On the Concircularity of the Centres of five Kiepert's Circles of a Convex Pentagon inscribed in a Circle

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(Received Oct. 21, 1963)

This paper forms a sequel of "On the Circularity of Kiepert's Point", hence the notations employed in this paper follows those of its preceding paper.

We are to call it briefly Kiepert's circle of a quadrilateral $A_1A_2A_3A_4$ when the circle passes through four points $P_{123}(\lambda)$, $P_{234}(\lambda)$, $P_{341}(\lambda)$ and $P_{412}(\lambda)$.

This paper concerns the concircularity of Kiepert's circle, and our main result is the following theorem.

Theorem. *Let $A_1A_2A_3A_4A_5$ be aconvex pentagon inscribed in a circle, and $S_{1234}(\lambda)$, $S_{2345}(\lambda)$, $S_{3451}(\lambda)$, $S_{4512}(\lambda)$ and $S_{5123}(\lambda)$ be centres of the Kiepert's circle of the five quadrilaterals $A_1A_2A_3A_4$, $A_2A_3A_4A_5$, $A_3A_4A_5A_1$, $A_4A_5A_1A_2$ and $A_5A_1A_2A_3$, respectively. Then, these five points are concircular.*

Proof. Without loss of generality, we can assume that the circumscribed circle of the pentagon $A_1A_2A_3A_4A_5$ is unit circle. Then, if we denote five points A_1, A_2, A_3, A_4 and A_5 by the complex numbers t_1, t_2, t_3, t_4 and t_5, respectively, we have $|t_i|=1$, $t_i\bar{t}_i=1$, $(i=1,2,3,4,5)$.

Of all the triangles obtainable out of the pentagon $A_1A_2A_3A_4A_5$, for example, a triangle $A_1A_2A_3$ has the following expressions in relation to P_{23}, P_{31} and P_{12}, which are the complex numbers of three points P_{23}, P_{31} and P_{12}, respectively:

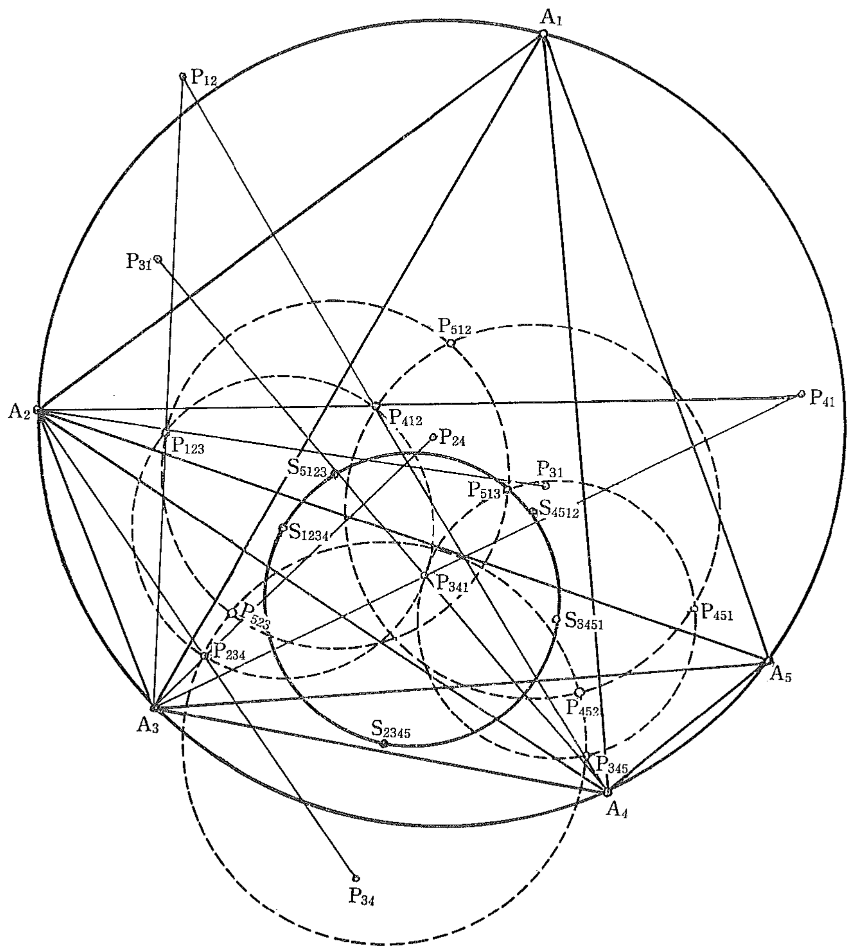
$$\bar{p}_{ij}=\frac{p_{ij}}{t_it_j},\quad p_{ij}+p_{ji}=t_i+t_j$$

$$p_{ij}=\frac{1}{2}(t_i+t_j)-\sqrt{-1}\,\lambda\,(t_j-t_i),\quad (\lambda\geqq 0),\quad (i\neq j,\ \ i,j=1,2,3).$$

The same expressions are obtained from all the other triangles.

Then by $Z_{1234}(\lambda)$ we denote the complex number of a centre of the Kiepert's circle which are constructed from a quadrilateral $A_1A_2A_3A_4$.

The Kiepert's points $P_{123}(\lambda)$, $P_{234}(\lambda)$, $P_{341}(\lambda)$ and $P_{412}(\lambda)$ of four triangles $A_1A_2A_3$, $A_2A_3A_4$, $A_3A_4A_1$ and $A_4A_1A_2$ is represented by the complex numbers $Z_{123}(\lambda)$, $Z_{234}(\lambda)$, $Z_{341}(\lambda)$ and $Z_{412}(\lambda)$, respectively, for which, as was shown in my preceding paper*, the following expressions hold.



$$Z_{123}(\lambda)=\frac{(t_1-p_{23})\ (\bar{t}_2p_{31}-t_2\bar{p}_{31})-(t_2-p_{31})\ (\bar{t}_1p_{23}-t_1\bar{p}_{23})}{(t_1-p_{23})\ (\bar{t}_2-\bar{p}_{31})-(\bar{t}_1-\bar{p}_{23})\ (t_2-p_{31})},$$

$$Z_{234}(\lambda)=\frac{(t_2-p_{34})\ (\bar{t}_3p_{42}-t_3\bar{p}_{42})-(t_3-p_{42})\ (\bar{t}_2p_{34}-t_2\bar{p}_{34})}{(t_2-p_{34})\ (\bar{t}_3-\bar{p}_{42})-(\bar{t}_2-\bar{p}_{34})\ (t_3-p_{42})},$$

$$Z_{341}(\lambda)=\frac{(t_3-p_{41})\ (\bar{t}_4p_{13}-t_4\bar{p}_{13})-(t_4-p_{13})\ (\bar{t}_3p_{41}-t_3\bar{p}_{41})}{(t_3-p_{41})\ (\bar{t}_4-\bar{p}_{13})-(\bar{t}_3-\bar{p}_{41})\ (t_4-p_{13})}, \tag{1}$$

$$Z_{412}(\lambda)=\frac{(t_4-p_{12})\ (\bar{t}_1p_{24}-t_1\bar{p}_{24})-(t_1-p_{24})\ (\bar{t}_4p_{12}-t_4\bar{p}_{12})}{(t_4-p_{12})\ (\bar{t}_1-\bar{p}_{24})-(\bar{t}_4-\bar{p}_{12})\ (t_1-p_{24})},$$

The same expression holds for the other four quadrilaterals $A_2A_3A_4A_5$, $A_3A_4A_5A_1$, $A_4A_5A_1A_2$ and $A_5A_1A_2A_3$.

We shall try now to get complex number $Z_{1234}(\lambda)$ which represents the centre of Kiepert's circle $S_{1234}(\lambda)$.

As it is, we try to get, first of all, the equation of the perpendicular bisector of a segment $P_{123}(\lambda)P_{341}(\lambda)$.

Then, we have the equation

$$\begin{vmatrix} Z_{123}(\lambda) & Z_{341}(\lambda) & Z \\ \bar{Z}_{341}(\lambda) & \bar{Z}_{123}(\lambda) & \bar{Z} \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

$$\{\bar{Z}_{341}(\lambda)-\bar{Z}_{123}(\lambda)\}Z+\{Z_{341}(\lambda)-Z_{123}(\lambda)\}\bar{Z}+\{Z_{123}(\lambda)\bar{Z}_{123}(\lambda)-Z_{341}(\lambda)\bar{Z}_{341}(\lambda)\}=0. \quad (2)$$

Similaly, we have the equation of the perpendicular bisector of a segment $P_{234}(\lambda)\,P_{412}(\lambda)$:

$$\{\bar{Z}_{412}(\lambda)-\bar{Z}_{234}(\lambda)\}Z+\{Z_{412}(\lambda)-Z_{234}(\lambda)\}\bar{Z}+\{Z_{234}(\lambda)\bar{Z}_{234}(\lambda)-Z_{412}(\lambda)\bar{Z}_{412}(\lambda)\}=0. \quad (3)$$

Complex number $Z_{1234}(\lambda)$ is equal to Z, which is obtained as the solution of two equations (2) and (3),
i. e.

$$\begin{vmatrix} \bar{Z}_{341}(\lambda)-\bar{Z}_{123}(\lambda) & Z_{341}(\lambda)-Z_{123}(\lambda) \\ \bar{Z}_{412}(\lambda)-\bar{Z}_{234}(\lambda) & Z_{412}(\lambda)-Z_{234}(\lambda) \end{vmatrix} Z$$

$$= \begin{vmatrix} Z_{341}(\lambda)\bar{Z}_{341}(\lambda)-Z_{123}(\lambda)\bar{Z}_{123}(\lambda) & Z_{341}(\lambda)-Z_{123}(\lambda) \\ Z_{412}(\lambda)\bar{Z}_{412}(\lambda)-Z_{234}(\lambda)\bar{Z}_{234}(\lambda) & Z_{412}(\lambda)-Z_{234}(\lambda) \end{vmatrix},$$

since the coefficient of Z is a pure imaginary number, we can put into the form

$$\begin{vmatrix} \bar{Z}_{341}(\lambda)-\bar{Z}_{123}(\lambda) & Z_{341}(\lambda)-Z_{123}(\lambda) \\ \bar{Z}_{412}(\lambda)-\bar{Z}_{234}(\lambda) & Z_{412}(\lambda)-Z_{234}(\lambda) \end{vmatrix} = \sqrt{-1}\,r_{1234},$$

where, r_{1234} is a real number.

Therefore,

$$\sqrt{-1}\,r_{1234}Z$$

$$=\{Z_{341}(\lambda)\bar{Z}_{341}-Z_{123}(\lambda)\bar{Z}_{123}(\lambda)\}\,\{Z_{412}(\lambda)-Z_{234}(\lambda)\}$$

$$-\{Z_{412}(\lambda)\bar{Z}_{412}(\lambda)-Z_{234}(\lambda)\bar{Z}_{234}(\lambda)\}\,\{Z_{341}(\lambda)-Z_{123}(\lambda)\}. \quad (4)$$

Both $Z_{341}(\lambda)\bar{Z}_{341}(\lambda)-Z_{123}(\lambda)\bar{Z}_{123}(\lambda)$ and $Z_{412}(\lambda)\bar{Z}_{412}(\lambda)-Z_{234}(\lambda)\bar{Z}_{234}(\lambda)$ which are in the right hand side are real numbers.
Then, we put

$$Z_{341}(\lambda)\bar{Z}_{341}(\lambda)-Z_{123}(\lambda)\bar{Z}_{123}(\lambda)=r^1{}_{1234},$$

$$Z_{412}(\lambda)\bar{Z}_{412}(\lambda)-Z_{234}(\lambda)\bar{Z}_{234}(\lambda)=r^2{}_{1234}.$$

The expressions (1) were put into the following form by the author in his paper*

$$Z_{123}(\lambda)=\frac{\alpha_1(t_1-p_{23})-\alpha_2(t_2-p_{31})}{\alpha},$$

$$Z_{234}(\lambda)=\frac{\beta_1(t_2-p_{34})-\beta_2(t_3-p_{42})}{\beta},$$

$$Z_{341}(\lambda)=\frac{\gamma_1(t_3-p_{41})-\gamma_2(t_4-p_{13})}{\gamma},$$

$$Z_{412}(\lambda)=\frac{\delta_1(t_4-p_{12})-\delta_2(t_1-p_{24})}{\delta},$$

where α, α_1, α_2, β, ……etc. are all real numbers.
It frollows fom (4) that

$$Z=\frac{r^1{}_{1234}}{\sqrt{-1}\,r_{1234}}\left\{\frac{\delta_1(t_4-p_{13})-\delta_2(t_1-p_{24})}{\delta}-\frac{\beta_1(t_2p_{34})-\beta_2(t_3-p_{42})}{\beta}\right\}$$

$$-\frac{r^2{}_{1234}}{\sqrt{-1}\,r_{1234}}\left\{\frac{\gamma_1(t_3-p_{41})-\gamma_2(t_4-p_{13})}{\gamma}-\frac{\alpha_1(t_1-p_{23})-\alpha_2(t_2-p_{31})}{\alpha}\right\}.$$

When we cronsider that

$$p_{42}=t_2+t_4-p_{24},\qquad t_{31}=t_1+t_3-p_{13},$$

the above expression can be represented in the following form:

$$Z=\frac{1}{\sqrt{-1}}(l_1t_1+m_1t_2+n_1t_3+s_1t_4+\lambda_1p_{12}+\mu_1p_{23}+\nu_1p_{34}+\xi_1p_{41}+\eta_1p_{13}+\zeta_1p_{24}),$$

where $\ell_1=\frac{1}{r_{1234}}\left(\frac{\alpha_1r^2{}_{1234}}{\alpha}-\frac{\delta_2r^1{}_{1234}}{\delta}\right)$, ……………………,

$\lambda_1=-\frac{\delta_1r^1{}_{1234}}{\delta r_{1234}}$, ……………etc. are all pure imaginary complex numbers.

Therefore,

$$Z_{1234}(\lambda)=\frac{1}{\sqrt{-1}}(l_1t_1+m_1t_2+n_1t_3+s_1t_4+\lambda_1p_{12}+\mu_1p_{23}+\nu_1p_{34}$$
$$+\xi_1p_{41}+\eta_1p_{13}+\zeta_1p_{24}),$$

$$Z_{2345}(\lambda)=\frac{1}{\sqrt{-1}}(l_2t_2+m_2t_3+n_2t_4+s_2t_5+\lambda_2p_{23}+\mu_2p_{34}$$
$$+\nu_2p_{45}+\xi_2p_{52}+\eta_2p_{24}+\zeta_2p_{35}),$$

$$Z_{3451}(\lambda)=\frac{1}{\sqrt{-1}}(l_3t_3+m_3t_4+n_3t_5+s_3t_1+\lambda_3p_{34}+\mu_3p_{45}$$
$$+\nu_3p_{51}+\xi_3p_{13}+\eta_3p_{35}+\zeta_3p_{41}),$$

$$Z_{4512}(\lambda)=\frac{1}{\sqrt{-1}}(l_4t_4+m_4t_5+n_4t_1+s_4t_2+\lambda_4p_{45}+\mu_4p_{51}$$
$$+\nu_4p_{12}+\xi_4p_{24}+\eta_4p_{41}+\zeta_4p_{52}).$$

When we put

$$\frac{Z_{1234}(\lambda)-Z_{3451}(\lambda)}{Z_{2345}(\lambda)-Z_{3451}(\lambda)}\times\frac{Z_{2345}(\lambda)-Z_{4512}(\lambda)}{Z_{1234}(\lambda)-Z_{4512}(\lambda)}\equiv Z(\lambda),$$

and if we can establish $Z(\lambda)=\bar{Z}(\lambda)$ we conclude that the theorem was proved.

To establish $Z(\lambda)=\bar{Z}(\lambda)$ the following expression nust be proved,

$$Z(\lambda)-\bar{Z}(\lambda)$$

$$\frac{Z_{1234}(\lambda)-Z_{3451}(\lambda)}{Z_{2345}(\lambda)-Z_{3451}(\lambda)}\times\frac{Z_{2345}(\lambda)-Z_{4512}(\lambda)}{Z_{1234}(\lambda)-Z_{4512}(\lambda)}-\frac{\bar{Z}_{1234}(\lambda)-\bar{Z}_{3451}(\lambda)}{\bar{Z}_{2345}(\lambda)-\bar{Z}_{3451}(\lambda)}\times\frac{\bar{Z}_{2345}(\lambda)-\bar{Z}_{4512}(\lambda)}{\bar{Z}_{1234}(\lambda)-\bar{Z}_{4512}(\lambda)}=0.$$

Then, we have

$$Z(\lambda)-\bar{Z}(\lambda)$$

$$=\left\{\frac{(l_1t_1+m_1t_2+n_1t_3+s_1t_4+\lambda_1p_{12}+\mu_1p_{23}+\nu_1p_{34}+\xi_1p_{41}+\eta_1p_{13}+\zeta_1p_{24})}{(l_2t_2+m_2t_3+n_2t_4+s_2t_5+\lambda_2p_{23}+\mu_2p_{34}+\nu_2p_{45}+\xi_2p_{52}+\eta_2p_{24}+\zeta_2p_{35})}\right.$$

$$\left.\frac{-(l_3t_3+m_3t_4+n_3t_5+s_3t_1+\lambda_3p_{34}+\mu_3p_{45}+\nu_3p_{51}+\xi_3p_{13}+\eta_3p_{35}+\zeta_3p_{41})}{-(l_3t_3+m_3t_4+n_3t_5+s_3t_1+\lambda_3p_{34}+\mu_3p_{45}+\nu_3p_{51}+\xi_3p_{13}+\eta_3p_{35}+\zeta_3p_{41})}\right\}$$

$$\times\left\{\frac{(l_2t_2+m_2t_3+n_2t_4+s_2t_5+\lambda_2p_{23}+\mu_2p_{34}+\nu_2p_{45}+\xi_2p_{52}+\eta_2p_{24}+\zeta_2p_{35})}{(l_1t_1+m_1t_2+n_1t_3+s_1t_4+\lambda_1p_{12}+\mu_1p_{23}+\nu_1p_{34}+\xi_1p_{41}+\eta_1p_{13}+\zeta_1p_{24})}\right.$$

$$\left.\frac{-(l_4t_4+m_4t_5+n_4t_1+s_4t_2+\lambda_4p_{45}+\mu_4p_{51}+\nu_4p_{12}+\xi_4p_{24}+\eta_4p_{41}+\xi_4p_{52})}{-(l_4t_4+m_4t_5+n_4t_1+s_4t_2+\lambda_4p_{45}+\mu_4p_{51}+\nu_4p_{12}+\xi_4p_{24}+\eta_4p_{41}+\xi_4p_{52})}\right\}$$

$$-\left\{\frac{(l_1\bar{t}_1+m_1\bar{t}_2+n_1\bar{t}_3+s_1\bar{t}_4+\lambda_1\bar{p}_{12}+\mu_1\bar{p}_{23}+\nu_1\bar{p}_{34}+\xi_1\bar{p}_{41}+\eta_1\bar{p}_{13}+\zeta_1\bar{p}_{24})}{(l_2\bar{t}_2+m_2\bar{t}_3+n_2\bar{t}_4+s_2\bar{t}_5+\lambda_2\bar{p}_{23}+\mu_2\bar{p}_{34}+\nu_2\bar{p}_{45}+\xi_2\bar{p}_{52}+\eta_2\bar{p}_{24}+\zeta_2\bar{p}_{35})}\right.$$

$$\left.\frac{-(l_3\bar{t}_3+m_3\bar{t}_4+n_3\bar{t}_5+s_3\bar{t}_1+\lambda_3\bar{p}_{34}+\mu_3\bar{p}_{45}+\nu_3\bar{p}_{51}+\xi_3\bar{p}_{13}+\eta_3\bar{p}_{35}+\zeta_3\bar{p}_{41})}{-(l_3\bar{t}_3+m_3\bar{t}_4+n_3\bar{t}_5+s_3\bar{t}_1+\lambda_3\bar{p}_{34}+\mu_3\bar{p}_{45}+\nu_3\bar{p}_{51}+\xi_3\bar{p}_{13}+\eta_3\bar{p}_{35}+\zeta_3\bar{p}_{41})}\right\}$$

$$\times\left\{\frac{(l_2\bar{t}_2+m_2\bar{t}_3+n_2\bar{t}_4+s_2\bar{t}_5+\lambda_2\bar{p}_{23}+\mu_2\bar{p}_{34}+\nu_2\bar{p}_{45}+\xi_2\bar{p}_{52}+\eta_2\bar{p}_{24}+\zeta_2\bar{p}_{35})}{(l_1\bar{t}_1+m_1\bar{t}_2+n_1\bar{t}_3+s_1\bar{t}_4+\lambda_1\bar{p}_{12}+\mu_1\bar{p}_{23}+\nu_1\bar{p}_{34}+\xi_1\bar{p}_{41}+\eta_1\bar{p}_{13}+\zeta_1\bar{p}_{24})}\right.$$

$$\left.\frac{-(l_4\bar{t}_4+m_4\bar{t}_5+n_4\bar{t}_1+s_4\bar{t}_2+\lambda_4\bar{p}_{45}+\mu_4\bar{p}_{51}+\nu_4\bar{p}_{12}+\xi_4\bar{p}_{24}+\eta_4\bar{p}_{41}+\xi_4\bar{p}_{52})}{-(l_4\bar{t}_4+m_4\bar{t}_5+n_4\bar{t}_1+s_4\bar{t}_2+\lambda_4\bar{p}_{45}+\mu_4\bar{p}_{51}+\nu_4\bar{p}_{12}+\xi_4\bar{p}_{24}+\eta_4\bar{p}_{41}+\xi_4\bar{p}_{52})}\right\}$$

$$=\frac{(l_1t_1+m_1t_2+\cdots\cdots+\lambda_1p_{12}+\cdots\cdots)-(\cdots\cdots\cdots\cdots+s_3t_1+\cdots\cdots\cdots\cdots)}{(l_2t_2+\cdots\cdots\cdots\cdots\cdots+\cdots\cdots\cdots\cdots\cdots)-(\cdots\cdots\cdots\cdots+s_3t_1+\cdots\cdots\cdots\cdots)}$$

$$\times\frac{(l_2t_2+\cdots\cdots\cdots\cdots\cdots)-(\cdots\cdots\cdots\cdots\cdots+n_4t_1+s_4t_2+\cdots\cdots\cdots\cdots\cdots)}{(l_1t_1+m_1t_2+\cdots\cdots)-(\cdots\cdots\cdots\cdots\cdots+n_4t_1+s_4t_2+\cdots\cdots\cdots\cdots\cdots)}$$

$$-\frac{(l_1\bar{t}_1+m_1\bar{t}_2+\cdots\cdots+\lambda_1\bar{p}_{12}+\cdots\cdots\cdots)-(\cdots\cdots\cdots+s_3\bar{t}_1+\cdots\cdots\cdots)}{(l_2t_2+\cdots\cdots+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots)-(\cdots\cdots\cdots+s_3t_1+\cdots\cdots\cdots)}$$

$$\times\frac{(l_2\bar{t}_2+\cdots\cdots\cdots\cdots\cdots)-(\cdots\cdots\cdots\cdots\cdots+n_4\bar{t}_1+s_4\bar{t}_2+\cdots\cdots\cdots\cdots\cdots)}{(l_1\bar{t}_1+m_1\bar{t}_2+\cdots\cdots)-(\cdots\cdots\cdots\cdots\cdots+n_4\bar{t}_1+s_4\bar{t}_2+\cdots\cdots\cdots\cdots\cdots)}$$

$$=\frac{(l_1-s_3)t_1+m_1t_2+\cdots\cdots\cdots\cdots\cdots}{-s_3t_1+l_2t_2+\cdots\cdots\cdots\cdots\cdots\cdots\cdots}\times\frac{-n_4t_1+(l_2-s_4)t_2+\cdots\cdots\cdots\cdots}{(l_1-n_4)t_1+(m_1-s_4)t_2+\cdots\cdots}$$

$$-\frac{(l_1-s_3)\bar{t}_1+m_1\bar{t}_2+\cdots\cdots\cdots\cdots}{s_3\bar{t}_1+l_2\bar{t}_2+\cdots\cdots\cdots\cdots\cdots\cdots\cdots}\times\frac{-n_4\bar{t}_1+(l_2-s_4)\bar{t}_2+\cdots\cdots\cdots\cdots}{(l_1-n_4)\bar{t}_1+(m_1-s_4)\bar{t}_2+\cdots\cdots}$$

$$=\frac{-n_4(l_1-s_3)t^2{}_1+\{-m_1n_4+(l_1-s_3)(l_2-s_4)\}t_1t_2+\cdots\cdots\cdots\cdots\cdots}{-s_3(l_1-n_4)t^2{}_1+\{l_2l_1-n_4)-s_3(m_1-s_4)\}t_1t_2+\cdots\cdots\cdots\cdots\cdots\cdots}$$

$$-\frac{-n_4(l_1-s_3)\bar{t}^2{}_1+\{-m_1n_4+(l_1-s_3)(l_2-s_4)\}\bar{t}_1\bar{t}_2+\cdots\cdots\cdots\cdots}{-s_3(l_1-n_4)\bar{t}^2{}_1+\{l_2(l_1-n_4)-s_3(m_1-s_4)\}\bar{t}_1\bar{t}_2+\cdots\cdots\cdots\cdots\cdots}$$

Reducing the above fractions to a common demoninator, we have

$$\text{the numerator}=[-n_4(l_1-s_3)t^2{}_1+\{-m_1n_4+(l_1-s_3)(l_2-s_4)\}t_1t_2+\cdots\cdots\cdots\cdots\cdots]$$

$$\times[-s_3l_1-n_4)\bar{t}^2{}_1+\{l_2(l_1-n_4)-s_3(m_1-s_4)\}\bar{t}_1\bar{t}_2+\cdots\cdots\cdots\cdots\cdots]$$

$$-[-n_4(l_1-s_3)\bar{t}^2{}_1+\{-m_1n_4+(l_1-s_3)(l_2-s_4)\}\bar{t}_1\bar{t}_2+\cdots\cdots\cdots]$$

$$\times[-s_3(l_1-n_4)t^2{}_1+\{l_2(l_1-n_4)-s_3(m_1-s_4)\}t_1t_2+\cdots\cdots\cdots\cdots\cdots\cdots]$$

$$=0.$$

Therfore $Z(\lambda)=\bar{Z}(\lambda)$, which shows that four points $S_{1234}(\lambda)$, $S_{2345}(\lambda)$, $S_{3451}(\lambda)$ and $S_{4512}(\lambda)$ are concircular.

Similarly $S_{2345}(\lambda)$, $S_{3451}(\lambda)$, $S_{4512}(\lambda)$ and $S_{5123}(\lambda)$ are also concircular. Hence the five points are concircular.

When we consider that $\lambda=0$, $\lambda=\infty$, we have the following.

Corollary 1. *From each set of five quadrilaterals* $A_1A_2A_3A_4$, $A_2A_3A_4A_5$, $A_3A_4A_5A_1$, $A_4A_5A_1A_2$ *and* $A_5A_1A_2A_3$ *out of a convex pentagon* $A_1A_2A_3A_4A_5$ *inscribed in a circle are obtained four triangles, then the centres* $S_{1234}(0)$, $S_{2345}(0)$, $S_{3451}(0)$, $S_{4512}(0)$ *and* $S_{5123}(0)$ *of five circles passing through the centro-ides of these four triangles in each set are concircular.*

Corollary 2. *The above theorem holds as well for orthogocenters of all the triangles in tha Corollary* 1.

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