A Study on the Multifarious N-sided Polygons

By Naoo SAKURA

Department of Mathematics, Facullty of Liberal Arts and Science, Shinshu University

(Received Oct, 30, 1962)

In this treatise the present writer will make a study of the properties of multifarious n-sided polygons in the complex plane.

1. First we have following theorem on three n-sided polygons.

Theorem 1. *There are given three n-sided polygons $a_1\ a_2 \cdots\cdots a_n$, $b_1\ b_2 \cdots\cdots b_n$ and $c_1\ c_2 \cdots\cdots c_n$. And let $a_1\ a_2 \cdots\cdots a_n$ be fixed; $b_1 b_2 \cdots\cdots\ b_n$ be the polygon inversely similar to $a_1\ a_2 \cdots\cdots a_n$; let $c_1\ c_2 \cdots\cdots c_n$ be put to a parallel translation into $b_1\ b_2 \cdots\cdots b_n$ in any position, then we find invariably the sum of the areas of the n triangles constant, which are constructed by a_1, a_2 and the centre of gravity of b_1, b_2, c_1 and c_2; a_2, a_3 and the centre of gravity of b_2, b_3, c_2 and c_3; $\cdots\cdots$; a_n, a_1 and centre of gravity of b_n, b_1, c_n and c_1.*

Proof. Let S be the required area of the n triangles and g_{ij} be the centre of gravity of b_i, b_j, c_i and c_j ($j=i+1$, $i=1, 2, \cdots\cdots, n-1$ and $j=1$ if $i=n$),

$$(1) \qquad S=\frac{\sqrt{-1}}{4}\sum_{i=1}^{n}\begin{vmatrix} g_{ij} & \bar{g}_{ij} & 1 \\ a_i & \bar{a}_i & 1 \\ a_j & \bar{a}_j & 1 \end{vmatrix}$$

$$=\frac{\sqrt{-1}}{4}\left[\sum (g_j-a_i)\bar{g}_{j\,i}-\sum(\bar{a}_j-\bar{a}_i)g_{i\,j}+\sum\left|\begin{matrix} a_i & \bar{a}_i \\ a_j & \bar{a}_j \end{matrix}\right|\right].$$

Since $b_1\ b_2 \cdots\cdots b_n$ is inversely similar to $a_1\ a_2 \cdots\cdots\ a_n$, and $c_1\ c_2 \cdots\cdots\ c_n$ is the result of the above parallel translation of $b_1\ b_2 \cdots\cdots b_n$, we have

$$(2) \qquad \frac{\bar{a}_2-\bar{a}_1}{b_2-b_1}=\frac{\bar{a}_3-\bar{a}_2}{b_3-b_2}=\cdots\cdots\cdots=\frac{\bar{a}_1-\bar{a}_n}{b_1-b_n}=k,$$

$$(3) \qquad \frac{\bar{a}_2-\bar{a}_1}{c_2-c_1}=\frac{\bar{a}_3-\bar{a}_2}{c_3-b_2}=\cdots\cdots\cdots=\frac{\bar{a}_1-\bar{a}_n}{c_1-c_n}=k,$$

where k is a constant complex number. Since g_{ij} is the centre of gravity b_i, b_j, c_i and c_j, we have

$$(4) \qquad 4g_{ij}=b_i+c_i+b_j+c_j.$$

From (2), (3) and (4) we have

$$\frac{8(\bar{a}_2-\bar{a}_1)\,g_{12}}{(b_2+c_2)^2-(b_1+c_1)^2}=\frac{8(\bar{a}_3-\bar{a}_2)\,g_{23}}{(b_3+c_3)^2-(b_2+c_2)^2}=\cdots\cdots\cdots\cdots\cdots\cdots$$

$$=\frac{8(\bar{a}_1-\bar{a}_n)\,g_{n1}}{(b_1+c_1)^2-(b_n+c_n)^2}=k,$$

Hence

$$\sum(\bar{a}_j-\bar{a}_i)g_{ij}=\frac{k}{8}\sum\left[(b_j+c_j)^2-(b_i+c_i)^2\right]=0,$$

$$\sum(a_j-a_i)\bar{g}_{ij}=0.$$

S is, therefore, equal to the area of the polygon $a_1\ a_2\cdots\cdots a_n$.

2. The following theorem concerning two n-sided regular polygons is now established.

Theorem 2. *There are given the two n-sided regular polygons $a_1\ a_2\cdots\cdots a_n$, $b_1 b_2\cdots\cdots b_n$ which have all their corresponding vertices in the same direction. Let a_j', b_j' be the points which devide the corresponding sides $a_j a_{j+1}$, $b_j b_{j+1}$ of the regular polygons to the ratio $t_1 : 1-t_1$, $t_2 : 1-t_2$ respectively, and further let c_j be the point dividing $a_j' b_j'$ to the ratio $t : 1-t$, then the polygon $c_1\ c_2\cdots\cdots c_n$ is also a regular n-sided polygon in this case.*

Proof. Let a be the complex number of the centre of the n-sided regular polygon $a_1\ a_2\cdots\cdots a_n$, and the radius of the circumcircle be r_a, the amplitude of a_1 be θ_a.

Denoting

$$r_a(\cos\,\theta_a+i\,\sin\,\theta_a)=a',$$

$$\cos\frac{2\pi}{n}+i\,\sin\frac{2\pi}{n}=\omega,$$

we have

$$(1)\qquad a_j=a+a'\omega^{j-1},\qquad (j=1,2,\cdots\cdots,n).$$

Similarly, we have

$$(2)\qquad b_j=b+b'\omega^{j-1},\qquad (j=1,2,\cdots\cdots,n).$$

The point a_j' which divides $a_j\ a_{j+1}$ to a given ratio $t_1 : 1-t_1$ is derived as follows; from (1).

$$a_j'=a_{j+1}t_1+a_j(1-t_1)$$

$$=(a+a'\omega^j)t_1+(a+a'\omega^{j-1})(1-t_1)$$

$$=a+a'\omega^{j-1}(\omega t_1+1-t_1),$$

where ωt_1+1-t_1 is a constant complex number, and is put k_1, we have

$$(3)\qquad a'_j=a+a'k_1\omega^{j-1}.$$

Similarly, we have

$$b'_j = b + b' k_2\, \omega^{j-1}, \qquad (4)$$

where

$$k_2 = \omega\, t_2 + 1 - t_2.$$

Further the point c_j which divides $a'_j\ b'_j$ to a given ratio $t\ :\ 1-t$ is given by (3) and (4) as follows;

$$\begin{aligned} c_j &= b_j' t + a_j'(1-t) \\ &= (b + b' k_2\, \omega^{j-1})\, t + (a + a' k_1\, \omega^{j-1})(1-t) \\ &= [bt + a(1-t)] + [b' k_1 t + a' k_2 (1-t)]\, \omega^{j-1}. \end{aligned}$$

Then putting

$$b\,t + a\,(1-t) = c, \qquad b' k_1 t + a' k_2 (1-t) = c',$$

we have

$$c_j = c + c' \omega^{j-1}.$$

It is, therefore, obvious that $c_1\ c_2 \cdots\cdots c_n$ is a n-sided regular polygon. As a speeial case of this theorem, I get the following corollary.

Corollary 1. *There are given the two n-sided regular polygons $a_1\ a_2 \cdots \cdots a_n, b_1\ b_2 \cdots\cdots b_n$ which have all their vertices in the same direction. Now, let g_i be the centroids of n triangles $a_j\ b_j\ b_{j+1}$, then the n-sided polygon $g_1\ g_2 \cdots\cdots g_n$ is also found a n-sided regular polygon.*

The another proof of the Theorem 2 is easily furnished by the application of the following theorem intrduced in 1940 by *Jesse, Douglas.

Theorem. *The necessary and sufficient condition in which $a_1\ a_2 \cdots\cdots a_n$ represents a regular polygon is*

$$a_1 + \omega^p a_2 + \cdots\cdots + \omega^{(n-1)p} a_n = 0, \qquad (p = 1, 2, \cdots\cdots,\ n-1)$$

where ω is the primitive root of $x^n = 1$.

Next, I will give the distinct proof of the above Theorem 2 by means of this theorem.

Proof. Since both $a_1\ a_2 \cdots\cdots a_n$ and $b_1\ b_2 \cdots\cdots b_n$ are n-sided regular polygons, from Douglas' theorem we can obtain

$$\begin{aligned} &a_1 + a_2\, \omega + \cdots\cdots + a_n \omega^{n-1} = 0, \\ &b_1 + b_2\, \omega + \cdots\cdots + b_n\, \omega^{n-1} = 0, \qquad (p = 1). \end{aligned}$$

Now

$$a'_j = t_1 a_{j+1} + (1-t_1)\, a_j, \qquad b_j' = t_2\, b_{j+1} + (1-t_2) b_j.$$

Consequently we have

$$a_1' + a_2' \omega + \cdots\cdots + a_n' \omega^{n-1} = 0,$$

$$b_1'+b_2'\omega+\cdots\cdots+b_n'\omega^{n-1}=0,$$

$$c_1+c_2\omega+\cdots\cdots+c_n\omega^{n-1}$$
$$=[tb_1'+(1-t)a_1']+[tb_2'+(1-t)a_2']\omega+\cdots\cdots+[tb_n'-(1-t)a_n']\omega^{n-1}$$
$$=t(b_1'+b_2'\omega+\cdots\cdots+b_n'\omega^{n-1})+(1-t)(a_1'+a_2'\omega+\cdots\cdots+a_n'\omega^{n-1})=0.$$

Hence $c_1\ c_2\cdots\cdots c_n$ is a n-sided regular polygon.

3. This time I will launch my study into several properties of directly similar n n-sided polygons.

Theorem 3. *There are given n n-sided polygons $a_1\ a_2\cdots\cdots a_n$, $b_1\ b_2\cdots\cdots b_n$, $\cdots\cdots$, $l_1\ l_2\cdots\cdots l_n$, and if these are directly similar to the fixed polygon $\alpha_1\ \alpha_2\cdots\cdots\alpha_n$, the n-sided polygon $g_1\ g_2\cdots\cdots\cdots g_n$ which denote the centroids g_i of n n-sided polygons $a_i\ b_i\cdots\cdots l_i$ $(i=1, 2, \cdots\cdots, n)$ is also directly similar to $\alpha_1\ \alpha_2\cdots\cdots\alpha_n$.*

Proof. Since n n-sided polygons $a_1\ a_2\cdots\cdots a_n, b_1\ b_2\cdots\cdots b_n$, $\cdots\cdots$, $l_1\ l_2\cdots\cdots l_n$ are directly similar to the fixed polygon $\alpha_1\ \alpha_2\cdots\cdots\alpha_n$,
we have

$$a_2-a_1=k_a(\alpha_2-\alpha_1),\qquad b_2-b_1=k_b(\alpha_2-\alpha_1),\quad \cdots\cdots\cdots,$$
$$a_3-a_2=k_a(\alpha_3-\alpha_2),\qquad b_3-b_2=k_b(\alpha_3-\alpha_2),\quad \cdots\cdots\cdots,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\quad \cdots\cdots\cdots,$$
$$a_1-a_n=k_a(\alpha_1-\alpha_n),\qquad b_1-b_n=k_b(\alpha_1-\alpha_n),\quad \cdots\cdots\cdots,$$
$$l_2-l_1=k_l(\alpha_2-\alpha_1),$$
$$l_3-l_2=k_l(\alpha_3-\alpha_2),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$l_1-l_n=k_l(\alpha_1-\alpha_n),$$

where all k_a, k_b, $\cdots\cdots$, k_l are complex numbers.
We have

$$g_{j+1}-g_j=\frac{1}{n}(a_{j+1}+b_{j+1}+\cdots\cdots+l_{j+1})-\frac{1}{n}(a_j+b_j+\cdots\cdots+l_j)$$
$$=\frac{1}{n}(k_a+k_b+\cdots\cdots+k_l)(\alpha_{j+1}-\alpha_j).$$

The above expression denotes that n-sided polygon $g_1\ g_2\cdots\cdots g_n$ is directly similar to the n-sided polygon $\alpha_1\alpha_2\cdots\cdots\alpha_n$.
Then we have the following corollary as a special case of this theorem.

Corollary 2. *There are given n n-sided polygons $a_1\ a_2\cdots\cdots a_n$, $b_1\ b_2\cdots\cdots b_n$, $\cdots\cdots\cdots$, $l_1\ l_2\cdots\cdots l_n$ which have all their vertices in the same direction. The n-sided polygon $g_1\ g_2\cdots\cdots g_n$ which denote the centroids g_i of n n-sided polygons $a_i\ b_i\cdots\cdots l_i$ $(i=1,\ 2, \cdots\cdots, n)$ is also a n-sided regular polygon.*

Corollary 3. *There are given n n-sided polygons $a_1\ a_2\cdots\cdots a_n$, $b_1\ b_2\cdots\cdots b_n$, $\cdots\cdots$, $l_1\ l_2\cdots\cdots l_n$ any two of which are directly similar to each other. When the directly similar right-angled triangles be given which have the hypotenuses* *that are made of the corresponding sides $a_i\ a_{i+1}$, $b_i\ b_{i+1}$, ……, $l_i\ l_{i+1}$ with the 3rd vertices a_i', b_i', ……, l_i' respectively, and furthermore let a_i'', b_i'', ……, l_i'' be the symmetric points of a_i', b_i', ………, l_i 'with respect to the corresponding hypotenuses, and g_i', g_i'', g_i be the centroids of the n-sided polygons a_i' b_i'………l_i', a_i'' b_i''………l_i'', a_i b_i………l_i $(i=1,\ 2,\ \cdots\cdots, n)$ respectively, then the arithmetic mean of the areas of g_1' g_2'………g_n' and g_1'' g_2''………g_n'' is equal to that of g_1 g_2………g_n.*

This theorem is evident from the Theorem 2 and Iwata's theorem.

Corollary 4. *There are given n n-sided polygons a_1 a_2………a_n, b_1b_2………b_n, ………, l_1 l_2………l_n any two of which are directly similar to each other. When there are the corresponding sides $a_i\ a_{i+1}$, $b_i\ b_{i+1}$, ………, $l_i\ l_{i+1}$, and are made, on the outes sides (or the insides) of them all, the triangles $a_i\ a_i'\ a_{i+1}$, $b_i\ b_i'\ b_{i+1}$, ………, $l_i\ l_i'\ l_{i+1}$ any two of which are directly similar to each other, then the centroid of n^2 vertices a_i, b_i, ………, l_i coincides with that of the origieal n^2 vertices a_i, b_i, ………, l_i.*

Proof. Since n n-sided polygons a_1 a_2………a_n, b_1 b_2………b_n, ………, l_1 l_2………l_n are directly similar to each other, we have

$$\frac{a_1'-a_1}{a_2-a_1}=\frac{a_2'-a_2}{a_3-a_2}=\cdots\cdots\cdots\cdots=\frac{a_n'-a_n}{a_1-a_n},$$

$$\frac{b_1'-b_1}{b_2-b_1}=\frac{b_2'-b_2}{a_3-a_2}=\cdots\cdots\cdots\cdots=\frac{b_n'-b_n}{b_1-b_n},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{l_1'-l_1}{l_2-l_1}=\frac{l_2'-l_2}{l_3-l_2}=\cdots\cdots\cdots\cdots=\frac{l_n'-l_n}{l_1-l_n}.$$

Consequently we obtain

$$a_1'+a_2'+\cdots\cdots\cdots+a_n'=a_1+a_2+\cdots\cdots\cdots+a_n,$$

$$b_1'+b_2'+\cdots\cdots\cdots+b_n'=b_1+b_2+\cdots\cdots\cdots+b_n,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$l_1'+l_2'+\cdots\cdots\cdots+l_n'=l_1+l_2+\cdots\cdots\cdots+l_n,$$

Therefore from these expressions, we have

$$\frac{1}{n^2}(\sum a_1'+\sum b_i'+\cdots\cdots+\sum l_i')=\frac{1}{n}(\sum a_i+\sum b_i+\cdots\cdots+\sum l_i)$$

Corollary 5. *There are given n ovals any two of which are directly similar to each other, and are also given the corresponding vertices of each oval as a, b, ………, l, with the centroid of the resultant n-sided polygon in g. When the n-sided polygon move about while taking their corresponding points the locus of g is found to be the oval directly similar to the original ovals.*

Finally there is obtained the following theorem on the areas of n n-sided regular polygons.

Theorem 3. *There are given n n-sided regular polygons* $a_1\ a_2\cdots\cdots a_n$, $b_1\ b_2\cdots\cdots b_n$, $\cdots\cdots\cdots$, $l_1\ l_2\cdots\cdots l_n$ *which have their vertices in the same direction. When there be given the areas* S_1, S_2, $\cdots\cdots\cdots$, S_n *made of* $a_1\ a_2\cdots\cdots a_n$, $b_1\ b_2\cdots\cdots b_n$, $\cdots\cdots\cdots$, *and* $l_1\ l_2\cdots\cdots\cdots l_n$ *respectively, and also be given* g_i *as the centroids of n n-sided polygons* $a_i\ b_i\cdots\cdots\cdots l_i$ $(i=1, 2, \cdots\cdots, n)$ *with S which is the resultant area of n-sided regular polygon* $g_1\ g_2\cdots\cdots g_n$, *then we get the following inequality,*

$$\sum_{i=1}^{n} \sqrt{S_i} \geqq n\sqrt{S}.$$

Proof. We have

$$\frac{S_1}{|a_{i+1}-a_i|^2}=\frac{S_2}{|b_{i+1}-b_i|^2}=\cdots\cdots\cdots=\frac{S_n}{|l_{i+1}-l_i|^2}$$

$$=\frac{S}{\left|\dfrac{a_{i+1}+b_{i+1}+\cdots\cdots+l_{i+1}}{n}-\dfrac{a_i+b_i+\cdots\cdots+l_i}{n}\right|^2},$$

consequently we get

$$\frac{\sqrt{S_1}+\sqrt{S_2}+\cdots\cdots+\sqrt{S_n}}{|a_{i+1}-a_i|+|b_{i+1}-b_i|+\cdots\cdots\cdots+|l_{i+1}-l_i|}$$

$$=\frac{n\sqrt{S}}{|(a_{i+1}+b_{i+1}+\cdots\cdots+l_{i+1})-(a_i+b_i+\cdots\cdots+l_i)|}.$$

Now, since we have the following relation

$$|a_{i+1}-a_i|+|b_{i+1}-b_i|+\cdots\cdots\cdots+|l_{i+1}-l_i| \geqq$$
$$|(a_{i+1}+b_{i+1}+\cdots\cdots+l_{i+1})-(a_i+b_i+\cdots\cdots+l_i)|,$$

we get the inequality

$$\sum_{i=1}^{n} \sqrt{S_i} \geqq n\sqrt{S}.$$

References

* DOUGLAS, J. (1940) Geometry of the polygons in the Complex Plane. *Journal of Mathematics and Physics* vol. 19, p. 93–130.

(1) IWATA, S. Two kinds of Generalizations of the Theorems Concering the Triangle. *The Science Report of Faculty of Liberal Arts and Education, Gifu University. (Natural Science)* vol. 2, no. 5 (1961).

(2) KOBAYASHI, M. (1954) Geometry of Complex numbers. Kyoritsu Shuppan, Tokyo, (in Japanese).

(3) KUBOTA, T. (1940) Special Theory of Elementary Geometry. Kyoritsu Shuppan, Tokyo, (in Japanese).

(4) WASEDA, Y., KITO, S., HONDA, H. (1952) Application of Geometry of complex numbers to Engineering. Ōmusha Shuppan, Kyoto, (in Japanese).