroup $Q \in \mathscr{A}(G, P, H)$ and a kQ-module S, the kQ-module ${}_Q W \otimes_{kQ} S$ is relatively $\{\, Q \cap {}^g Q \mid g \in G \setminus H \,\}$-projective.*

(ii) *Assume that a direct summand Z of the $k[G \times D^{\mathrm{op}}]$-module kG is relatively $\mathscr{X}(G \times D^{\mathrm{op}}, \Delta D, H \times D^{\mathrm{op}})$-projective. Then for a subgroup $Q \in \mathscr{A}(G, P, H)$ and a kQ-module S, the kQ-module ${}_Q Z \otimes_{kQ} S$ is relatively $\{\, Q \cap {}^g Q \mid g \in G \setminus H \,\}$-projective.*

Proof. Let $\mathscr{T} = \{\, T \leqslant Q \times Q^{\mathrm{op}} \cap {}^{(s,a)}\Delta D \mid (s, a) \in G \times D^{\mathrm{op}} \setminus H \times D^{\mathrm{op}} \,\}$.

(i) Let $\mathscr{Y} = \mathscr{Y}(G \times D^{\mathrm{op}}, \Delta D, H \times D^{\mathrm{op}})$. The $k[Q \times Q^{\mathrm{op}}]$-module ${}_{Q \times Q^{\mathrm{op}}} W$ is relatively $Q \times Q^{\mathrm{op}} \wedge_{H \times D^{\mathrm{op}}} \mathscr{Y}$-projective. We show that

$$Q \times Q^{\mathrm{op}} \wedge_{H \times D^{\mathrm{op}}} \mathscr{Y} \subseteq \mathscr{T}.$$

Take a subgroup $R \in \mathscr{Y}$. There exists an element $(s', a') \in G \times D^{\mathrm{op}} \smallsetminus H \times D^{\mathrm{op}}$ for which $R \leqslant H \times D^{\mathrm{op}} \cap {}^{(s',a')}\Delta D$. Then we have for an element $(t, b) \in H \times D^{\mathrm{op}}$ that

$$Q \times Q^{\mathrm{op}} \cap {}^{(t,b)}R \leqslant Q \times Q^{\mathrm{op}} \cap {}^{(t,b)(s',a')}\Delta D.$$

Since $(t, b)(s', a') \in G \times D^{\mathrm{op}} \smallsetminus H \times D^{\mathrm{op}}$, we obtain for some element $(s, a) \in G \times D^{\mathrm{op}} \smallsetminus H \times D^{\mathrm{op}}$ that

$$Q \times Q^{\mathrm{op}} \cap {}^{(t,b)}R \leqslant Q \times Q^{\mathrm{op}} \cap {}^{(s,a)}\Delta D.$$

Thus we see that $Q \times Q^{\mathrm{op}} \wedge_{H \times D^{\mathrm{op}}} \mathscr{Y} \subseteq \mathscr{T}$.

Notice for an element $(s, a) \in G \times D^{\mathrm{op}} \smallsetminus H \times D^{\mathrm{op}}$ that

$$Q \times Q^{\mathrm{op}} \cap {}^{(s,a)}\Delta D = {}^{(s,a)}\Delta(Q^s \cap {}^aQ).$$

Now an indecomposable direct summand W' of the $k[Q \times Q^{\mathrm{op}}]$-module ${}_{Q \times Q^{\mathrm{op}}}W$ is isomorphic for some element $x \in H$ to $k[QxQ]$, which has ${}^{(1,x)}\Delta(Q \cap {}^xQ)$ as a vertex. Because W' is relatively $\mathscr{T}$-projective, there exists an element $(s, a) \in G \times D^{\mathrm{op}} \smallsetminus H \times D^{\mathrm{op}}$ for which

$${}^{(1,x)}\Delta(Q \cap {}^xQ) \leqslant_{Q \times Q^{\mathrm{op}}} Q \times Q^{\mathrm{op}} \cap {}^{(s,a)}\Delta D = {}^{(s,a)}\Delta(Q^s \cap {}^aQ).$$

Hence we have that $Q \cap {}^xQ \leqslant_Q Q \cap {}^{sa}Q$. Since $sa \in G \smallsetminus H$, we have that

$$Q \cap {}^xQ \leqslant_Q Q \cap {}^gQ$$

for some element $g \in G \smallsetminus H$.

Now, because $W' \otimes_{kQ} S \simeq k[QxQ] \otimes_{kQ} S \simeq kQ \otimes_{k[Q \cap {}^xQ]} x \otimes S$, the kQ-module $W' \otimes_{kQ} S$ is relatively $Q \cap {}^xQ$-projective. Thus we conclude that $W' \otimes_{kQ} S$ is relatively $Q \cap {}^gQ$-projective for some element $g \in G \smallsetminus H$.

(ii) We set $\mathscr{X} = \mathscr{X}(G \times D^{\mathrm{op}}, \Delta D, H \times D^{\mathrm{op}})$. The $k[Q \times Q^{\mathrm{op}}]$-module ${}_{Q \times Q^{\mathrm{op}}}Z$ is relatively $Q \times Q^{\mathrm{op}} \wedge_{G \times D^{\mathrm{op}}} \mathscr{X}$-projective. Take a subgroup $R \in \mathscr{X}$. Then there exists an element $(s', a') \in G \times D^{\mathrm{op}} \smallsetminus H \times D^{\mathrm{op}}$ for which $R \leqslant \Delta D \cap {}^{(s',a')}\Delta D$. For an element $(t, b) \in H \times D^{\mathrm{op}}$, we have

$$Q \times Q^{\mathrm{op}} \cap {}^{(t,b)}R \leqslant Q \times Q^{\mathrm{op}} \cap {}^{(t,b)}\Delta D \cap {}^{(t,b)(s',a')}\Delta D.$$

Either (t, b) or $(t, b)(s', a')$ does not belong to $H \times D^{\mathrm{op}}$ so that we have for some element $(s, a) \in G \times D^{\mathrm{op}} \smallsetminus H \times D^{\mathrm{op}}$ that

$$Q \times Q^{\mathrm{op}} \cap {}^{(t,b)}R \leqslant Q \times Q^{\mathrm{op}} \cap {}^{(s,a)}\Delta D.$$

Namely we see that $Q \times Q^{\mathrm{op}} \wedge_{G \times D^{\mathrm{op}}} \mathscr{X} \subseteq \mathscr{T}$.

Now an indecomposable direct summand Z' of ${}_{Q \times Q^{\mathrm{op}}}Z$ is isomorphic for some element $x \in G$ to $k[QxQ]$, which has ${}^{(1,x)}\Delta(Q \cap {}^xQ)$ as a vertex. The module Z' is relatively $\mathscr{T}$-projective; hence we have that

$${}^{(1,x)}\Delta(Q \cap {}^xQ) \leqslant_{Q \times Q^{\mathrm{op}}} Q \times Q^{\mathrm{op}} \cap {}^{(s,a)}\Delta D = {}^{(s,a)}\Delta(Q^s \cap {}^aQ)$$

for some element $(s, a) \in G \times D^{\mathrm{op}} \setminus H \times D^{\mathrm{op}}$. Therefore, it follows that $Q \cap {}^{x}Q \leqslant_Q Q \cap {}^{sa}Q$. Since $sa \in G \setminus H$, we see that

$$Q \cap {}^{x}Q \leqslant_Q Q \cap {}^{g}Q$$

for some element $g \in G \setminus H$.

Now, because $Z' \otimes_{kQ} S \simeq k[QxQ] \otimes_{kQ} S = kQ \otimes_{k[Q \cap {}^{x}Q]} x \otimes S$, the tensor product $Z' \otimes_{kQ} S$ is relatively $Q \cap {}^{x}Q$-projective. Consequently the tensor product $Z' \otimes_{kQ} S$ is relatively $Q \cap {}^{g}Q$-projective for some element $g \in G \setminus H$.

Proposition 4.4 *We let a source module X of the block ideal B and a source module Y of the block ideal C be in the Green correspondence with respect to $(G \times D^{\mathrm{op}}, \Delta D, H \times D^{\mathrm{op}})$. Assume that an indecomposable kG-module U lying in B and an indecomposable kH-module V lying in C have vertices in $\mathscr{A}(G, P, H)$ and are in the Green correspondence with respect to (G, P, H). Then the following hold.*

(i) *If $Q \in \mathscr{A}(G, P, H)$ is a Y-vertex of V and S is a (Q, Y)-source of V, then Q is an X-vertex of U and S is a (Q, X)-source of U.*

(ii) *If $Q \in \mathscr{A}(G, P, H)$ is an X-vertex of U and S is a (Q, X)-source of U, then Q is a Y-vertex of V and S is a (Q, Y)-source of V.*

Proof. We first note for any $g \in G \setminus H$ that $Q \cap {}^{g}Q$ is a proper subgroup of Q.

Now we let

$$\begin{aligned} L^* \otimes_B X &\simeq Y \oplus Y_0, & Y_0 &\equiv O(\mathscr{Y}(G \times D^{\mathrm{op}}, \Delta D, H \times D^{\mathrm{op}})),\\ L \otimes_C Y &\simeq X \oplus X_0, & X_0 &\equiv O(\mathscr{X}(G \times D^{\mathrm{op}}, \Delta D, H \times D^{\mathrm{op}})),\\ L^* \otimes_B U &\simeq V \oplus V_0, & V_0 &\equiv O(\mathscr{Y}(G, P, H)),\\ L \otimes_C V &\simeq U \oplus U_0, & U_0 &\equiv O(\mathscr{X}(G, P, H)). \end{aligned}$$

(i) Let us show that $S \mid {}_{Q}X^* \otimes_B U$. Because

$$\begin{aligned} X^* \otimes_B L \otimes_C V &\simeq X^* \otimes_B U \oplus X^* \otimes_B U_0,\\ X^* \otimes_B L \otimes_C V &\simeq Y^* \otimes_C V \oplus Y_0^* \otimes_C V \end{aligned}$$

and $S \mid {}_{Q}Y^* \otimes_C V$, the module S is isomorphic with an indecomposable direct summand of either ${}_{Q}X^* \otimes_B U$ or ${}_{Q}X^* \otimes_B U_0$. The kQ-module ${}_{Q}X^* \otimes_B U_0$ is a direct summand of the kQ-module ${}_{Q}U_0$, which is projective relative to the family $Q \wedge_G \mathscr{X}(G, P, H)$. We see for an element $g \in G$ and an element $t \in G \setminus H$ that $Q \cap {}^{g}(P \cap {}^{t}P) \leqslant Q \cap {}^{gt}P \leqslant P \cap {}^{gt}P$ so that the kQ-module ${}_{Q}X^* \otimes_B U_0$ has no indecomposable direct summand having Q as a vertex. Therefore we obtain that $S \mid {}_{Q}X^* \otimes_B U$, as claimed.

Next we show that Q is an X-vertex of U. It follows that

$$U \mid L \otimes_C V \mid L \otimes_C Y \otimes_{kQ} S \simeq X \otimes_{kQ} S \oplus X_0 \otimes_{kQ} S.$$

If $U \mid X_0 \otimes_{kQ} S$, then we would have

$$S \mid {}_QX^* \otimes_B U \mid {}_QU \mid {}_QX_0 \otimes_{kQ} S.$$

However, Lemma 4.3 (ii) says that ${}_QX_0 \otimes_{kQ} S$ is projective relative to the family $\{ Q \cap {}^gQ \mid g \in G \setminus H \}$, a contradiction.

(ii) We show that $S \mid {}_{kQ}Y^* \otimes_C V$. Because

$$Y^* \otimes_C L^* \otimes_B U \simeq Y^* \otimes_C V \oplus Y^* \otimes_C V_0,$$
$$Y^* \otimes_C L^* \otimes_B U \simeq X^* \otimes_B U \oplus X_0^* \otimes_B U$$

and $S \mid {}_QX^* \otimes_B U$, the kQ-module S is isomorphic to a direct summand of either ${}_QY^* \otimes_C V$ or ${}_QY^* \otimes_C V_0$. The kQ-module ${}_QY^* \otimes_C V_0$ is projective relative to the family $Q \wedge_H \mathscr{Y}(G, P, H)$. We see for an element $s \in H$ and an element $t \in G \setminus H$ that $Q \cap {}^s(H \cap {}^tP) \leqslant Q \cap {}^{st}P \leqslant P \cap {}^{st}P$ so that the kQ-module ${}_QY^* \otimes_C V_0$ has no indecomposable direct summand having Q as a vertex. Therefore we see that $S \mid {}_QY^* \otimes_C V$, as desired.

Next we prove that Q is a Y-vertex of V. Because

$$V \mid L^* \otimes_B U \mid L^* \otimes_B X \otimes_{kQ} S \simeq Y \otimes_{kQ} S \oplus Y_0 \otimes_{kQ} S$$

the kH-module V is a direct summand of either $Y \otimes_{kQ} S$ or $Y_0 \otimes_{kQ} S$. If $V \mid Y_0 \otimes_{kQ} S$, then we would have

$$S \mid {}_QY^* \otimes_C V \mid {}_QV \mid {}_QY_0 \otimes_{kQ} S.$$

However, Lemma 4.3 (i) says that ${}_QY_0 \otimes_{kQ} S$ is projective relative to the family $\{ Q \cap {}^gQ \mid g \in G \setminus H \}$, a contradiction.

Since $L \mid {}_{G \times H^{\mathrm{op}}}B$ and L has ΔD as a vertex, an indecomposable direct summand of ${}_{G \times D^{\mathrm{op}}}L$ has ΔD as a vertex, which is a source module of the block B.

Proposition 4.5 *We take a source module X of the block ideal B as a direct summand of the $k[G \times D^{\mathrm{op}}]$-module ${}_{G \times D^{\mathrm{op}}}L$. Assume that an indecomposable B-module U has an X-vertex belonging to $\mathscr{A}(G, P, H)$. Then the Green correspondent V of U to H with respect to (G, P, H) lies in the block C.*

Proof. Let $Q \in \mathscr{A}(G, P, H)$ be an X-vertex of U and S a (Q, X)-source of U. Our choice of the source module X implies that

$$S \mid {}_QX^* \otimes_B U \mid {}_QL^* \otimes_B U.$$

Therefore an indecomposable direct summand V' of $L^* \otimes_B U$ has Q as vertex, which must be isomorphic with V by Green correspondence:

$$V \simeq V' \,\Big|\, L^* \otimes_B U$$

and that the Green correspondent V lies in the block C.

Proof of Theorem 1.3. (i) Let $Q \in \mathscr{A}(G, P, H)$ be an X-vertex of U. Then Proposition 4.5 says that the Green correspondent V lies in the block C. Proposition 4.4 says that Q is a Y-vertex of V and we can take a common (Q, X)-source of U and (Q, Y)-source of V, which we denote by S. We write the restrictions of $\mathrm{res}^D_Q : H^*(D, k) \to H^*(Q, k)$ to $H^*(G, B; X)$ and $H^*(H, C; Y)$ as follows:

$$r_{B,Q} : H^*(G, B; X) \to H^*(Q, k), \quad r_{C,Q} : H^*(H, C; Y) \to H^*(Q, k).$$

Then Benson and Linckelmann [2, Theorem 1.1] implies that

$$V_{G,B}(U) = {r_{B,Q}}^*(V_Q(S)), \quad V_{H,C}(V) = {r_{C,Q}}^*(V_Q(S)).$$

Since ${r_{B,Q}}^* = \iota^* \circ {r_{C,Q}}^*$, we obtain that

$$V_{G,B}(U) = \iota^* V_{H,C}(V).$$

(ii) It is well known that the Green correspondent U of V lies in the block $B = C^G$ without any assumption on Y-vertices. Let $Q \in \mathscr{A}(G, P, H)$ be a Y-vertex of V. Then Proposition 4.4 says that Q is an X-vertex of U and we can take a common (Q, X)-source of U and (Q, Y)-source of V. Thus the same argument as in the above applies.

Example. (cf Benson and Linckelmann [2, Corollary 1.4]) Let B be a block ideal of kG and $D \leqslant G$ a defect group of B. Let $X = kGi$ be a source module of B. Let U be an indecomposable B-module and Q an X-vertex of U and S a (Q, X)-source of U.

Assume that Q is normal in D. Let $H = N_G(Q)$. Let $P \leqslant D$ and assume that H contains $N_G(P)$ and that $Q \in \mathscr{A}(G, P, H)$.

Let (D, b_D) be a Sylow B-subpair such that $b_D X(D) = X(D)$ and let $(Q, b_Q) \leqslant (D, b_D)$. Let C be a unique block ideal of kH covering the block ideal b_Q of $kQC_G(Q)$.

Because Q is normal in D, we see that $H = N_G(Q) \geqslant DC_G(D)$. We also know that $C^G = B$ and D is a defect group of the block C and (D, b_D) is a Sylow C-subpair.

We let $K = DC_G(D)$. The $k[K \times D^{\mathrm{op}}]$-module $Z = kK\,\mathrm{Br}_D(i)$ is a source module of b_D and the source module X of B is the Green correspondent of Z with respect to $(G \times D^{\mathrm{op}}, \Delta D, K \times D^{\mathrm{op}})$. Let Y be the Green correspondent of

Z with respect to $(H \times D^{\mathrm{op}}, \Delta D, K \times D^{\mathrm{op}})$; Y is a source module of the block $C = b_D{}^H$ by Proposition 2.3; X and Y are in the Green correspondence with respect to $(G \times D^{\mathrm{op}}, \Delta D, H \times D^{\mathrm{op}})$.

Theorem 1.2 (ii) implies that $H^*(G, B; X) \subseteq H^*(H, C; Y)$.

Let V be the Green correspondent of U with respect to (G, P, H). Then we have from Theorem 1.3 that

$$V_{G,B}(U) = \iota^* V_{H,C}(V).$$

5. Appendix–relative projectivity

In this appendix, following Broué [3], we shall state briefly the theory on projectivity relative to bimodules over symmetric algebras. We also add some facts.

Let A, B and C be symmetric k-algebras of finite dimension. We let denote by ${}_A\mathsf{mod}_C$ the category of finitely generated (A, C)-bimodules whose left and right k-module structures coincide. For $L, L' \in {}_A\mathsf{mod}_C$ we shall denote by ${}_A(L, L')_C$ the set of (A, C)-homomorphisms from L to L'. Similarly, the symbol ${}_A(L, L')$ means the set of A-homomorphisms from the left A-module ${}_AL$ to the left A-module ${}_AL'$.

Let X be an (A, B)-bimodule such that the left A-module ${}_AX$ and the right B-module X_B are both finitely generated and projecive. Then the functors

$$\begin{aligned} {}_XS &: {}_B\mathsf{mod}_C \to {}_A\mathsf{mod}_C; M \mapsto X \otimes_B M, \\ {}_XT &: {}_A\mathsf{mod}_C \to {}_B\mathsf{mod}_C; L \mapsto X^* \otimes_A L \end{aligned}$$

are biadjoint pair. Namely, for arbitrary $M \in {}_B\mathsf{mod}_C$ and $L \in {}_A\mathsf{mod}_C$, we have natural transformations

$$\begin{aligned} \varphi_{L,M} &: {}_A(X \otimes_B M, L)_C \xrightarrow{\sim} {}_B(M, X^* \otimes_A L)_C \\ \psi_{M,L} &: {}_B(X^* \otimes_A L, M)_C \xrightarrow{\sim} {}_A(L, X \otimes_B M)_C. \end{aligned}$$

We should notice that the natural transformations above depend on the choice of symmetrizing forms of the algebras A and B.

We define $\eta_{X^*,X} : B \to X^* \otimes_A X$ and $\varepsilon_{X^*,X} : X^* \otimes_A X \to B$ by the following:

$$\begin{aligned} \varphi_{X,B} &: {}_A(X, X)_B \xrightarrow{\sim} {}_B(B, X^* \otimes_A X)_B \\ &\qquad \mathrm{Id}_X \mapsto \eta_{X^*,X}, \\ \psi_{B,X} &: {}_B(X^* \otimes_A X, B)_B \xrightarrow{\sim} {}_A(X, X)_B \\ &\qquad \varepsilon_{X^*,X} \mapsto \mathrm{Id}_X. \end{aligned}$$

Similary, considering the k-dual X^*, we define

$$\begin{aligned}\varphi_{X^*,A} : {}_B(X^*, X^*)_A &\xrightarrow{\sim} {}_A(A, X\otimes_B X^*)_A\\ \mathrm{Id}_{X^*} &\mapsto \eta_{X,X^*},\\ \psi_{A,X^*} : {}_A(X\otimes_B X^*, A)_A &\xrightarrow{\sim} {}_B(X^*, X^*)_A\\ \varepsilon_{X,X^*} &\mapsto \mathrm{Id}_{X^*}.\end{aligned}$$

The maps $\eta_{X^*,X}$ and ε_{X,X^*} represent the unit $\mathrm{Id}_{{}_B\mathsf{mod}_C} \to {}_XT \circ {}_XS$ and the counit ${}_XS \circ {}_XT \to \mathrm{Id}_{{}_A\mathsf{mod}_C}$ of an adjunction for the adjoint pair $({}_XS, {}_XT)$; the maps η_{X,X^*} and $\varepsilon_{X^*,X}$ represent the unit $\mathrm{Id}_{{}_A\mathsf{mod}_C} \to {}_XS \circ {}_XT$ and the counit ${}_XT \circ {}_XS \to \mathrm{Id}_{{}_B\mathsf{mod}_C}$ of an adjunction for the adjoint pair $({}_XT, {}_XS)$. Hereafter we shall denote by η_X, η_{X^*}, ε_{X^*} and ε_X the maps $\eta_{X^*,X}$, η_{X,X^*}, $\varepsilon_{X^*,X}$ and ε_{X,X^*}, respectively.

For the (A, B)-bimodule X the relative trace map, which we would like to denote by ${}^X\mathrm{Tr}$, is defined as follows: for L, $L' \in {}_A\mathsf{mod}_C$

$${}^X\mathrm{Tr} : {}_B(X^*\otimes_A L, X^*\otimes_A L')_C \to {}_A(L, L')_C; \alpha \mapsto \varepsilon_X \circ (\mathrm{Id}_X\otimes \alpha) \circ \eta_{X^*}.$$

Proposition 5.1 *We have for homomorphisms $\beta : X^* \otimes_A L \to X^* \otimes_A L'$, $\alpha : L_1 \to L$ and $\alpha' : L' \to L'_1$ that*

$$\alpha' \circ {}^X\mathrm{Tr}(\beta) \circ \alpha = {}^X\mathrm{Tr}\left((\mathrm{Id}_{X^*} \otimes \alpha') \circ \beta \circ (\mathrm{Id}_{X^*} \otimes \alpha)\right).$$

Proposition 5.2 *Let X be an (A, B)-bimodule and assume that ${}_AX$ and X_B are finitely generated and projective.*

(i) *Let Y be a (B, C)-bimodule and assume that ${}_BY$ and Y_C are finitely generated and projective. Then we obtain* ${}^{X\otimes_B Y}\mathrm{Tr} = {}^X\mathrm{Tr} \circ {}^Y\mathrm{Tr}$.

(ii) *Let X' be an (A, B)-bimodule and assume that ${}_AX'$ and X'_B are finitely generated and projective. Then we have* $\mathrm{Im}\,{}^{X\oplus X'}\mathrm{Tr} = \mathrm{Im}\,{}^X\mathrm{Tr} + \mathrm{Im}\,{}^{X'}\mathrm{Tr}$.

Theorem 5.3 *Let X be an (A, B)-bimodule such that ${}_AX$ and X_B are finitely generated and projecive. For $L \in {}_A\mathsf{mod}_C$ the followings are equivalent to each other.*

(i) *There exist (A, C)-homomorphisms $i : L \to X \otimes_B X^* \otimes_A L$ and $q : X \otimes_B X^* \otimes_A L \to L$ such that $q \circ i = \mathrm{Id}_L$.*

(ii) *There exists a module $M \in {}_B\mathsf{mod}_C$ and (A, C)-homomorphisms $i : L \to X \otimes_B M$ and $q : X \otimes_B M \to L$ such that $q \circ i = \mathrm{Id}_L$.*

(iii) *There exists a (B, C)-homomorphism $\beta : X^* \otimes_A L \to X^* \otimes_A L$ such that* ${}^X\mathrm{Tr}(\beta) = \mathrm{Id}_L$.

(iv) *There exist an* (A,C)*-homomorphism* $q : X \otimes_B X^* \otimes_A L \to L$ *such that* $q \circ (\eta_{X^*} \otimes \mathrm{Id}_L) = \mathrm{Id}_L$.

(v) *There exist an* (A,C)*-homomorphism* $i : L \to X \otimes_B X^* \otimes_A L$ *such that* $(\varepsilon_{X^*} \otimes \mathrm{Id}_L) \circ i = \mathrm{Id}_L$.

(v') *(relative projectivity of L) For* (A,C)*-homomorphisms* $\alpha : L \to L'$ *and* $\pi : L'' \to L'$*, if there exists a* (B,C)*-homomorphism* $\beta : X^* \otimes_A L' \to X^* \otimes_A L''$ *such that* $(\mathrm{Id}_{X^*} \otimes \pi) \circ \beta = \mathrm{Id}_{X^* \otimes_A L'}$*, then there exists an* (A,C)*-homomorphism* $\widehat{\alpha} : L \to L''$ *such that* $\pi \circ \widehat{\alpha} = \alpha$.

(iv') *(relative injectivity of L) For* (A,C)*-homomorphisms* $\alpha : L' \to L$ *and* $\iota : L' \to L''$*, if there exists a* (B,C)*-homomorphism* $\beta : X^* \otimes_A L'' \to X^* \otimes_A L'$ *such that* $\beta \circ (\mathrm{Id}_{X^*} \otimes \iota) = \mathrm{Id}_{X^* \otimes_A L'}$*, then there exists an* (A,C)*-homomorphism* $\widehat{\alpha} : L'' \to L$ *such that* $\widehat{\alpha} \circ \iota = \alpha$.

A module $L \in {}_A\mathsf{mod}_C$ satisfying one (and then all) of the conditions in Theorem 5.3 is said to be *relatively* ${}_XS$*-projective* or simply *relatively* X*-projective*.

A homomorphism $\alpha : L \to L'$, where $L, L' \in {}_A\mathsf{mod}_C$ is said to be *relatively* ${}_XS$*-projective* or simply *relatively* X*-projective* if it factors through $X \otimes_B T$ for some $T \in {}_B\mathsf{mod}_C$. Theorem 5.3 implies that a homomorphism $\alpha : L \to L'$ is relatively X-projective if and only if it is the image under the relative trace map ${}^X\mathrm{Tr} : {}_B(X^* \otimes_A L, X^* \otimes_A L')_C \to {}_A(L, L')_C$.

Remark. Although in [3] a relatively X-projective module is said to be ${}_XS$*-split* and relatively X-projective homomorphism is said to be ${}_XS$*-split*, we would like to use the term "relatively projective".

By definition we have

$$ {}^X\mathrm{Tr}(\mathrm{Id}_{X^*}) = \varepsilon_X \circ \eta_{X^*} : A \to A, $$

which we denote by $\widehat{\pi}_X$. The homomorphism $\widehat{\pi}_X$ is relatively X-projective.

The element $\pi_X = \widehat{\pi}_X(1_A) \in Z(A)$ is called a relatively X-projecive element and the element $\pi_{X^*} = \widehat{\pi}_{X^*}(1_B) \in Z(B)$ is called a relatively X^*-projective element. The map $\widehat{\pi}_X : A \to A$ is given by multiplication by the element $\pi_X \in Z(A)$ and the map $\widehat{\pi}_{X^*} : B \to B$ is given by multiplication by the element $\pi_{X^*} \in Z(B)$.

Theorem 5.4 *Let* X *be an* (A,B)*-bimodule and assume that* ${}_AX$ *and* X_B *are finitely generated and projective. Let* Y *be a* (B,C)*-bimodule and assume that* ${}_BY$ *and* Y_C *are finitely generated and projective. Assume that an* (A,C)*-bimodule* L *is isomorphic to a direct summand of the tensor product* $X \otimes_B Y$*. Then the modules* ${}_AL$ *and* L_C *are finitely generated and projective, respectively, and the following hold.*

(i) *The module L is relatively X-projective and the homomorphism $\widehat{\pi}_L : A \to A$ is relatively X-projective.*

(ii) *The module L^* is relatively Y^*-projective and the homomorphism $\widehat{\pi}_{L^*} : C \to C$ is relatively Y^*-projective.*

Proof. By definition the module L is relatively X-projective and the homomorphism $\widehat{\pi}_L$ factors through $X \otimes_B Y \otimes L^*$. Taking duals we have the assertion (ii).

By Proposition 5.1 we see

Proposition 5.5 *Let X be an (A, B)-bimodule and assume that ${}_AX$ and X_B are finitely generated and projective. For modules $L,\ L' \in {}_A\mathsf{mod}_C$ we let*

$$_{X^*}\mathrm{Res} : {}_A(L, L')_C \to {}_B(X^* \otimes_A L, X^* \otimes_A L')_C; f \mapsto \mathrm{Id}_{X^*} \otimes f.$$

Then we have

$$^X\mathrm{Tr} \circ {}_{X^*}\mathrm{Res} = \widehat{\pi}_X (= \textit{multiplication by } \pi_X).$$

Theorem 5.6 *Let X be an (A, B)-bimodule and assume that ${}_AX$ and X_B are finitely generated and projective. If the relatively projective element $\pi_X \in Z(A)$ is invertible, then arbitrary $L \in {}_A\mathsf{mod}_C$ is relatively X-projective.*

Proof. The assumption that $\pi_X \in Z(A)$ is invertible implies that the relative trace map $^X\mathrm{Tr} : {}_B(X^* \otimes_A L, X^* \otimes_A L)_C \to {}_A(L, L)_C$ is epimoriphic because of the previous proposition.

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