

STRESS-FUNCTIONS FOR THE SOLID OF REVOLUTION

By

Bennoyuke TANIMOTO*

(Civil Engineering Department, Faculty of Engineering.¹⁾)

1. Symmetrical strain in the solid of revolution has sometimes been met in the theory of elasticity and in technological problems. When the solid is subjected to surface tractions only, stress-function has been given²⁾ and widely known. But so far as I am aware, nothing has been worked out in the case in which the solid is subjected to body-forces, together with surface tractions, the solid is in the state of motion, or the solid is governed by stress-strain relations other than Hooke's law.

In certain cases of these, stress-functions can be found, and they will play an elementary role in treating problems within the range of the original simultaneous partial differential equations. From the mathematical point of view such stress-functions are solutions or 'the' solutions of the original simultaneous equations, in the meaning that the simultaneous equations have been reduced to a single partial differential equation which is satisfied by the stress-function. That is to say, on having eliminated the simultaneity of the differential equations, the problem reduces to the attack on a single partial differential equation. Displacements and stresses are then given by certain operations performed on the stress-function.

2. Fundamental equations in terms of stresses referred to cylindrical coordinates are

$$\frac{\partial \widehat{r}_r}{\partial r} + \frac{\partial \widehat{r}_z}{\partial z} + \frac{\widehat{r}_r - \widehat{\theta}\theta}{r} + \rho R = 0, \quad \frac{\partial \widehat{r}_z}{\partial r} + \frac{\partial \widehat{z}_z}{\partial z} + \frac{\widehat{r}_z}{r} + \rho Z = 0 \dots \dots \dots (1)$$

For the convenience of the succeeding calculation, we write

$$\rho R = -\frac{\partial U}{\partial r}, \quad \rho Z = -\frac{\partial W}{\partial z},$$

where U , W are some functions of r , z , t ; t representing time.

* Professor of Shinshu University.

Strain-components are defined by

$$\left. \begin{aligned} e_{rr} &= \frac{\partial u}{\partial r}, & e_{\theta\theta} &= \frac{u}{r}, & e_{zz} &= \frac{\partial w}{\partial z}, & e_{rz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}, \\ \Delta &= e_{rr} + e_{\theta\theta} + e_{zz} = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z}, \end{aligned} \right\}$$

u, w being displacements in the directions of r and z .

3. In the first place, we suppose that the solid is governed by Hooke's law, which is written

$$\bar{r}\bar{r} = \lambda\Delta + 2\mu e_{rr}, \quad \bar{\theta}\bar{\theta} = \lambda\Delta + 2\mu e_{\theta\theta}, \quad \bar{z}\bar{z} = \lambda\Delta + 2\mu e_{zz}, \quad \bar{r}\bar{z} = \mu e_{rz}.$$

Then on proceeding just as Love developed²⁾, we have, after some amount of calculation,

i) for the stress-components

$$\left. \begin{aligned} \bar{r}\bar{r} &= \frac{\partial}{\partial z} \left\{ \sigma r^2 \chi - \frac{\partial^2 \chi}{\partial r^2} \right\} + \left\{ U + (1-\sigma) r \frac{\partial U}{\partial r} - \sigma \frac{\partial^2}{\partial z^2} \int r U dr \right\}, \\ \bar{\theta}\bar{\theta} &= \frac{\partial}{\partial z} \left\{ \sigma r^2 \chi - \frac{1}{r} \frac{\partial \chi}{\partial r} \right\} + \left\{ U - \sigma r \frac{\partial U}{\partial r} - \sigma \frac{\partial^2}{\partial z^2} \int r U dr \right\}, \\ \bar{z}\bar{z} &= \frac{\partial}{\partial z} \left\{ (2-\sigma) r^2 \chi - \frac{\partial^2 \chi}{\partial z^2} \right\} - \left\{ 2U + (2-\sigma) r \frac{\partial U}{\partial r} + (1-\sigma) \frac{\partial^2}{\partial z^2} \int r U dr \right\}, \\ \bar{r}\bar{z} &= \frac{\partial}{\partial r} \left\{ (1-\sigma) r^2 \chi - \frac{\partial^2 \chi}{\partial z^2} \right\} - \int \left\{ (2-\sigma) \frac{\partial U}{\partial r} + (1-\sigma) r \frac{\partial^2 U}{\partial r^2} - \sigma r \frac{\partial^2 U}{\partial z^2} \right\} dz; \end{aligned} \right\} \dots\dots\dots (2)$$

ii) for the displacement-components

$$\left. \begin{aligned} u &= -\frac{1+\sigma}{E} \frac{\partial^2 \chi}{\partial r \partial z} + \frac{1+\sigma}{E} r U, \\ w &= \frac{1+\sigma}{E} \left\{ 2(1-\sigma) r^2 \chi - \frac{\partial^2 \chi}{\partial z^2} \right\} - \frac{1+\sigma}{E} \int \left\{ 2U + 2(1-\sigma) r \frac{\partial U}{\partial r} \right. \\ &\quad \left. + (1-2\sigma) \frac{\partial^2}{\partial z^2} \int r U dr \right\} dz; \end{aligned} \right\} \dots\dots\dots (3)$$

iii) for the differential equation satisfied by the stress-function χ

$$\begin{aligned} (1-\sigma) r^4 \chi &= \int \left\{ (2-\sigma) \frac{1}{r} \frac{\partial U}{\partial r} + (4-3\sigma) \frac{\partial^2 U}{\partial r^2} + 2(1-\sigma) \frac{\partial^2 U}{\partial z^2} + \frac{\partial^2 W}{\partial z^2} \right. \\ &\quad \left. + (1-\sigma) r \frac{\partial^3 U}{\partial r^3} + 2(1-\sigma) r \frac{\partial^3 U}{\partial r \partial z^2} + (1-\sigma) \frac{\partial^4}{\partial z^4} \int r U dr \right\} dz; \dots (4) \end{aligned}$$

where

$$r^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \quad r^4 = r^2 r^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right)^2.$$

4. To obtain formulas expressed in terms of a single stress-function for

the solid of revolution, in which case the solid is subjected to uniform gravitation, it is sufficient to take

$$W=U \quad \text{and} \quad \rho Z = \rho g = -\frac{\partial U}{\partial z}, \quad \rho R = -\frac{\partial U}{\partial r} = 0,$$

ρg being in the positive direction of the z -axis; and then we obtain

$$U = -\rho g z.$$

Then equations (2), (3) and (4) reduce to:

i) for the stress-components

$$\left. \begin{aligned} \bar{r}\bar{r} &= \frac{\partial}{\partial z} \left\{ \sigma \nabla^2 \chi - \frac{\partial^2 \chi}{\partial r^2} \right\} - \rho g z, \\ \bar{\theta}\bar{\theta} &= \frac{\partial}{\partial z} \left\{ \sigma \nabla^2 \chi - \frac{1}{r} \frac{\partial \chi}{\partial r} \right\} - \rho g z, \\ \bar{z}\bar{z} &= \frac{\partial}{\partial z} \left\{ (2-\sigma) \nabla^2 \chi - \frac{\partial^2 \chi}{\partial z^2} \right\} + 2\rho g z, \\ \bar{r}\bar{z} &= \frac{\partial}{\partial r} \left\{ (1-\sigma) \nabla^2 \chi - \frac{\partial^2 \chi}{\partial z^2} \right\} - \rho g \sigma r; \end{aligned} \right\} \dots\dots\dots (5)$$

ii) for the displacement-components

$$\left. \begin{aligned} u &= -\frac{1+\sigma}{E} \frac{\partial^2 \chi}{\partial r \partial z} - \frac{1+\sigma}{E} \rho g r z, \\ w &= \frac{1+\sigma}{E} \left\{ 2(1-\sigma) \nabla^2 \chi - \frac{\partial^2 \chi}{\partial z^2} \right\} + \frac{1+\sigma}{E} \rho g \left\{ \frac{1}{2} (1-2\sigma) r^2 + z^2 \right\}; \end{aligned} \right\} \dots\dots\dots (6)$$

iii) for the differential equation satisfied by the stress-function χ

$$\nabla^4 \chi = -\frac{3-2\sigma}{1-\sigma} \rho g. \quad \dots\dots\dots (7)$$

For the purpose of actual calculation in prescribed boundary problem, it will be more convenient to take the function χ in (7) to be divided into two parts; i.e.

$$\chi = \chi_0 + \chi_1,$$

in which χ_0 and χ_1 satisfy respectively the equations

$$\nabla^4 \chi_0 = 0, \quad \text{and} \quad \nabla^4 \chi_1 = -\frac{3-2\sigma}{1-\sigma} \rho g; \quad \dots\dots\dots (8)$$

χ_1 then being taken to be particular integrals of (7). Accordingly χ_1 may be found to be

$$\chi_1 = -\frac{1}{8(8\alpha+2\beta+3\gamma)} \frac{3-2\sigma}{1-\sigma} \rho g (\alpha r^4 + \beta r^2 z^2 + \gamma z^4). \quad (\alpha, \beta, \gamma \text{ being constants.})$$

We thus obtain:

i) for the stress-components

$$\begin{aligned}
\widehat{r}r &= \frac{\partial}{\partial z} \left\{ \sigma r^2 \chi_0 - \frac{\partial^2 \chi_0}{\partial r^2} \right\} - \frac{\rho g}{2(8\alpha + 2\beta + 3\gamma)} \frac{3-2\sigma}{1-\sigma} \\
&\quad \times \left\{ -(1-2\sigma)\beta + 6\sigma\gamma \right\} z - \rho g z, \\
\widehat{\theta}\theta &= \frac{\partial}{\partial z} \left\{ \sigma r^2 \chi_0 - \frac{1}{r} \frac{\partial \chi_0}{\partial r} \right\} - \frac{\rho g}{2(8\alpha + 2\beta + 3\gamma)} \frac{3-2\sigma}{1-\sigma} \\
&\quad \times \left\{ -(1-2\sigma)\beta + 6\sigma\gamma \right\} z - \rho g z, \\
\widehat{z}z &= \frac{\partial}{\partial z} \left\{ (2-\sigma) r^2 \chi_0 - \frac{\partial^2 \chi_0}{\partial z^2} \right\} - \frac{\rho g}{2(8\alpha + 2\beta + 3\gamma)} \frac{3-2\sigma}{1-\sigma} \\
&\quad \times \left\{ 2(2-\sigma)\beta + 6(1-\sigma)\gamma \right\} z + 2\rho g z, \\
\widehat{r}z &= \frac{\partial}{\partial r} \left\{ (1-\sigma) r^2 \chi_0 - \frac{\partial^2 \chi_0}{\partial z^2} \right\} - \frac{\rho g}{2(8\alpha + 2\beta + 3\gamma)} \frac{3-2\sigma}{1-\sigma} \\
&\quad \times \left\{ 8(1-\sigma)\alpha - \sigma\beta \right\} \gamma - \rho g \sigma r;
\end{aligned} \tag{10}$$

ii) for the displacement-components

$$\begin{aligned}
u &= -\frac{1+\sigma}{E} \frac{\partial^2 \chi_0}{\partial r \partial z} + \frac{1+\sigma}{E} \frac{\rho g}{2(8\alpha + 2\beta + 3\gamma)} \frac{3-2\sigma}{1-\sigma} \beta r z - \frac{1+\sigma}{E} \rho g r z, \\
w &= \frac{1+\sigma}{E} \left\{ 2(1-\sigma) r^2 \chi_0 - \frac{\partial^2 \chi_0}{\partial z^2} \right\} - \frac{1+\sigma}{E} \frac{\rho g}{2(8\alpha + 2\beta + 3\gamma)} \frac{3-2\sigma}{1-\sigma} \\
&\quad \times \left\{ [8(1-\sigma)\alpha + \frac{1}{2}(1-2\sigma)\beta] r^2 + [2(1-\sigma)\beta + 3(1-2\sigma)\gamma] z^2 \right\} \\
&\quad + \frac{1+\sigma}{E} \rho g \left\{ \frac{1}{2}(1-2\sigma) r^2 + z^2 \right\},
\end{aligned} \tag{11}$$

χ_0 being biharmonic functions.

5. To obtain formulas expressed in terms of a single stress-function, in which case the solid of revolution is subjected to centrifugal force due to the rotation about the z -axis, it is sufficient to take

$$\rho R = \rho \omega^2 r = -\frac{\partial U}{\partial r}, \quad \rho Z = -\frac{\partial U}{\partial z} = 0,$$

ω being a uniform angular velocity of the solid; we then obtain

$$U = -\frac{1}{2} \rho \omega^2 r^2.$$

Then from equations (2), (3) and (4), we have:

i) for the stress-components

$$\left. \begin{aligned} \bar{r}r &= \frac{\partial}{\partial z} \left\{ \sigma r^2 \chi - \frac{\partial^2 \chi}{\partial r^2} \right\} - \frac{1}{2} (3-2\sigma) \rho \omega^2 r^2, \\ \bar{\theta}\theta &= \frac{\partial}{\partial z} \left\{ \sigma r^2 \chi - \frac{1}{r} \frac{\partial \chi}{\partial r} \right\} - \frac{1}{2} (1-2\sigma) \rho \omega^2 r^2, \\ \bar{z}z &= \frac{\partial}{\partial z} \left\{ (2-\sigma) r^2 \chi - \frac{\partial^2 \chi}{\partial z^2} \right\} + (3-\sigma) \rho \omega^2 r^2, \\ \bar{r}z &= \frac{\partial}{\partial r} \left\{ (1-\sigma) r^2 \chi - \frac{\partial^2 \chi}{\partial z^2} \right\} + (3-2\sigma) \rho \omega^2 r z; \end{aligned} \right\} \dots\dots\dots (12)$$

ii) for the displacement-components

$$\left. \begin{aligned} u &= -\frac{1+\sigma}{E} \frac{\partial^2 \chi}{\partial r \partial z} - \frac{1+\sigma}{E} \frac{1}{2} \rho \omega^2 r^3, \\ w &= \frac{1+\sigma}{E} \left\{ 2(1-\sigma) r^2 \chi - \frac{\partial^2 \chi}{\partial z^2} \right\} + \frac{1+\sigma}{E} (3-2\sigma) \rho \omega^2 r^2 z; \end{aligned} \right\} \dots\dots\dots (13)$$

iii) for the differential equation satisfied by the stress-function χ

$$r^4 \chi = -2 \frac{3-2\sigma}{1-\sigma} \rho \omega^2 z. \dots\dots\dots (14)$$

As in the preceding article, we write

$$\chi = \chi_0 + \chi_1,$$

in which χ_0 and χ_1 satisfy the equations

$$r^4 \chi_0 = 0, \quad \text{and} \quad r^4 \chi_1 = -2 \frac{3-2\sigma}{1-\sigma} \rho \omega^2 z,$$

χ_1 thus being taken to be particular integrals, which may be found to be

$$\chi_1 = -\frac{\rho \omega^2}{4(8\alpha+6\beta+15\gamma)} \frac{3-2\sigma}{1-\sigma} (\alpha r^4 z + \beta r^2 z^3 + \gamma z^5). \quad (\alpha, \beta, \gamma \text{ being constants.})$$

We then obtain:

i) for the stress-components

$$\left. \begin{aligned} \bar{r}r &= \frac{\partial}{\partial z} \left\{ \sigma r^2 \chi_0 - \frac{\partial^2 \chi_0}{\partial r^2} \right\} - \frac{\rho \omega^2}{2(8\alpha+6\beta+15\gamma)} \frac{3-2\sigma}{1-\sigma} \\ &\quad \times \left\{ [-2(3-4\sigma)\alpha + 3\sigma\beta] r^2 + 3[-(1-2\sigma)\beta + 10\sigma\gamma] z^2 \right\} - \frac{1}{2} (3-2\sigma) \rho \omega^2 r^2, \\ \bar{\theta}\theta &= \frac{\partial}{\partial z} \left\{ \sigma r^2 \chi_0 - \frac{1}{r} \frac{\partial \chi_0}{\partial r} \right\} - \frac{\rho \omega^2}{2(8\alpha+6\beta+15\gamma)} \frac{3-2\sigma}{1-\sigma} \\ &\quad \times \left\{ [-2(1-4\sigma)\alpha + 3\sigma\beta] r^2 + 3[-(1-2\sigma)\beta + 10\sigma\gamma] z^2 \right\} - \frac{1}{2} (1-2\sigma) \rho \omega^2 r^2, \\ \bar{z}z &= \frac{\partial}{\partial z} \left\{ (2-\sigma) r^2 \chi_0 - \frac{\partial^2 \chi_0}{\partial z^2} \right\} - \frac{\rho \omega^2}{2(8\alpha+6\beta+15\gamma)} \frac{3-2\sigma}{1-\sigma} \\ &\quad \times \left\{ [8(2-\sigma)\alpha + 3(1-\sigma)\beta] r^2 + 3[2(2-\sigma)\beta + 10(1-\sigma)\gamma] z^2 \right\} + (3-\sigma) \rho \omega^2 r^2, \end{aligned} \right\}$$

$$\begin{aligned} \widehat{r}z = \frac{\partial}{\partial r} \left\{ (1-\sigma) r^2 \chi_0 - \frac{\partial^2 \chi_0}{\partial z^2} \right\} - \frac{\rho \omega^2}{(8\alpha + 6\beta + 15\gamma)} \frac{3-2\sigma}{1-\sigma} \\ \times [8(1-\sigma)\alpha - 3\sigma\beta] \gamma z + (3-2\sigma) \rho \omega^2 r z; \end{aligned} \quad \dots\dots\dots (15)$$

iii) for the displacement-components

$$\begin{aligned} u = -\frac{1+\sigma}{E} \frac{\partial^2 \chi_0}{\partial r \partial z} + \frac{1+\sigma}{E} \frac{\rho \omega^2}{2(8\alpha + 6\beta + 15\gamma)} \frac{3-2\sigma}{1-\sigma} \left\{ 2\alpha r^3 + 3\beta r z^2 \right\} \\ - \frac{1+\sigma}{E} \frac{1}{2} \rho \omega^2 r^3, \\ w = \frac{1+\sigma}{E} \left\{ 2(1-\sigma) r^2 \chi_0 - \frac{\partial^2 \chi_0}{\partial z^2} \right\} \\ - \frac{1+\sigma}{E} \frac{\rho \omega^2}{2(8\alpha + 6\beta + 15\gamma)} \frac{3-2\sigma}{1-\sigma} \left\{ [16(1-\sigma)\alpha + 3(1-2\sigma)\beta] r^2 z \right. \\ \left. + [4(1-\sigma)\beta + 10(1-2\sigma)\gamma] z^3 \right\} + \frac{1+\sigma}{E} (3-2\sigma) \rho \omega^2 r^2 z. \end{aligned} \quad \dots\dots\dots (16)$$

6. We consider the case in which the solid is subjected to uneven temperature distribution that is symmetrical about the z-axis. The stress-strain relations in this case are assumed to be of the forms

$$\widehat{r}r = \lambda \Delta + 2\mu e_{rr} - \kappa T, \quad \widehat{\theta}\theta = \lambda \Delta + 2\mu e_{\theta\theta} - \kappa T, \quad \widehat{z}z = \lambda \Delta + 2\mu e_{zz} - \kappa T, \quad \widehat{r}z = \mu e_{rz},$$

where T is an arbitrary temperature distribution in the solid and $\kappa = (3\lambda + 2\mu)c$; c being its linear coefficient of thermal expansion.

Then we find after some calculation that the following equations hold; i.e.

i) for the stress-components

$$\begin{aligned} \widehat{r}r = \frac{\partial}{\partial z} \left\{ \sigma r^2 \chi - \frac{\partial^2 \chi}{\partial r^2} \right\} - EcT, \\ \widehat{\theta}\theta = \frac{\partial}{\partial z} \left\{ \sigma r^2 \chi - \frac{1}{r} \frac{\partial \chi}{\partial r} \right\} - EcT, \\ \widehat{z}z = \frac{\partial}{\partial z} \left\{ (2-\sigma) r^2 \chi - \frac{\partial^2 \chi}{\partial z^2} \right\} + EcT, \\ \widehat{r}z = \frac{\partial}{\partial r} \left\{ (1-\sigma) r^2 \chi - \frac{\partial^2 \chi}{\partial z^2} \right\} + Ec \frac{\partial}{\partial r} \int T dz; \end{aligned} \quad \dots\dots\dots (17)$$

ii) for the displacement-components

$$\begin{aligned} u = -\frac{1+\sigma}{E} \frac{\partial^2 \chi}{\partial r \partial z}, \\ w = \frac{1+\sigma}{E} \left\{ 2(1-\sigma) r^2 \chi - \frac{\partial^2 \chi}{\partial z^2} \right\} + 2(1+\sigma)c \int T dz; \end{aligned} \quad \dots\dots\dots (18)$$

iii) for the differential equation satisfied by the stress-function χ

$$\nabla^4 \chi = -\frac{Ec}{1-\sigma} \nabla^2 \int T dz. \quad \dots\dots\dots (19)$$

It will be more convenient to take the function χ in (19) to be divided into two parts; i.e.

$$\chi = \chi_0 + \chi_1,$$

in which

$$\nabla^4 \chi_0 = 0, \quad \text{and} \quad \nabla^2 \chi_1 = -\frac{Ec}{1-\sigma} \int T dz. \quad \dots\dots\dots (20)$$

χ_1 then being taken to be particular integrals, since harmonic function can be included in χ_0 .

7. The general expressions obtained in (2), (3) and (4) can be transformed into compact forms. For this purpose we find it convenient to introduce the integral

$$\Omega = \int \int U r \, dr \, dz, \quad \dots\dots\dots (21)$$

which multiplied by 2π may be taken to express the whole potential energy due to the body force, provided the double integration be taken through the volume of the solid.

Then after a certain amount of reduction, the system of equations (2), (3) and (4) take the simple forms:

i) for the stress-components

$$\left. \begin{aligned} \widehat{rr} &= \frac{\partial}{\partial z} \left\{ \sigma \nabla^2 - \frac{\partial^2}{\partial r^2} \right\} (\chi - \Omega) + 2\sigma U, \\ \widehat{\theta\theta} &= \frac{\partial}{\partial z} \left\{ \sigma \nabla^2 - \frac{1}{r} \frac{\partial}{\partial r} \right\} (\chi - \Omega) + 2\sigma U, \\ \widehat{zz} &= \frac{\partial}{\partial z} \left\{ (2-\sigma) \nabla^2 - \frac{\partial^2}{\partial z^2} \right\} (\chi - \Omega) + 2(1-\sigma) U, \\ \widehat{rz} &= \frac{\partial}{\partial r} \left\{ (1-\sigma) \nabla^2 - \frac{\partial^2}{\partial z^2} \right\} (\chi - \Omega) + (1-2\sigma) \frac{\partial}{\partial r} \int U dz; \end{aligned} \right\} \dots\dots (22)$$

ii) for the displacement-components

$$\left. \begin{aligned} u &= -\frac{1+\sigma}{E} \frac{\partial^2}{\partial r \partial z} (\chi - \Omega), \\ w &= \frac{1+\sigma}{E} \left\{ 2(1-\sigma) \nabla^2 - \frac{\partial^2}{\partial z^2} \right\} (\chi - \Omega) + \frac{1+\sigma}{E} 2(1-2\sigma) \int U dz; \end{aligned} \right\} \dots\dots (23)$$

iii) for the differential equation satisfied by the stress-function

$$(1-\sigma) \nabla^4 (\chi - \Omega) = -(1-2\sigma) \nabla^2 \int U dz - \frac{\partial U}{\partial z} + \frac{\partial W}{\partial z}. \quad \dots\dots\dots (24)$$

These are of compact forms, and therefore will be far more convenient

for making general discussion concerning the body-force stress. Now since χ and \mathcal{Q} have always presented themselves as the combined form $(\chi - \mathcal{Q})$ in equations (22), (23, and (24), then we can replace it by a new function, χ' say, and these equations are written in the forms:

i) for the stress-components

$$\left. \begin{aligned} \bar{r}\bar{r} &= \frac{\partial}{\partial z} \left\{ \sigma \nabla^2 \chi' - \frac{\partial^2 \chi'}{\partial r^2} \right\} + 2\sigma U, \\ \bar{\theta}\bar{\theta} &= \frac{\partial}{\partial z} \left\{ \sigma \nabla^2 \chi' - \frac{1}{r} \frac{\partial \chi'}{\partial r} \right\} + 2\sigma U, \\ \bar{z}\bar{z} &= \frac{\partial}{\partial z} \left\{ (2-\sigma) \nabla^2 \chi' - \frac{\partial^2 \chi'}{\partial z^2} \right\} + 2(1-\sigma) U, \\ \bar{r}\bar{z} &= \frac{\partial}{\partial r} \left\{ (1-\sigma) \nabla^2 \chi' - \frac{\partial^2 \chi'}{\partial z^2} \right\} + (1-2\sigma) \frac{\partial}{\partial r} \int U dz; \end{aligned} \right\} \dots\dots\dots (25)$$

ii) for the displacement-components

$$\left. \begin{aligned} u &= -\frac{1+\sigma}{E} \frac{\partial^2 \chi'}{\partial r \partial z}, \\ w &= \frac{1+\sigma}{E} \left\{ 2(1-\sigma) \nabla^2 \chi' - \frac{\partial^2 \chi'}{\partial z^2} \right\} + \frac{1+\sigma}{E} 2(1-2\sigma) \int U dz; \end{aligned} \right\} \dots\dots\dots (26)$$

iii) for the differential equation satisfied by the stress-function χ'

$$(1-\sigma) \nabla^4 \chi' = -(1-2\sigma) \int U dz - \frac{\partial U}{\partial z} + \frac{\partial W}{\partial z}. \dots\dots\dots (27)$$

8. In the case of uniform gravitation, we may obtain another system of equations, by taking $U = W = -\rho g z$,

i) for the stress-components

$$\left. \begin{aligned} \bar{r}\bar{r} &= \frac{\partial}{\partial z} \left\{ \sigma \nabla^2 \chi' - \frac{\partial^2 \chi'}{\partial r^2} \right\} - 2\sigma \rho g z, \\ \bar{\theta}\bar{\theta} &= \frac{\partial}{\partial z} \left\{ \sigma \nabla^2 \chi' - \frac{1}{r} \frac{\partial \chi'}{\partial r} \right\} - 2\sigma \rho g z, \\ \bar{z}\bar{z} &= \frac{\partial}{\partial z} \left\{ (2-\sigma) \nabla^2 \chi' - \frac{\partial^2 \chi'}{\partial z^2} \right\} - 2(1-\sigma) \rho g z, \\ \bar{r}\bar{z} &= \frac{\partial}{\partial r} \left\{ (1-\sigma) \nabla^2 \chi' - \frac{\partial^2 \chi'}{\partial z^2} \right\}; \end{aligned} \right\}$$

ii) for the displacement-components

$$u = -\frac{1+\sigma}{E} \frac{\partial^2 \chi'}{\partial r \partial z}, \quad w = \frac{1+\sigma}{E} \left\{ 2(1-\sigma) \nabla^2 \chi' - \frac{\partial^2 \chi'}{\partial z^2} \right\} - \frac{1+\sigma}{E} (1-2\sigma) \rho g z^2;$$

iii) for the differential equation satisfied by the stress-function χ'

$$\nabla^4 \chi' = \frac{1-2\sigma}{1-\sigma} \rho g.$$

9. In the case of centrifugal force with a uniform angular velocity we may also obtain another set of equations, by taking $U=W=-\frac{1}{2}\rho\omega^2r^2$,

i) for the stress-components

$$\left. \begin{aligned} \widehat{rr} &= \frac{\partial}{\partial z} \left\{ \sigma \nabla^2 \chi - \frac{\partial^2 \chi}{\partial r^2} \right\} - \sigma \rho \omega^2 r^2, \\ \widehat{\theta\theta} &= \frac{\partial}{\partial z} \left\{ \sigma \nabla^2 \chi - \frac{1}{r} \frac{\partial \chi}{\partial r} \right\} - \sigma \rho \omega^2 r^2, \\ \widehat{zz} &= \frac{\partial}{\partial z} \left\{ (2-\sigma) \nabla^2 \chi - \frac{\partial^2 \chi}{\partial z^2} \right\} - (1-\sigma) \rho \omega^2 r^2, \\ \widehat{rz} &= \frac{\partial}{\partial r} \left\{ (1-\sigma) \nabla^2 \chi - \frac{\partial^2 \chi}{\partial z^2} \right\} - (1-2\sigma) \rho \omega^2 r z; \end{aligned} \right\}$$

ii) for the displacement-components

$$u = -\frac{1+\sigma}{E} \frac{\partial^2 \chi}{\partial r \partial z}, \quad w = \frac{1+\sigma}{E} \left\{ 2(1-\sigma) \nabla^2 \chi - \frac{\partial^2 \chi}{\partial z^2} \right\} - \frac{1+\sigma}{E} (1-2\sigma) \rho \omega^2 r^2 z;$$

iii) for the differential equation satisfied by the stress-function χ

$$\nabla^4 \chi = \frac{2(1-2\sigma)}{1-\sigma} \rho \omega^2 z.$$

10. We proceed to derive the dynamical stress-function, which is valid in the solid vibrating symmetrically about its axis. We then put

$$\left. \begin{aligned} \frac{\partial U}{\partial r} &= \rho \frac{\partial^2 u}{\partial t^2}, \quad \text{and} \quad \frac{\partial W}{\partial z} = \rho \frac{\partial^2 w}{\partial t^2}, \\ \text{and accordingly} \quad \rho R &= -\rho \frac{\partial^2 u}{\partial t^2}, \quad \text{and} \quad \rho Z = -\rho \frac{\partial^2 w}{\partial t^2}. \end{aligned} \right\} \dots\dots\dots (28)$$

In this way the system of equations (1), together with Hooke's law, represents the case of symmetrical vibration of the solid. Using (3) and (28), equation (27) becomes, after a simple rearrangement,

$$(1-\sigma) \nabla^4 \chi' - (3-4\sigma) \frac{1+\sigma}{E} \rho \frac{\partial^2}{\partial t^2} \nabla^2 \chi' + 2(1-2\sigma) \frac{(1+\sigma)^2}{E^2} \rho^2 \frac{\partial^4 \chi'}{\partial t^4} = 0;$$

or by using relations between elastic constants

$$\lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}, \quad \mu = \frac{E}{2(1+\sigma)},$$

the equation just written takes the form

$$\frac{\partial^4 \chi'}{\partial t^4} - \frac{\lambda+3\mu}{\rho} \frac{\partial^2}{\partial t^2} \nabla^2 \chi' + \frac{\lambda+2\mu}{\rho} \frac{\mu}{\rho} \nabla^4 \chi' = 0.$$

This equation may also be written in the form

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\mu}{\rho} \nabla^2 \right) \left(\frac{\partial^2}{\partial t^2} - \frac{\lambda+2\mu}{\rho} \nabla^2 \right) \chi' = 0.$$

We introduce the operators

$$\square_1 = \left(\nabla^2 - a^2 \frac{\partial^2}{\partial t^2} \right), \quad \square_2 = \left(\nabla^2 - b^2 \frac{\partial^2}{\partial t^2} \right),$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \quad a^2 = \frac{\rho}{\lambda + 2\mu}, \quad b^2 = \frac{\rho}{\mu},$$

a, b having been termed the wave-slownesses by W. R. Hamilton; and then the above equation is written

$$\square_1 \square_2 \chi' = 0. \quad \dots\dots\dots (29)$$

Again, in virtue of (3) and (28), we have from (25) and (26) :

i) for the stress-components

$$\left. \begin{aligned} \widehat{r\bar{r}} &= \frac{\partial}{\partial z} \left\{ \sigma \square_2 - \frac{\partial^2}{\partial r^2} \right\} \chi', \\ \widehat{\theta\theta} &= \frac{\partial}{\partial z} \left\{ \sigma \square_2 - \frac{1}{r} \frac{\partial}{\partial r} \right\} \chi', \\ \widehat{z\bar{z}} &= \frac{\partial}{\partial z} \left\{ \sigma \square_2 + 2(1+\sigma) \square_1 - \frac{\partial^2}{\partial z^2} \right\} \chi', \\ \widehat{r\bar{z}} &= \frac{\partial}{\partial r} \left\{ (1-\sigma) \square_1 - \frac{\partial^2}{\partial z^2} \right\} \chi'; \end{aligned} \right\}$$

ii) for the displacement-components

$$u = -\frac{1}{2\mu} \frac{\partial^2 \chi'}{\partial r \partial z}, \quad w = \frac{1}{2\mu} \left\{ 2(1-\sigma) \square_1 - \frac{\partial^2}{\partial z^2} \right\} \chi';$$

iii) for the dilatation and the rotation

$$\Delta = \frac{1-2\sigma}{2\mu} \frac{\partial}{\partial z} \square_2 \chi', \quad w = -\frac{1-\sigma}{2\mu} \frac{\partial}{\partial r} \square_1 \chi'.$$

11. The function which satisfies the above equation (29) is in general divided into three parts; i.e.

$$\chi' = \chi_1 + \chi_2 + \chi_3,$$

each of them satisfying the equations

$$\square_1 \chi_1 = 0, \quad \square_2 \chi_2 = 0, \quad \square_1 \square_2 \chi_3 = 0.$$

Of these the first two functions will at once be seen to be in accordance with those due to H. Lamb.

i) For χ_1 we may obtain

$$\left. \begin{aligned} \widehat{r\bar{r}}_1 &= \left\{ \sigma (a^2 - b^2) \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2} \right\} \chi_1, & \widehat{\theta\theta}_1 &= \left\{ \sigma (a^2 - b^2) \frac{\partial^2}{\partial t^2} - \frac{1}{r} \frac{\partial}{\partial r} \right\} \chi_1, \\ \widehat{z\bar{z}}_1 &= \left\{ \sigma (a^2 - b^2) \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2} \right\} \chi_1, & \widehat{r\bar{z}}_1 &= -\frac{\partial^2 \chi_1}{\partial r \partial z}, \\ u_1 &= -\frac{1}{2\mu} \frac{\partial \chi_1}{\partial r}, & w_1 &= -\frac{1}{2\mu} \frac{\partial \chi_1}{\partial z}, \end{aligned} \right\}$$

$$\Delta_1 = \frac{1-2\sigma}{2\mu} (a^2 - b^2) \frac{\partial^2 \chi_1}{\partial t^2}, \quad \omega_1 = 0,$$

where, for brevity, χ_1 is replaced by $\partial\chi_1/\partial z$.

ii) For χ_2 we may also obtain

$$\left. \begin{aligned} \widehat{r}r_2 &= \frac{\partial^3 \chi_2}{\partial r^2 \partial z}, & \widehat{\theta}\theta_2 &= \frac{1}{r} \frac{\partial^2 \chi_2}{\partial r \partial z}, & \widehat{z}z_2 &= \frac{\partial}{\partial z} \left\{ -b^2 \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial z^2} \right\} \chi_2, \\ \widehat{r}z_2 &= \frac{\partial}{\partial r} \left\{ -\frac{b^2}{2} \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial z^2} \right\} \chi_2, & & & & \\ u_2 &= \frac{1}{2\mu} \frac{\partial^2 \chi_2}{\partial r \partial z}, & u_2 &= \frac{1}{2\mu} \left\{ -b^2 \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial z^2} \right\} \chi_2, & & \\ \Delta_2 &= 0, & \omega_2 &= \frac{b^2}{4\mu} \frac{\partial^3 \chi_2}{\partial r \partial t^2}. \end{aligned} \right\}$$

iii) As for χ_3 if we assume the product form of functions, we have

$\chi_3 = t^3 \times$ (Harmonics), $t^2 \times$ (Harmonics), $t \times$ (Biharmonics), and (Biharmonics), and so these have no concern with the free vibration, but will be useful for the analysis of forced or transient system of the solid.

12. This article will be devoted to the derivation of the stress-function for the visco-elastic solid governed by Voigt's law. The equations of motion are as before, while the stress-strain relations become

$$\left. \begin{aligned} \widehat{r}r &= \lambda\Delta + 2\mu e_{rr} + \lambda' \frac{\partial \Delta}{\partial t} + 2\mu' \frac{\partial e_{rr}}{\partial t}, \\ \widehat{\theta}\theta &= \lambda\Delta + 2\mu e_{\theta\theta} + \lambda' \frac{\partial \Delta}{\partial t} + 2\mu' \frac{\partial e_{\theta\theta}}{\partial t}, \\ \widehat{z}z &= \lambda\Delta + 2\mu e_{zz} + \lambda' \frac{\partial \Delta}{\partial t} + 2\mu' \frac{\partial e_{zz}}{\partial t}, \\ \widehat{r}z &= \mu e_{rz} + \mu' \frac{\partial e_{rz}}{\partial t}. \end{aligned} \right\} \dots\dots\dots (30)$$

We then have the following displacement-equations:

$$\left. \begin{aligned} &\left\{ (\lambda + 2\mu) + (\lambda' + 2\mu') \frac{\partial}{\partial t} \right\} \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} \right) + \left(\mu + \mu' \frac{\partial}{\partial t} \right) \frac{\partial^2 u}{\partial z^2} \\ &\quad + \left\{ (\lambda + \mu) + (\lambda' + \mu') \frac{\partial}{\partial t} \right\} \frac{\partial^2 w}{\partial r \partial z} = \rho \frac{\partial^2 u}{\partial t^2}, \\ &\left\{ (\lambda + \mu) + (\lambda' + \mu') \frac{\partial}{\partial t} \right\} \left(\frac{\partial^2 u}{\partial r \partial z} + \frac{1}{r} \frac{\partial u}{\partial z} \right) + \left(\mu + \mu' \frac{\partial}{\partial t} \right) \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) \\ &\quad + \left\{ (\lambda + 2\mu) + (\lambda' + 2\mu') \frac{\partial}{\partial t} \right\} \frac{\partial^2 w}{\partial z^2} = \rho \frac{\partial^2 w}{\partial t^2}. \end{aligned} \right\} \dots\dots\dots (31)$$

Now it will be convenient to introduce the following operators:

$$\left. \begin{aligned} \square_1' &= -\frac{\rho}{\lambda+2\mu} \left\{ \frac{\partial^2}{\partial t^2} - \left(\frac{\lambda+2\mu}{\rho} + \frac{\lambda'+2\mu'}{\rho} \frac{\partial}{\partial t} \right) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right) \right\} \\ &= \left(1 + \frac{a^2}{a'^2} \frac{\partial}{\partial t} \right) \nabla^2 - a^2 \frac{\partial^2}{\partial t^2}, \\ \square_2' &= -\frac{\rho}{\mu} \left\{ \frac{\partial^2}{\partial t^2} - \left(\frac{\mu}{\rho} + \frac{\mu'}{\rho} \frac{\partial}{\partial t} \right) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right) \right\} \\ &= \left(1 + \frac{b^2}{b'^2} \frac{\partial}{\partial t} \right) \nabla^2 - b^2 \frac{\partial^2}{\partial t^2}, \end{aligned} \right\} \dots\dots\dots (32)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2},$$

$$a^2 = \frac{\rho}{\lambda+2\mu}, \quad b^2 = \frac{\rho}{\mu}, \quad a'^2 = \frac{\rho}{\lambda'+2\mu'}, \quad b'^2 = \frac{\rho}{\mu'}.$$

Assuming a stress-function χ for the system under consideration, we may put

$$u = -\frac{1}{2\mu} \left(1 + \kappa \frac{\partial}{\partial t} \right) \frac{\partial^2 \chi}{\partial r \partial z}, \quad w = \frac{1}{2\mu} \left\{ 2(1-\sigma) \square_1' - \left(1 + \kappa' \frac{\partial}{\partial t} \right) \frac{\partial^2}{\partial z^2} \right\} \chi, \dots\dots\dots (33)$$

where κ, κ' are constants to be determined. Then on substituting these equations into the first of (31) and rearranging the result, we have

$$\begin{aligned} &\frac{1}{2\mu} \frac{\partial^3}{\partial r \partial z \partial t} \left[\left(\frac{\lambda'+\mu'}{\lambda+\mu} - \kappa \right) \left\{ (\lambda+2\mu) + (\lambda'+2\mu') \frac{\partial}{\partial t} \right\} \nabla^2 \right. \\ &\quad \left. + (\kappa - \kappa') \left\{ (\lambda+\mu) + (\lambda'+\mu') \frac{\partial}{\partial t} \right\} \frac{\partial^2}{\partial z^2} + \left(\kappa - \frac{\lambda'+\mu'}{\lambda+\mu} \right) \rho \frac{\partial^2}{\partial t^2} \right] \chi = 0. \end{aligned}$$

This equation is identically satisfied if we take

$$\kappa = \kappa' = \frac{\lambda'+\mu'}{\lambda+\mu}. \dots\dots\dots (34)$$

In virtue of these values of κ and κ' , the second equation of (31) can be rearranged, with (33), to the form

$$\square_1' \square_2' \chi = 0, \dots\dots\dots (35)$$

and equations (33) become

$$u = -\frac{1}{2\mu} \left(1 + \kappa \frac{\partial}{\partial t} \right) \frac{\partial^2 \chi}{\partial r \partial z}, \quad w = \frac{1}{2\mu} \left\{ 2(1-\sigma) \square_1' - \left(1 + \kappa \frac{\partial}{\partial t} \right) \frac{\partial^2}{\partial z^2} \right\} \chi, \dots\dots\dots (36)$$

where, for shortness, κ is represented by $\kappa = \frac{\lambda'+\mu'}{\lambda+\mu}$.

Then the stress-components can be expressed:

$$\left. \begin{aligned}
 \bar{r}\bar{r} &= \frac{\partial}{\partial z} \left\{ \left(1 + \frac{\lambda'}{\lambda} \frac{\partial}{\partial t} \right) \sigma \square_{z'} - \left(1 + \frac{\mu'}{\mu} \frac{\partial}{\partial t} \right) \left(1 + \kappa \frac{\partial}{\partial t} \right) \frac{\partial^2}{\partial r^2} \right\} \chi, \\
 \bar{\theta}\bar{\theta} &= \frac{\partial}{\partial z} \left\{ \left(1 + \frac{\lambda'}{\lambda} \frac{\partial}{\partial t} \right) \sigma \square_{z'} - \left(1 + \frac{\mu'}{\mu} \frac{\partial}{\partial t} \right) \left(1 + \kappa \frac{\partial}{\partial t} \right) \frac{1}{r} \frac{\partial}{\partial r} \right\} \chi, \\
 \bar{z}\bar{z} &= \frac{\partial}{\partial z} \left\{ \left(1 + \frac{\lambda'}{\lambda} \frac{\partial}{\partial t} \right) \sigma \square_{z'} + \left(1 + \frac{\mu'}{\mu} \frac{\partial}{\partial t} \right) 2(1-\sigma) \square_{1'} \right. \\
 &\quad \left. - \left(1 + \frac{\mu'}{\mu} \frac{\partial}{\partial t} \right) \left(1 + \kappa \frac{\partial}{\partial t} \right) \frac{\partial^2}{\partial z^2} \right\} \chi, \\
 \bar{r}\bar{z} &= \frac{\partial}{\partial r} \left(1 + \frac{\mu'}{\mu} \frac{\partial}{\partial t} \right) \left\{ (1-\sigma) \square_{1'} - \left(1 + \kappa \frac{\partial}{\partial t} \right) \frac{\partial^2}{\partial z^2} \right\} \chi;
 \end{aligned} \right\} \dots (37)$$

and the dilatation and rotation become:

$$\Delta = \frac{1-2\sigma}{2\mu} \frac{\partial}{\partial z} \square_{z'} \chi, \quad w = -\frac{1-\sigma}{2\mu} \frac{\partial}{\partial r} \square_{1'} \chi.$$

13. This article will show that the known stress-function for the solid of revolution, subjected to surface tractions only, is compatible with any given boundary conditions. The equations of equilibrium for the solid, referred to the displacement-components, are

$$\left. \begin{aligned}
 2(1-\sigma) \left\{ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} \right\} + (1-2\sigma) \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial r \partial z} &= 0, \\
 \frac{\partial^2 u}{\partial r \partial z} + \frac{1}{r} \frac{\partial u}{\partial z} + (1-2\sigma) \left\{ \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right\} + 2(1-\sigma) \frac{\partial^2 w}{\partial z^2} &= 0.
 \end{aligned} \right\} \dots (38)$$

We know that these equations are satisfied by

$$u = -\frac{1}{2\mu} \frac{\partial^2 \chi}{\partial r \partial z}, \quad w = \frac{1}{2\mu} \left\{ 2(1-\sigma) r^2 - \frac{\partial^2}{\partial z^2} \right\} \chi, \dots (39)$$

where χ satisfies the equation

$$\nabla^4 \chi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right)^2 \chi = 0. \dots (40)$$

Now we take the first equation of (39); and let it be taken as a differential equation. Any boundary problem concerning the solid will have its solution, and the component u will take a definite form of functions for the boundary problem. We then may have for χ

$$\chi = -2\mu \iint U dr dz,$$

where no arbitrary functions of integration need be added, for such functions will have no concern with the elastic displacement under consideration, as well as with the stress distribution within the solid.

It thus follows that, for completing the compatibility of the stress-function χ with the given boundary condition, it is sufficient to write

$$u = -\frac{1}{2\mu} \frac{\partial^2 \chi}{\partial r \partial z}, \quad w = \frac{1}{2\mu} \left\{ 2(1-\sigma) r^2 - \frac{\partial^2}{\partial z^2} \right\} \chi + w', \quad \dots (41)$$

w' being some function of r and z . Then the substitution of (40) into the fundamental equations (38) will afford

$$\frac{\partial^2 w'}{\partial r \partial z} = 0, \quad (1-2\sigma) \left\{ \frac{\partial^2 w'}{\partial r^2} + \frac{1}{r} \frac{\partial w'}{\partial r} \right\} + 2(1-\sigma) \frac{\partial^2 w'}{\partial z^2} = 0. \quad \dots (42)$$

From the first of these equations, we at once have

$$w' = R(r) + Z(z), \quad \dots (43)$$

$R(r)$ and $Z(z)$ being functions of r and z respectively. On substituting this into the second equation of (42), we have

$$\frac{1-2\sigma}{2(1-\sigma)} \left(\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} \right) + \frac{d^2 Z}{dz^2} = 0;$$

and this equation holds only when

$$\frac{1-2\sigma}{2(1-\sigma)} \left(\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} \right) = 2\alpha, \quad \frac{d^2 Z}{dz^2} = -2\alpha,$$

α being constant. The solutions of these equations are respectively

$$R(r) = \frac{1-\sigma}{1-2\sigma} \alpha r^2 + a \log r + c', \quad Z(z) = -\alpha z^2 + bz + c''.$$

a, b, c', c'' are all constants. Thus from (42) we have

$$w' = \alpha \left(\frac{1-\sigma}{1-2\sigma} r^2 - z^2 \right) + a \log r + bz + c, \quad \dots (44)$$

c being for shortness written for $c' + c''$.

The stress-components are in general given by the equations

$$\begin{aligned} \widehat{r_r} &= \frac{2\mu}{1-2\sigma} \left\{ (1-\sigma) \frac{\partial u}{\partial r} + \sigma \left(\frac{u}{r} + \frac{\partial w}{\partial z} \right) \right\}, \quad \widehat{\theta\theta} = \frac{2\mu}{1-2\sigma} \left\{ (1-\sigma) \frac{u}{r} + \sigma \left(\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} \right) \right\}, \\ \widehat{z_z} &= \frac{2\mu}{1-2\sigma} \left\{ (1-\sigma) \frac{\partial w}{\partial z} + \sigma \left(\frac{\partial u}{\partial r} + \frac{u}{r} \right) \right\}, \quad \widehat{r_z} = \mu \left\{ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right\}; \end{aligned}$$

and hence (44) affords the following stress-components

$$\widehat{r_r} = Az + C, \quad \widehat{\theta\theta} = Az + C, \quad \widehat{z_z} = \frac{1-\sigma}{\sigma} (Az + C), \quad \widehat{r_z} = -\frac{1-\sigma}{2\sigma} Ar + \frac{B}{r}, \quad \dots (45)$$

where, for convenience sake,

$$A = -\frac{4\mu\sigma}{1-2\sigma} \alpha, \quad B = \mu a, \quad C = \frac{2\mu\sigma}{1-2\sigma} b.$$

Also, the displacement-components take the forms

$$u' = 0, \quad w' = \frac{A}{4\mu\sigma} \left\{ -(1-\sigma) r^2 + (1-2\sigma) z^2 \right\} + \frac{B}{\mu} \log r + \frac{1-2\sigma}{2\mu\sigma} Cz. \quad \dots (46)$$

We shall examine that these functions can be derived from the stress-function which satisfies the biharmonic equation (40). We know that in terms of the stress-function the stress-components are given by the operations

$$\left. \begin{aligned} \bar{r}\bar{r} &= \frac{\partial}{\partial z} \left\{ \sigma r^2 - \frac{\partial^2}{\partial r^2} \right\} \chi, & \theta\theta &= \frac{\partial}{\partial z} \left\{ \sigma r^2 - \frac{1}{r} \frac{\partial}{\partial r} \right\} \chi, \\ \bar{z}\bar{z} &= \frac{\partial}{\partial z} \left\{ (2-\sigma) r^2 - \frac{\partial^2}{\partial z^2} \right\} \chi, & \bar{r}z &= \frac{\partial}{\partial r} \left\{ (1-\sigma) r^2 - \frac{\partial^2}{\partial z^2} \right\} \chi. \end{aligned} \right\} \dots\dots\dots (47)$$

Firstly, as to the function whose constant is *C* it can easily be seen that it is the same as the function $\chi = z^3$, which is a particular solution of (40).

Secondly, as to the function whose constant is *B*, we have, from the fourth of (47),

$$\frac{\partial}{\partial r} \left\{ (1-\sigma) r^2 - \frac{\partial^2}{\partial z^2} \right\} \chi = \frac{1}{r},$$

which will be equivalent to

$$(1-\sigma) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) \chi = \log r.$$

The particular solution of this equation is

$$\chi = \frac{1}{4(1-\sigma)} (r^2 \log r - r^2),$$

and this is a biharmonic function as before.

Thirdly, we take the function whose constant is *A*. If this stress system can be derived from the stress-function χ , we must have from the first of (41)

$$\chi = R(r) + Z(z), \dots\dots\dots (48)$$

R(*r*) and *Z*(*z*) being functions of *r* and *z* only respectively. Then from the second equation of (46) and that of (39), we have

$$2(1-\sigma) \left\{ \frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \frac{1}{4\sigma} r^2 \right\} + (1-2\sigma) \left\{ \frac{d^2 Z}{dz^2} - \frac{1}{2\sigma} z^2 \right\} = 0,$$

which holds only when

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \frac{1}{4\sigma} r^2 = \frac{c}{2(1-\sigma)}, \quad \frac{d^2 Z}{dz^2} - \frac{1}{2\sigma} z^2 = -\frac{c}{1-2\sigma};$$

and the general solution of these equations are

$$R(r) = -\frac{1}{64\sigma} r^4 + \frac{c}{8(1-\sigma)} r^2 + c_1 \log r + c', \quad Z(z) = \frac{1}{24\sigma} z^4 - \frac{c}{2(1-2\sigma)} z^2 + c_2 z + c'',$$

*c*₁, *c*₂, *c*' , *c*'' being constant. Therefore (48) results in

$$\chi = -\frac{1}{64\sigma} r^4 + \frac{1}{24\sigma} z^4 + \frac{c}{8(1-\sigma)} r^2 - \frac{c}{2(1-2\sigma)} z^2 + c_1 \log r + c_2 z + c_3,$$

where *c*₃ is written for *c*' + *c*'' . It is evident here that in this equation the terms in *c*, *c*₁, *c*₂, *c*₃ have no importance as to the stress distribution, since they are included in biharmonic functions. Hence we may take, without loss of generality,

$$\chi = \frac{A}{8\sigma} \left(-\frac{1}{8}r^4 + \frac{1}{3}z^4 \right),$$

and this is also a biharmonic function.

Thus it can be concluded that the additional function u' given in (41) may be included in the biharmonic function χ , and accordingly that the w' does by no means play any significant role on the stress- and displacement-distributions within the solid; and hence that equations (39), provided that the function χ in these equations satisfies the differential equation (40), is equivalent to the original simultaneous equations (38), or in other words that the equations (39) and (40) are the solution of the original simultaneous equations (38).

14. As a simple application of the stress-function given in Article 5, which is for the solid of revolution rotating about its axis with uniform angular velocity ω , a rotating hollow shaft will be treated. Let the length of the shaft be $2l$, the outer radius R , and the inner radius R' ; the origin of coordinates being at the center of gravity of the solid.

Boundary conditions are taken to be

$$\left. \begin{aligned} \widehat{rz} = \widehat{r'r} = 0 \text{ for } r=R \text{ and } r=R', \\ \widehat{rz} = 0 \text{ and } \int_{R'}^R \widehat{z\bar{z}} r dr = 0 \text{ for } z = \pm l. \end{aligned} \right\}$$

Fundamental equations to be referred are equations (15) and (16), in which χ_0 is biharmonic. Particular solutions suitable for the present problem are

$$\chi_0 = Az \log r + Br^2z + Cz^3,$$

and in addition we may have $\beta = \gamma = 0$.

Then we can arrive at the following results:

$$\left. \begin{aligned} \widehat{r'r} &= \frac{\rho\omega^2 3 - 2\sigma}{8} \frac{1}{1-\sigma} (R^2 - r^2) - \frac{\rho\omega^2 3 - 2\sigma}{8} \frac{1}{1-\sigma} R'^2 \left(\frac{R^2}{r^2} - 1 \right), \\ \widehat{\theta\theta} &= \frac{\rho\omega^2}{8} \frac{1}{1-\sigma} \left\{ (3-2\sigma)R^2 - (1+2\sigma)r^2 \right\} + \frac{\rho\omega^2 3 - 2\sigma}{8} \frac{1}{1-\sigma} R'^2 \left(\frac{R^2}{r^2} + 1 \right), \\ \widehat{z\bar{z}} &= \frac{\rho\omega^2}{4} \frac{\sigma}{1-\sigma} (R^2 - 2r^2) + \frac{\rho\omega^2}{4} \frac{\sigma}{1-\sigma} R'^2, \quad \widehat{rz} = 0, \\ u &= \frac{\rho\omega^2 3 - 5\sigma}{8E} \frac{1}{1-\sigma} \left\{ R^2 - \frac{(1+\sigma)(1-2\sigma)}{3-5\sigma} r^2 \right\} r \\ &\quad + \frac{\rho\omega^2 3 - 5\sigma}{8E} \frac{1}{1-\sigma} R'^2 \left\{ \frac{(3-2\sigma)(1+\sigma)}{3-5\sigma} \frac{R^2}{r^2} + 1 \right\} r, \\ w &= -\frac{\rho\omega^2}{2E} \sigma R^2 z - \frac{\rho\omega^2}{2E} \sigma R'^2 z. \end{aligned} \right\}$$

It can be seen that if we put $R=0$, the above solution reduces to the known one for the solid shaft³⁾.

Summary

The present article gives stress-functions for the following respective cases:

1. General expressions of the stress-function for the stress-equations (1) are given, provided Hooke's law holds for the stress-strain relations.

2. As the first application of the above reduction 1., a stress-function will at once follow, for the solid of revolution subjected to uniform gravitation parallel to the z -axis.

3. As the second application of 1., stress-function is derived for the solid of revolution rotating with uniform angular velocity so that centrifugal force may exert upon it.

4. If in equations (1) we put

$$\rho R = -\rho \frac{\partial^2 u}{\partial t^2}, \quad \rho Z = -\rho \frac{\partial^2 w}{\partial t^2},$$

the reduction made in 1. transforms to the dynamical system of equations, viz., the longitudinal vibration of the solid. In this case also a stress-function can be derived by a slight modification of the results obtained in 1.

5. As a further development of the above item 4., stress-function for visco-elastic solid which obeys Voigt's law can be obtained.

6. When the solid is strained by an uneven temperature distribution in it, stress-function is also derived.

As to the differential equations in the above two items, 5. and 6., it would seem to call for a further scrutiny.

Several applications of the above stress-functions to elementary boundary problems have been worked out, the results of which are of course in accordance with those treated otherwise, when available³⁾; but they are not given here.

Acknowledgements

I should like to express my hearty thanks to Dr. Yutaka TANAKA, member of Japan Academy and Honorary Professor of Tokyo University, for

his grateful and constant supervision. My thanks are also due to Dr. Tomoyasu YUKI, Dean and Professor of Faculty of Engineering, Shinshu University, for his kind consideration in publishing this article. Again, I acknowledge the Ministry of Education for giving me a financial aid.

References

- 1) 500 Wakazato, Nagano City, Honshu, Japan.
- 2) Love, *Elasticity*, 4th ed., pp. 274-276.
- 3) Loc. cit. 2), pp. 146-148, pp. 289-290, etc.; S. Timoshenko, *Theory of Elasticity*, 1934, Chapter XI.