# ON STIELTJÉS INTEGRAL IN SEMI-ORDERED LINEAR SPACE[1]

By

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H. Nakano defined Stieltjés integral in a semi-ordered linear space as follow in his book.[2]

Let R be a continuous semi-ordered linear space, if $a_\lambda (\alpha \leqq \lambda \leqq \beta)$ are monotone increasing and $\varphi(\lambda)$ is a continuous function in $[\alpha, \beta]$, then the limit

$$\lim_{\varepsilon \to 0} \sum_{\nu=1}^{n} \varphi(\tau_\nu)(a_{\lambda_\nu} - a_{\lambda_{\nu-1}})$$

exists for every partition of the interval $[\alpha, \beta]$:

$$\alpha = \lambda_0 < \lambda_1 < \cdots\cdots < \lambda_n = \beta, \qquad \lambda_\nu - \lambda_{\nu-1} \leqq \varepsilon \quad (\nu = 1, 2, \cdots, n)$$

$$\lambda_{\nu-1} \leqq \tau_\nu \leqq \lambda_\nu$$

This limiting value is written as $\int_\alpha^\beta \varphi(\lambda)\, da_\lambda$ and called the Stieltjés integral of $\varphi(\lambda)$ in $[\alpha, \beta]$.

When the semi-ordered linear space is regularly complete,[3] we can extend the definition of Stieltjés integral for a totally additive class of functions containing continuous functions, by the method used by H. Nakano in his book.[4] This is the purpose of my paper.

## § I Almost continuous functions

A function $\varphi(\lambda)$ is defined in a point set M. Let $a_\lambda \in R (\alpha \leqq \lambda \leqq \beta)$ be a monotone increasing system. For an interval $I = (\alpha, \beta)$, we define its length $|I|$ to mean $|I| = a_{\beta-0} - a_{\alpha+0}$ where $a_{\beta-0} = \lim_{\lambda \to \beta-0} a_\lambda = \bigcup_{\lambda < \beta} a_\lambda$ and $a_{\alpha+0} = \lim_{\lambda \to \alpha+0} a_\lambda = \bigcap_{\lambda > \alpha} a_\lambda$

For a function $\varphi(\lambda)$, if we can find a double sequence of intervals $I_{\nu.\mu}$ $(\nu.\mu = 1, 2, \cdots)$ such that

$$b_\nu = |I_{\nu.1}| + |I_{\nu.2}| + \cdots \downarrow_{\nu=1}^{\infty} 0$$

and $\varphi(\lambda)$ is continuous in $M(I_{\nu.1} \dot{+} I_{\nu.2} + \cdots)'$ for every $\nu = 1, 2, \cdots$, then $\varphi(\lambda)$ is said to be almost continuous in $M$.

*Definition.*[5] If for any $a_{\mu.\nu} \downarrow_{\nu=1}^{\infty} 0 (\mu=1, 2, \cdots)$ there exists $\nu_\mu \uparrow_{\mu=1}^{\infty} +\infty$ such that $a_{\mu.\nu_\mu}$ $(\mu=1, 2, \cdots)$ have an upper bound, then $R$ is said to be *regularly complete.*

*Lemma* I. If $R$ is regularly complete, then for any $a_{\mu.\nu} \downarrow_{\nu=1}^{\infty} 0$ $(\mu=1, 2, \cdots)$ we can find $\nu_\mu (\mu=1, 2, \cdots)$ for which $\sum_{\mu=1}^{\infty} a_{\mu.}\nu_\mu$ is convergent.

*Proof.* As $a_{\mu.\nu} \downarrow_{\nu=1}^{\infty} 0 (\mu=1, 2, \cdots)$ by assumption, we have

$$2^\mu a_{\mu.\nu} \downarrow_{\nu=1}^{\infty} 0 \quad (\mu=1, 2, \cdots)$$

Since $R$ is regularly complete, we obtain hence

$$2^\mu a_{\mu.}\nu_\mu \leqq l \quad \text{that is,} \quad 0 \leqq a_{\mu.}\nu_\mu \leqq \frac{1}{2^\mu} l$$

Therefore $\sum_{\mu=1}^{\infty} a_{\mu.}\nu_\mu$ is convergent, because $\sum_{\mu=1}^{\infty} \frac{1}{2^\mu} l = l.$

*Lemma* II. If $R$ is regularly complete, then for any $a_{\mu.\nu} \downarrow_{\nu=1}^{\infty} 0$ $(\mu=1. 2, \cdots)$ we can find $\nu_\mu{}^\rho (\mu.\rho=1, 2, \cdots)$ such that $\sum_{\mu=1}^{\infty} a_{\mu.}\nu_\mu{}^\rho \downarrow_{\rho=1}^{\infty} 0$

*Proof.* As $a_{\mu.\nu} \downarrow_{\nu=1}^{\infty} 0$ $(\mu=1, 2, \cdots)$ by assumption we have

$$2^\rho a_{\mu.\nu} \downarrow_{\nu=1}^{\infty} 0 \quad (\rho.\mu=1, 2, \cdots)$$

and by lemma I, we can find $\nu_\mu{}^\rho$ for which $\sum_{\mu.\rho} 2^\rho a_{\mu,}\, \nu_\mu{}^\rho$ is convergent. Thus we can find an element $m$ such that

$$m \geqq 2^\rho \sum_{\mu=1}^{\infty} a_{\mu,}\, \nu_\mu{}^\rho \qquad \text{for every } \rho=1, 2, \cdots$$

we have then $\quad \sum_{\mu=1}^{\infty} a_{\mu.}\, \nu_\mu{}^\rho \leqq \frac{1}{2^\rho} m \downarrow_{\rho=1}^{\infty} 0$

*Lemma* III. If $b_{\mu,\nu} \downarrow_{\mu=1}^{\infty} 0$ $(\nu=1, 2, \cdots)$, and $\sum_{\nu=1}^{\infty} b_{1.\nu}$ is convergent, then we have $\quad \sum_{\nu=1}^{\infty} b_{\mu.\nu} \downarrow_{\mu=1}^{\infty} 0$

*Proof.* Putting $\quad c = \lim_{\mu\to\infty} \sum_{\nu=1}^{\infty} b_{\mu.\nu}$ We have

$$0 \leqq c \leqq \sum_{\nu=1}^{\lambda-1} b_{\mu.\nu} + \sum_{\nu=\lambda}^{\infty} b_{1.\nu}$$

For fixed $\lambda$, making $\mu\to\infty$, we obtain by assumption

$$0 \leqq c \leqq \sum_{\nu=\lambda}^{\infty} b_{1.\nu}$$

On the other hand $\sum_{\nu=1}^{\infty} b_{1.\nu}$ is convergent by assumption and hence $\lim_{\lambda\to\infty} \sum_{\nu=\lambda}^{\infty} b_{1.\nu}=0$, consequently we obtain $c=0$.

*Theorem* I. If a function $\varphi(\lambda)$ is upper semi-continuous or lower semi-continuous in a point set $M$, then $\varphi(\lambda)$ is almost continuous in $M$.

*Proof.* In the first place, we consider the case where $M$ is bounded, that is, the interval $(\alpha, \beta)$ includes $M$, and $\varphi(\lambda)$ is upper semi-continuous in $M$. As the whole rational numbers are countable, they may be written as $\alpha_1, \alpha_2, \cdots$. The point set $\{\lambda : \varphi(\lambda) < \alpha_n\}$ $(n=1, 2, \cdots)$ is an open set with regard to $M$, that is, it is the intersection of $M$ and an open set.

While on a straight line, an open set is sum of at most countable intervals, they may be written as $I_{n.1} \dot{+} I_{n.2} \dot{+} \cdots$ $I_{n.\mu} I_{n.\rho} = 0$ $(\mu \neq \rho)$ and since they are included by $(\alpha, \beta)$, sum of their lengthes is convergent because

$$a_{n.1} = |I_{n.1}| + |I_{n.2}| + \cdots \leqq a_\beta - a_\alpha$$

and for a point $\lambda$ of $M$ which is not included by any of $I_{n.1}, I_{n.2}, \cdots$ we have $\varphi(\lambda) \geqq \alpha_n$ for $\lambda \in M(I_{n.1} \dot{+} I_{n.2} \dot{+} \cdots)'$ and

$$|I_{n.n'}| + |I_{n.n'+1}| + \cdots \downarrow_{n'=1}^{\infty} 0$$

For $I_{n.\mu} = (\alpha_{n.\mu}, \beta_{n.\mu})$ $(\mu=1, 2, \cdots)$, if we define

$$I_{n.\rho} = I_{n.1.\rho} = I_{n.2.\rho} = \cdots = I_{n.\rho.\rho} \qquad (\rho=1, 2, \cdots)$$

and $I_{n.v.\rho} = (\alpha_{n.\rho}, \alpha_{n.\rho} + \frac{\beta_{n.\rho} - \alpha_{n.\rho}}{2^{v-\rho+1}})$, $J_{n.v.\rho} = (\beta_{n.\rho} - \frac{\beta_{n.\rho} - \alpha_{n.\rho}}{2^{v-\rho+1}}, \beta_{n.\rho})$ $(v > \rho)$ $(\rho=1,2,\cdots)$

then, putting $b_{n.v} = \sum_{\kappa=1}^{\infty} |I_{n.v.\kappa}| + \sum_{\kappa=1}^{v-1} |J_{n.v.\kappa}|$, we obtain $b_{n.v} \downarrow_{v=1}^{\infty} 0$ by lemma III. In the set $M_n$ which excludes $\sum_{\kappa=1}^{v-1} I_{n.v.\kappa} + \sum_{\kappa=1}^{v-1} J_{n.v.\kappa} + \sum_{\kappa=v}^{\infty} I_{n.v.\kappa}$ from $M$, if we consider a point $\lambda_0$ which satisfies $\varphi(\lambda_0) \geqq \alpha_n$, then for any point of $M_n$ in the interval $(\lambda_0 - \delta, \lambda_0 + \delta)$, we have always $\varphi(\lambda) \geqq \alpha_n$, where $\delta = \min_{\rho=1,2,\cdots,v} \frac{\beta_{n.\rho} - \alpha_{n.\rho}}{2^{v-\rho+1}}$

If we apply the same method to $n=1, 2, \cdots$, then we have

$$b_{\mu.v} \downarrow_{v=1}^{\infty} 0 \qquad (\mu=1, 2, \cdots)$$

By lemma II $\sum_{\mu=1}^{\infty} b_{\mu.v_{\mu\rho}} = \sum_{\mu=1}^{\infty} \sum_{\kappa=1}^{\infty} |I_{\mu.v_{\mu\rho}.\kappa}| \downarrow_{\rho=1}^{\infty} 0$ $\qquad$ (*)

In $M_0$ which excludes (*) from $M$, for any rational number $\alpha$, if we consider a point $\lambda_0$ which satisfies $\varphi(\lambda_0) > \alpha$ and choose suitable positive number $\delta$, then a point of $M_0$ in the interval $(\lambda_0 - \delta, \lambda_0 + \delta)$ satisfies $\varphi(\lambda) > \alpha$. Therefore, for any point $\lambda_0$ of $M_0$, if we consider a rational number $\alpha$ which satisfies $\varphi(\lambda_0) > \alpha$, then we obtain the same conclusion. Consequently in $M_0$ $\underline{\lim}_{\lambda \to \lambda_0} {}_{M_0} \varphi(\lambda) \geqq \alpha$. Hence for the limit $\alpha \to \varphi(\lambda_0)$ $\underline{\lim}_{\lambda \to \lambda_0} {}_{M_0} \varphi(\lambda) \geqq \varphi(\lambda_0)$ with regard to $M_0$.

Therefore $\varphi(\lambda)$ is lower semi-continuous in $M_0$, and $\varphi(\lambda)$ is upper semi-continuous in $M$, therefore $\varphi(\lambda)$ is almost continuous in $M$. If $\varphi(\lambda)$ is lower semi-continuous, then $-\varphi(\lambda)$ is upper semi-continuous, and so by the same method, $\varphi(\lambda)$ is almost continuous in $M$.

Next we consider the case where $M$ is not bounded, if we suppose $M_\beta$ is the intersection of $M$ and an interval $[-\beta, \beta]$, then $M_\beta$ is a bounded set, and $\varphi(\lambda)$ is upper semi-continuous in $M_\beta$, therefore in a set $M_\beta'$ which excludes

$$c_{\beta.v} = |I_{\beta.v1}| + |I_{\beta.v.2}| + \cdots \downarrow_{v=1}^{\infty} 0 \quad (v=1, 2, \cdots)$$

from $M_\beta$, $\varphi(\lambda)$ is continuous, if we consider every natural number $\beta$, then $c_{\mu.v} \downarrow_{v=1}^{\infty} 0 \quad (\mu=1, 2, \cdots)$

And by lemma II $\sum_{\mu=1}^{\infty} c_{\mu.} v_\mu{}^\rho \downarrow_{\rho=1}^{\infty} 0$ $\quad (\divideontimes)$

consequently $\varphi(\lambda)$ is continuous in a set which excludes points of at most countable intervals $(\divideontimes)$ from $M$, that is, $\varphi(\lambda)$ is almost continuous in $M$.

We obtain at once by definition.

*Theorem* II. If $\varphi(\lambda)$ and $\psi(\lambda)$ are almost continuous in a point set $M$, then $\varphi(\lambda) \pm \psi(\lambda)$, $\varphi(\lambda) \cdot \psi(\lambda)$, $\max[\varphi(\lambda), \psi(\lambda)]$ and $\min[\varphi(\lambda), \psi(\lambda)]$ are almost continuous, and if $\psi(\lambda) \neq 0$ in $M$, then the same conclusion holds for $\varphi(\lambda)/\psi(\lambda)$.

*Theorem* III. If all functions $\varphi_1(\lambda)$, $\varphi_2(\lambda)$, $\cdots$ are almost continuous in a point set $M$, and $\overline{\varphi}(\lambda) = \overline{\lim_{n\to\infty}}\varphi_n(\lambda)$, $\underline{\varphi}(\lambda) = \underline{\lim_{n\to\infty}}\varphi_n(\lambda)$ are finite functions, then $\overline{\varphi}(\lambda)$ and $\underline{\varphi}(\lambda)$ are almost continuous in $M$.

*Proof*. Before the proof of the general case we consider firstly the following special case. $\varphi_1(\lambda) \leqq \varphi_2(\lambda) \leqq \cdots$, if we put $\varphi_0(\lambda) = \lim_{n\to\infty} \varphi_n(\lambda)$, then by assumption $\varphi_0(\lambda)$ is a finite function, and $\varphi_n(\lambda)$ is almost continuous in $M$; and hence we can find $I_{n.v.\kappa}$ $(n.v.\kappa=1, 2, \cdots)$ such that

$$b_{n.v} = |I_{n.v.1}| + |I_{n.v.2}| + \cdots = \sum_{\kappa=1}^{\infty} |I_{n.v.\kappa}| \downarrow_{v=1}^{\infty} 0$$

and $\varphi_n(\lambda)$ is continuous in $M(I_{n.v.1}+I_{n.v.2}+\cdots)'$. By virtue of lemma II, we can find $v_\mu^\rho$ such that

$$b_\rho = \sum_{\mu=1}^{\infty} b_{\mu.v_\mu\rho} = \sum_{\mu=1}^{\infty} \sum_{\kappa=1}^{\infty} |I_{\mu.v_\mu\rho.\kappa}| \downarrow_{\rho=1}^{\infty} 0$$

and all functions $\varphi_1(\lambda)$, $\varphi_2(\lambda)$, $\cdots$ are continuous in the set $M_1 = M \times (\sum_{\mu=1}^{\infty} \sum_{\kappa=1}^{\infty} I_{\mu.v_\mu\rho.\kappa})'$. Thus $\varphi_0(\lambda)$ is lower semi-continuous in $M_1$, and consequently almost continuous in $M_1$ by theorem I. Therefore $\varphi_0(\lambda)$ is continuous in $M_2$ which excludes at most countable intervals, the sum of whose lengthes is decreasing and tends to zero.

After all, $\varphi_0(\lambda)$ is continuous in $M_2$ which excludes at most countable intervals, the sum of whose lengthes tends to zero, that is, almost continuous in $M$.

If $\varphi_1(\lambda) \geqq \varphi_2(\lambda) \geqq \cdots$ then $-\varphi_1(\lambda) \leqq -\varphi_2(\lambda) \leqq \cdots$ and therefore

$$\varphi_0(\lambda) = \lim_{n\to\infty} \varphi_n(\lambda) = -\lim_{n\to\infty}(-\varphi_n(\lambda)) \quad \text{is almost continuous in } M.$$

Next we prove the general case. As

$$\overline{\lim_{n\to\infty}} \varphi_n(\lambda) = \inf[\sup\{\varphi_n(\lambda), \varphi_{n+1}(\lambda), \cdots\}] \quad \text{if we put}$$

$$\varphi_{n.m}(\lambda) = \max[\varphi_n(\lambda), \varphi_{n+1}(\lambda), \ldots, \varphi_m(\lambda)] \quad (m>n),$$

then by theorem II, $\varphi_{n.m}(\lambda)$ is almost continuous in $M$, and

$$\varphi_{n.n}(\lambda) \leqq \varphi_{n.n+1}(\lambda) \leqq \cdots$$

$$\lim_{m\to\infty} \varphi_{n.m}(\lambda) = \sup\{\varphi_n(\lambda), \varphi_{n+1}(\lambda), \cdots\}$$

therefore $\sup\{\varphi_n(\lambda), \varphi_{n+1}(\lambda), \cdots\}$ is almost continuous in $M$.

If we put $g_n(\lambda) = \sup\{\varphi_n(\lambda), \varphi_{n+1}(\lambda), \cdots\}$
then $g_1(\lambda) \geqq g_2(\lambda) \geqq \cdots$

and
$$\lim_{n\to\infty} g_n(\lambda) = \overline{\lim_{n\to\infty}} \varphi_n(\lambda) = \overline{\varphi}(\lambda)$$

and therefore $\overline{\varphi}(\lambda)$ is almost continuous in $M$, and as

$$\underline{\varphi}(\lambda) = \underline{\lim_{n\to\infty}} \varphi_n(\lambda) = -\overline{\lim_{n\to\infty}}(-\varphi_n(\lambda))$$

and $-\varphi_n(\lambda)$ is almost continuous in $M$, and so $\underline{\varphi}(\lambda)$ is also continuous in $M$.

§ 2. The integral of bounded almost continuous functions

If a function $\varphi(\lambda)$ is bounded and almost continuous in a closed interval $[\alpha, \beta]$, that is, $\sup_{\alpha\leqq\lambda\leqq\beta}|\varphi(\lambda)| \leqq \gamma$, and $\varphi(\lambda)$ is continuous in the closed intervals which exclude at most countable intervals $I_{v.1}$, $I_{v.2}$, $\cdots$, the sum of whose lengthes $b_v$ is decreasing and converges to zero, then there exists bounded continuous function $\varphi_{b_v}(\lambda)$ which satisfies $\varphi(\lambda) = \varphi_{b_v}(\lambda)$, $|\varphi_{b_v}(\lambda)| \leqq \gamma$ in $[\alpha, \beta]$ $(I_{v.1} \dotplus I_{v.2} \dotplus \cdots)'$.

The above function $\varphi_{b_v}(\lambda)$ is called $\gamma$ bounded $b_v$ approximated continuous function of $\varphi(\lambda)$.

$\varphi_{b_v}(\lambda)$ and $\varphi_{c_v}(\lambda)$ are continuous in $[\alpha, \beta]$ and the set of $\lambda$

$$\{\lambda: \varphi_{b_v}(\lambda) \neq \varphi_{c_v}(\lambda),\ \alpha \leqq \lambda \leqq \beta\}$$

are covered with at most countable intervals, the sum of whose lengthes tends to zero.

But the function $|\varphi_{b_v}(\lambda) - \varphi_{c_v}(\lambda)|$ is continuous in $[\alpha, \beta]$, for any positive number $\delta$, the set of $\lambda$

$$\{\lambda: |\varphi_{b_v}(\lambda) - \varphi_{c_v}(\lambda)| \geqq \delta,\ \alpha \leqq \lambda \leqq \beta\}$$

is a closed set. (with respect to $[\alpha, \beta]$, and $[\alpha, \beta]$ is a closed set, hence the above set is a closed set).

Therefore by Heine-Borel's covering theorem, the above set are covered with finite intervals, the sum of whose lengthes $b_v+c_v \downarrow_{v=1}^{\infty} 0.$

This union of finite intervals may be written as
$(\alpha_1, \beta_1) \dotplus (\alpha_2, \beta_2) \dotplus \cdots \dotplus (\alpha_n, \beta_n)$ where $\alpha_1<\beta_1<\alpha_2<\beta_2<\cdots<\alpha_n<\beta_n$
therefore by the property of Stieltjés integral of continuous function

$$\left| \int_\alpha^\beta \varphi_{b_v}(\lambda)\, da_\lambda - \int_\alpha^\beta \varphi_{c_v}(\lambda)\, da_\lambda \right| \leqq \int_\alpha^\beta \left| \varphi_{b_v}(\lambda) - \varphi_{c_v}(\lambda) \right| da_\lambda$$

$$= \sum_{i=1}^n \int_{\alpha^i}^{\beta^i} \left| \varphi_{b_v}(\lambda) - \varphi_{c_v}(\lambda) \right| da_\lambda + \sum_{i=0}^n \int_{\beta^i}^{\alpha^{i+1}} \left| \varphi_{b_v}(\lambda) - \varphi_{c_v}(\lambda) \right| da_\lambda \leqq \sum_{i=1}^n \int_{\alpha^i}^{\beta^i} 2\gamma\, da_\lambda$$

$$+ \sum_{i=0}^n \int_{\beta i}^{\alpha^i+1} \delta\, da_\lambda \leqq 2\gamma \sum_{i=1}^n (a_{\beta i} - a_{\alpha_i}) + \delta (a_\beta - a_\alpha)$$

where $\beta_0 = \alpha$, $\alpha_{n+1} = \beta$, hence

$$\left| \int_\alpha^\beta \varphi_{b_v}(\lambda)\, da_\lambda - \int_\alpha^\beta \varphi_{c_v}(\lambda)\, da_\lambda \right| \leqq 2\gamma (b_v + c_v) + \gamma (a_\beta - a_\alpha)$$

as $\delta$ is any positive number

$$\left| \int_\alpha^\beta \varphi_{b_v}(\lambda)\, da_\lambda - \int_\alpha^\beta \varphi_{c_v}(\lambda)\, da_\lambda \right| \leqq 2\gamma (b_v + c_v) \downarrow_{v=1}^{\infty} 0$$

Therefore by the generalized Cauchy's convergent theorem,[6] for any $\gamma$ bounded, $b_v$ approximated continuous function $\varphi_{b_v}(\lambda)$, the finite limiting value $\lim_{b_v \to 0} \int_\alpha^\beta \varphi_{b_v}(\lambda)\, da_\lambda$ exists, this limiting value is defined as the Stieltjés integral of bounded almost continuous function $\varphi(\lambda)$ in $[\alpha, \beta]$ and is denoted by $\int_\alpha^\beta \varphi(\lambda)\, da_\lambda$ similar to continuous function.

If functions are bounded and almost continuous in $[\alpha, \beta]$, then we can prove the following properties.

1° $\int_\alpha^\beta \varphi(\lambda)\, da_\lambda = \int_\alpha^\gamma \varphi(\lambda)\, da_\lambda + \int_\lambda^\beta \varphi(\lambda)\, da_\lambda \qquad (\alpha \leqq \gamma \leqq \beta)$

2° $\int_\alpha^\beta \{\varphi(\lambda) + \psi(\lambda)\}\, da_\lambda = \int_\alpha^\beta \varphi(\lambda)\, da_\lambda + \int_\alpha^\beta \psi(\lambda)\, da_\lambda$

3° If $\gamma$ is a constant, then $\int_\alpha^\beta \gamma\varphi(\lambda)\, da_\lambda = \gamma \int_\alpha^\beta \varphi(\lambda)\, da_\lambda$

*Definition.* A point set, which can be covered with at most countable intervals, the sum of whose lengthes is decreasing and tends to zero, is said to be a set of *measure zero.*

4° If $\varphi(\lambda) \geqq \psi(\lambda)$ in $[\alpha, \beta]$ except at most set of measure zero, then

$$\int_\alpha^\beta \varphi(\lambda)\, da_\lambda \geqq \int_\alpha^\beta \psi(\lambda)\, da_\lambda$$

5° $\left| \int_\alpha^\beta \varphi(\lambda)\, da_\lambda \right| \leqq \int_\alpha^\beta |\varphi(\lambda)|\, da_\lambda \qquad (\beta > \alpha)$

We also can prove by usual method.

*Theorem.* If function $\varphi_n(\lambda)$ is almost continuous and uniformly bounded in the closed interval $[\alpha, \beta]$ then

$$\int_\alpha^\beta \underline{\lim}_{n\to\infty}\varphi_n(\lambda)\, da_\lambda \leqq \underline{\lim}_{n\to\infty}\int_\alpha^\beta \varphi_n(\lambda)\, da_\lambda \leqq \overline{\lim}_{n\to\infty}\int_\alpha^\beta \varphi_n(\lambda)\, da_\lambda \leqq \int_\alpha^\beta \overline{\lim}_{n\to\infty}\varphi_n(\lambda)\, da_\lambda$$

hence if specialy $\lim_{n\to\infty} \varphi_n(\lambda) = \varphi(\lambda)$ then

$$\lim_{n\to\infty}\int_\alpha^\beta \varphi_n(\lambda)\, da_\lambda = \int_\alpha^\beta \varphi(\lambda)\, da_\lambda$$

§ 3. Integral of positive and almost continuous functions

$\varphi(\lambda)$ is positive and almost continuous in a closed interval $[\alpha, \beta]$, For any $\gamma > 0$, if we put

$$\varphi_\gamma(\lambda) = \min\,[\varphi(\lambda),\ \gamma] \qquad (\alpha \leqq \lambda \leqq \beta)$$

then $\varphi_\gamma(\lambda)$ is upper bounded, positive and almost continuous, and if the limiting value

$$\lim_{\gamma\to\infty}\int_\alpha^\beta \varphi_\gamma(\lambda)\, da_\lambda$$

exists, then $\varphi(\lambda)$ is said to be *integrable*, and the above value is defined as

$$\int_\alpha^\beta \varphi(\lambda)\, da_\lambda$$

If such a limit does not exist, then we define this integral as $+\infty$.

Propertiës of the Stieltjés integral of positive and almost continuous function hold similarly to bounded function, and we have the following theorems.

*Theorem* I. Integral of positive and almost continuous function $\varphi(\lambda)$ is equal to $\lim_{\nu\to\infty}\int_\alpha^\beta \varphi_{b_\nu}(\lambda)\, da_\lambda$ where $\varphi_{b_\nu}(\lambda)$ is any positive $b_\nu$ approximated continuous function.

*Theorem* II. For a sequence of positive and almost continuous functions $\varphi_1 \leqq \varphi_2 \leqq \cdots$, if we put $\varphi(\lambda) = \lim_{n\to\infty} \varphi_n(\lambda)$, then we have

$$\int_\alpha^\beta \varphi(\lambda)\, da_\lambda = \lim_{n\to\infty}\int_\alpha^\beta \varphi_n(\lambda)\, da_\lambda$$

(corresponds to Lebesgue's theorem)

*Theorem* III. If a sequence of positive functions $\varphi_n(\lambda)$ $(n=1, 2, \cdots)$ are almost continuous in a closed interval $[\alpha, \beta]$, then we have

$$\underline{\lim}_{n\to\infty}\int_\alpha^\beta \varphi_n(\lambda)\, da_\lambda \geqq \int_\alpha^\beta \underline{\lim}_{n\to\infty}\varphi_n(\lambda)\, da_\lambda$$

(corresponds to Fatou's theorem)

§ 4. Absolutely convergent integral.

$\varphi(\lambda)$ is not necessarily bounded and almost continuous in $[\alpha, \beta]$, if we put

$$\varphi^+(\lambda) = \max\,[\varphi(\lambda),\ 0]; \quad \varphi^-(\lambda) = \max\,[-\varphi(\lambda),\ 0]$$

then $\varphi^+(\lambda)$ and $\varphi^-(\lambda)$ are positive and almost continuous, and

$$\varphi(\lambda) = \varphi^+(\lambda) - \varphi^-(\lambda)$$
$$|\varphi(\lambda)| = \varphi^+(\lambda) + \varphi^-(\lambda)$$

If both $\varphi^+(\lambda)$ and $\varphi^-(\lambda)$ are integrable in $[\alpha, \beta]$, then $\varphi(\lambda)$ is said to be integrable, and its integral is defined by

$$\int_\alpha^\beta \varphi(\lambda)\, da_\lambda = \int_\alpha^\beta \varphi^+(\lambda)\, da_\lambda - \int_\alpha^\beta \varphi^-(\lambda)\, da_\lambda$$

We have the following theorems.

*Theorem* I. The necessary and sufficient condition of integrability of almost continuous function $\varphi(\lambda)$ in $[\alpha, \beta]$ is that the positive and almost continuous function $|\varphi(\lambda)|$ is integrable, and in this case we have the following relation.

$$\left| \int_\alpha^\beta \varphi(\lambda)\, da_\lambda \right| \leqq \int_\alpha^\beta |\varphi(\lambda)|\, da_\lambda$$

*Theorem* II. If $\varphi(\lambda)$ is integrable in $[\alpha, \beta]$, then for any positive number $\Gamma$, there exists such $c_v$ as

$$\left| \int_\alpha^\beta \varphi(\lambda)\, \psi(\lambda)\, da_\lambda \right| \leqq c_v \downarrow_{v=1}^{\infty} 0$$

for $\Gamma$ bounded almost continuous function $\psi(\lambda)$ which satisfies

$$\int_\alpha^\beta |\psi(\lambda)|\, da_\lambda \leqq d_v \downarrow_{v=1}^{\infty} 0$$

Lastly we have the theorem which corresponds to Lebesgue's theorem.

*Theorem* III. If a sequence of functions $\varphi_n(\lambda)$ $(n=1, 2, \cdots)$ are almost continuous in $[\alpha, \beta]$ and converge to $\varphi_0(\lambda)$, and when

$$|\varphi_n(\lambda)| \leqq \Phi(\lambda) \qquad (n=1, 2, \cdots)$$

where $\Phi(\lambda)$ is a proper integrable and positive function in $[\alpha, \beta]$, then $\varphi_n(\lambda)$ and $\varphi_0(\lambda)$ are integrable and

$$\lim_{n\to\infty} \int_\alpha^\beta \varphi_n(\lambda)\, da_\lambda = \int_\alpha^\beta \varphi_0(\lambda)\, da_\lambda$$

## References.


(1) It was published in Mathematics (in Japanese) Vol. 2 No. 4 November 1950 compiled by the Mathematical Society of Japan.

(2) H. Nakano: Modern Spectral Theory. Tokyo Mathematical Book Series Vol. II §9.

(3) H. Nakano: Modulared Semi-ordered Linear Spaces. Tokyo Mathematical Book Series Vol. I §17.

(4) H. Nakano: Theory of Classical Integration (in Japanese) 1949. Tokyo.

(5) cf. (3).

(6) cf. (2) Theorem 6.4.