

On Some Properties of N -sided Skew Polygon

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The aim of this paper is to study some properties of 4-sided skew polygon, and one salient particular property of n -sided skew polygon.

1. 4-sided skew polygon

The 3-dimensional space expansion of CEVA's Theorem for 2-dimensional space is known as the following theorem.

Theorem. *When four points P_1, P_2, P_3, P_4 are taken on four sides $A_1A_2, A_2A_3, A_3A_4, A_4A_1$ of 4-sided skew polygon respectively, the necessary and sufficient condition for these four points to be coplanar is*

$$\frac{\vec{A_1P_1}}{\vec{P_1A_2}} \cdot \frac{\vec{A_2P_2}}{\vec{P_2A_3}} \cdot \frac{\vec{A_3P_3}}{\vec{P_3A_4}} \cdot \frac{\vec{A_4P_4}}{\vec{P_4A_1}} = 1,$$

where each expression with arrows represents a vector.

In this paper this theorem is to be constantly referred to as 'The Expansion of CEVA's Theorem'.

The present writer deduced the following theorem from 'The Expansion of Ceva's Theorem'.

Theorem 1. *Let $\vec{A_1A_2} : \vec{A_1A_4} = \vec{A_3A_2} : \vec{A_3A_4}$ in 4-sided skew polygon.*

Let in the triangle $A_1A_2A_4$ the two intersecting points for the bisectors of two angles $A_1A_2A_4, A_2A_4A_1$ and for the two sides A_1A_4, A_1A_2 be P_4, P_1 respectively. Further, in the triangle $A_2A_3A_4$ the two intersecting points for the two bisectors of two angles $A_4A_2A_3, A_3A_4A_2$ and for the two sides A_3A_4, A_2A_3 be A_3, A_2 respectively. Then four points P_1, P_2, P_3, P_4 are coplanar.

Theorem 2. *If in the 4-sided skew polygon $A_1A_2A_3A_4$, each inscribed circle of the triangles $A_1A_2A_4$ and $A_2A_3A_4$ contacts with the same point in side A_2A_4 , then the other four points of contact are coplanar.*

The above two theorems are easily deduced from 'The Expansion of CEVA's Theorem'.

2. n -sided skew polygon

In order to expand further 'The Expansion of CEVA's Theorem' for 4-sided

skew polygon to n -sided skew polygon, we will define as follows.

Definition. If in the n -sided skew polygon $A_1A_2\cdots A_n$ in 3-dimensional space, $n-2$ triangles $A_nA_1A_2$, $A_nA_2A_3$, \cdots , $A_nA_{n-2}A_{n-1}$ construct different planes, the present writer call this n -sided polygon *n -sided skew polygon*.

Now we get the following theorem.

Theorem 3. *If some of n points P_1, P_2, \cdots, P_n in n -sided skew polygon $A_1A_2A_3\cdots A_n$ are taken on some sides of n sides $A_1A_2, A_2A_3, \cdots, A_nA_1$ the remaining points are taken on the productions of the remaining sides of the same polygon. Then the necessary and sufficient condition for these n points to be coplanar is*

$$\frac{\overrightarrow{A_1P_1}}{\overrightarrow{P_1A_2}} \cdot \frac{\overrightarrow{A_2P_2}}{\overrightarrow{P_2A_3}} \cdot \frac{\overrightarrow{A_3P_3}}{\overrightarrow{P_3A_4}} \cdot \cdots \cdot \frac{\overrightarrow{A_nP_n}}{\overrightarrow{P_nA_1}} = 1.$$

Proof. The present writer make us of the mathematical induction to proof the above Theorem 3.

(i). If $n=4$, then this theorem is evident from 'The Expansion of CEVA'S Theorem'.

If $n=5$, then suppose that some of five points are taken on some sides of 5-sided skew polygon and the remaining points are taken on the remaining sides of the same polygon. Now suppose that such five points are in the same plane, then if the intersecting point of the plane π and the line A_4A_1 is denoted by $\pi \cap A_4A_1 = X_1$, for the 4-sided skew polygon $A_1A_2A_3A_4$ we get

$$(1) \quad \frac{\overrightarrow{A_1P_1}}{\overrightarrow{P_1A_2}} \cdot \frac{\overrightarrow{A_2P_2}}{\overrightarrow{P_2A_3}} \cdot \frac{\overrightarrow{A_3P_3}}{\overrightarrow{P_3A_4}} \cdot \frac{\overrightarrow{A_4X}}{\overrightarrow{XA_1}} = 1.$$

Since three points P_4, P_5, X are coplanar, (1) using MENELAUS' Theorem deduces

$$(2) \quad \frac{\overrightarrow{A_1P_1}}{\overrightarrow{P_1A_2}} \cdot \frac{\overrightarrow{A_2P_2}}{\overrightarrow{P_2A_3}} \cdot \frac{\overrightarrow{A_3P_3}}{\overrightarrow{P_3A_4}} \cdot \frac{\overrightarrow{A_4P_4}}{\overrightarrow{P_4A_5}} \cdot \frac{\overrightarrow{A_5P_5}}{\overrightarrow{P_5A_1}} = 1.$$

Conversely, if (2) is satisfied it follows from the converse of MENELAUS' Theorem that such X' as $\frac{\overrightarrow{A_1P_1}}{\overrightarrow{P_1A_2}} \cdot \frac{\overrightarrow{A_2P_2}}{\overrightarrow{P_2A_3}} \cdot \frac{\overrightarrow{A_3P_3}}{\overrightarrow{P_3A_4}} \cdot \frac{\overrightarrow{A_4X'}}{\overrightarrow{X'A_1}} = 1$ is on the line A_4A_1 , and the three points P_4, P_5, X' are coplanar.

We now say that four points P_1, P_2, P_3, X' are coplanar from 'The Expansion of CEVA'S Theorem',

Let this plane be π , then since both π and π' have three points P_1, P_2, P_3 in common, $\pi \equiv \pi'$. Namely if (2) is satisfied, five points P_1, P_2, P_3, P_4, P_5 are coplanar.

(ii). Now suppose that in a r -sided skew polygon some of points $P_1, P_2,$

....., P_{r-1} , Y are on some of its r -sides A_1A_2 , A_2A_3 ,, A_rA_1 , respectively, and the remaining points are on the production of its remaining sides.

Furthermore, suppose that the necessary and sufficient condition for such point to be coplanar is

$$(3) \quad \frac{\overrightarrow{A_1P_1}}{\overrightarrow{P_1A_2}} \cdot \frac{\overrightarrow{A_2P_2}}{\overrightarrow{P_2A_3}} \cdot \dots \cdot \frac{\overrightarrow{A_rY}}{\overrightarrow{YA_1}} = 1.$$

If we denote the plane where the above points to be coplanar by σ , then we obtain two points P_r, P_{r+1} such that

$$\sigma \cap A_rA_{r+1} = P_r, \quad \sigma \cap A_{r+1}A_1 = P_{r+1}.$$

Using MENELAUS' Theorem the following is deduced from (3)

$$(4) \quad \frac{\overrightarrow{A_1P_1}}{\overrightarrow{P_1A_1}} \cdot \frac{\overrightarrow{A_2P_2}}{\overrightarrow{P_2A_3}} \cdot \dots \cdot \frac{\overrightarrow{A_rP_r}}{\overrightarrow{P_rA_{r+1}}} \cdot \frac{\overrightarrow{A_{r+1}P_{r+1}}}{\overrightarrow{P_{r+1}A_1}} = 1.$$

Conversely, if (4) is satisfied it follows from the converse of MENELAUS' Theorem that such Y' as

$$(5) \quad \frac{\overrightarrow{A_1P_1}}{\overrightarrow{P_1A_2}} \cdot \frac{\overrightarrow{A_2P_2}}{\overrightarrow{P_2A_3}} \cdot \dots \cdot \frac{\overrightarrow{A_{r-1}P_{r-1}}}{\overrightarrow{P_{r-1}A_r}} \cdot \frac{\overrightarrow{A_rY'}}{\overrightarrow{Y'A_1}} = 1$$

is on the line A_rA_1 , and the three points P_r, P_{r+1}, Y' are collinear, now since r points $P_1, P_2, \dots, P_{r-1}, Y$ are coplanar, and that $r+1$ points $P_1, P_2, \dots, P_r, P_{r+1}$ are coplanar. Therefore, if $n=r+1$ still the theorem is held. Hence the proof of the theorem is concluded.

References

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