Note on c. s. s. Notions

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Introduction. The notion of c. s. s. complexes was originally introduced by EILENBERG–ZILBER [4] as a precise abstract approach to the topological spaces. In the early development, it was rather homological. However, by the recent contributions of MILNOR [9] and KAN [6], the c. s. s. complexes have attained the homotopical (geometrical) aspect.

The c. s. s. analogues of certain topological notions have already been established, and the c. s. s. notions have been able to simplify the proofs of many propositions. However, it seems that c. s. s. notions can provide us with a direct ladder from group-theoretic concepts to geometric configurations (cf. [2] and [3]). For the solution of this problem will be required more expositions of topological concepts in terms of the c. s. s. theory.

In the present note, we shall define the relative homotopy groups $\pi_n(K, L; a^0)$ for the Kan-pair (K, L, a^0), and then prove certain properties of them.

1. ***Definition of*** $\pi_n(K, L; a^0)$.

A *c. s. s. complex* K is a collection of elements x's which are called simplicies, associated with the following three functions:

(i) The *dimension function* $D : K \longrightarrow Z_+ =$ non-negative integers. A simplex x for which $D(x) = n$ is said to be n-dimensional, and we denote the totality of n-simplices by K_n.

(ii) The i-th *face operator* $\partial_i : K_n \longrightarrow K_{n-1}$, $0 \leqq i \leqq n$, $n \geqq 1$.

(iii) The j-th *degeneracy operator* $s_j : K_n \longrightarrow K_{n+1}$, $0 \leqq j \leqq n$, $n \geqq 0$.

Moreover, face operators ∂_i's and degeneracy operators s_j's satisfy the following commutation laws :

$$\partial_i \partial_j = \partial_{j-1} \partial_i \qquad \text{for } i < j,$$

$$s_j s_j = s_{j+1} s_i \qquad \text{for } i \leqq j,$$

$$\partial_i s_j = \begin{cases} s_{j-1} \partial_i & \text{for } i < j, \\ \text{identities} & \text{for } i = j \text{ and } j+1, \\ s_j \partial_{i-1} & \text{for } i > j+1. \end{cases}$$

An *equation* in the c. s. s. complex K is a set of n $(n-1)$-simplices $x_0, x_1, \cdots\cdots, x_{k-1}, x_{k+1}, \cdots\cdots, x_n$ $(0 \leqq k \leqq n)$ satisfying $\partial_{j-1} x_i = \partial_i x_j$ for all $i < j$ and $i \neq k \neq j$, which is denoted by $[x_0, \cdots\cdots, x_{k-1}, *, x_{k+1}, \cdots\cdots, x_n]$. If there exists such an

n-simplex x as $\partial_i x = x_i$ for all $i \neq k$, then the equation $[x_0, \cdots\cdots, x_{k-1}, *, x_{k+1}, \cdots\cdots, x_n]$ is said to be *solvable*, and we call x a *solvent* and $\partial_k x$ a *solution* of this equation. A c. s. s. complex K is called a *Kan-complex*, provided that all the equations in K are always solvable.

Now, if a 0-simplex a^0 is distinguished in K, then the pair of K and a^0 is called the c. s. s. complex with the base point a^0 and is denoted by (K, a^0).

Two n-simplices x and y of K are to be *homotopic* in K (in notation $x \sim y$), provided that

(i) $\partial_i x = \partial_i y$ for all $0 \leqq i \leqq n$,

(ii) there exists such an $(n+1)$-simplex z, which is called a homotopy from x to y, in K as

(iia) $\partial_i z = s_{n-1}\, \partial_i x = s_{n-1}\, \partial_i y$ for $0 \leqq i \leqq n-1$,

(iib) $\partial_n z = x$ and $\partial_{n+1} z = y$.

Lemma 1. *In a Kan-complex, the homotopy relation is an equivalence relation.*

A non-empty subset L of K is a *subcomplex* of K, provided that L is a c. s. s. complex in the c. s. s. structure of K. If the base point a^0 is chosen in L, then the triple K, L and a^0 is called the pair of c. s. s. complexes relative to a^0, and denoted by $(K, L; a^0)$. If both K and L are Kan-complexes, we call $(K, L; a^0)$ the *Kan-pair* (relative to a^0).

Let $\Gamma_n(K, L; a^0)$ be the collection of n-simplices in K satisfying (i) $\partial_0 x \epsilon L$, and (ii) $\partial_i x = a^{n-1}$ for $0 < i \leqq n$, where $a^{n-1} = s_{n-2} \cdots\cdots s_0 a^0$.

Definition 1. Two n-simplices x_1 and x_2 of $\Gamma_n(K, L; a^0)$ are *homotopic* in $\Gamma_n(K, L; a^0)$ (in notation $x_1 \underset{y}{\sim} x_2$ in $\Gamma_n(K, L; a^0)$ or simply $x_1 \sim x_2$) provided that

(i) $\partial_0 x_1$ and $\partial_0 x_2$ are homotopic in L,

(ii) there exists such an $(n+1)$-simplex y (which is called a homotopy from x_1 to x_2) as

(iia) $\partial_0 y$ is a homotopy in L from $\partial_0 x_1$ to $\partial_0 x_2$,

(iib) $\partial_i y = a^n$ for $0 < i < n$,

(iic) $\partial_n y = x_1$ and $\partial_{n+1} y = x_2$.

Lemma 2. $x_1 \underset{y}{\sim} x_2$ *in* $\Gamma_n(K, L; a^0)$ *implies* $\partial_0 x_1 \sim \partial_0 x_2$ *in* L.

Lemma 3. *If* $(K, L; a^0)$ *is a Kan-pair, then the homotopy relation is an equivalence relation in* $\Gamma_n(K, L; a^0)$.

Proof. *Reflexivity.* Let x be an element of $\Gamma_n(K, L; a^0)$. Then, the direct computation shows that $y = s_n x$ is a homotopy in $\Gamma_n(K, L; a^0)$ from x to x.

Symmetry and Transitivity. Suppose that $x_1 \underset{y_n}{\sim} x_2$ and $x_1 \underset{y_{n+1}}{\sim} x_3$ in $\Gamma_n(K, L; a^0)$. Let y_0 be a solvent of the equation $[y_{00}, \cdots\cdots, y_{0n}, *]$, where $y_{0i} = a^n$ for $0 \leqq i < n-1$, $y_{0, n-1} = \partial_0 y_n$, and $y_{0n} = \partial_0 y_{n+1}$, and z a solution of the equation $[y_0, \cdots\cdots, y_n, y_{n+1}, *]$, where y_0, y_n and y_{n+1} are already determined simplices and $y_i = a^{n+1}$ for $0 < i < n$. Then, as easily seen, $x_2 \underset{z}{\sim} x_3$ in $\Gamma_n(K, L; a^0)$.

In the sequel of this note, we assume that $(K, L; a^0)$ is the fixed Kan-pair.

Now, we shall define a multiplication in $\Gamma_n(K, L; a^0)$. Let x and y be any two simplices of $\Gamma_n(K, L; a^0)$ $(n \geqq 2)$, x_0 a solvent of the equation $[a^{n-1}, \cdots\cdots,$

a^{n-1}, $\partial_0 x$, $*$, $\partial_0 y$] in L, and z a solution of the equation $[x_0, a^n, \cdots\cdots, a^n, x, *, y]$. And, we define $x \cdot y = z$, which is called a product of x and y. Obviously, z is not unique.

Lemma 4. *Any two products of x and y are homotopic in $\Gamma_n(K, L; a^0)$.*

Proof. If we have two solvents x_0 and x_0' of the equation $[a^{n-1}, \cdots\cdots, a^{n-1}, \partial_0 x, *, \partial_0 y]$, let w be a solvent of the equation $[x_0, a^n, \cdots\cdots, a^n, x, *, y]$ and z be the solution of the same equation defined by $\partial_n w = z$, and in a similar way w' and z'. Define w_0 as a solvent of the equation $[a^n, \cdots\cdots, a^n, s_{n-1}\partial_0 x, *, x_0, x_0']$, then $\partial_{n-1} w_0$ gives a homotopy from $\partial_{n-1} x_0 = \partial_0 z$ to $\partial_{n-1} x' = \partial_0 z'$. Next, let v be a solution of the equation $[w_0, \cdots\cdots, w_{n-1}, *, w_{n+1}, w_{n+2}]$, where w_0 is the above defined, $w_i = a^{n+1}$ for $0 < i < n-1$, $w_{n-1} = s_n x$, $w_{n+1} = w$ and $w_{n+2} = w'$. Then, v is a homotopy from z to z' in $\Gamma_n(K, L; a^0)$.

Lemma 5. *Let x, x' and y be three elements in $\Gamma_n(K, L; a^0)$, $(n \geqq 2)$, and assume $x \sim x'$ in $\Gamma_n(K, L; a^0)$, then we have $x \cdot y \sim x' \cdot y$.*

Proof is similar to the preceding one.

Lemma 6. *Two elements x and y of $\Gamma_n(K, L; a^0)$ are homotopic, if and only if there exists an $(n+1)$-simplex z satisfying*

$$\partial_i z = \begin{cases} \text{an } n\text{-simplex of } L & \text{if } i = 0, \\ x & \text{if } i = n-1, \\ y & \text{if } i = n, \\ a^n & \text{otherwise.} \end{cases}$$

Proof. Suppose we have an $(n+1)$-simplex z satisfying the conditions. Define w as a solvent of the equation $[a^n, \cdots\cdots, a^n, *, s_{n-1}\partial_0 y, s_{n-2}\partial_0 y, \partial_0 z]$, then $\partial_{n-2} w$ is a homotopy from $\partial_0 y$ to $\partial_0 x$. Next, let v be a solution of the equation $[w, a^{n-1}, \cdots\cdots, a^{n-1}, *, s_n y, s_{n+1} y, z]$, then v is a homotopy from y to x. Therefore, Lemma 3 ensures our required result. The converse is quite similar, so the proof will be omitted.

Lemma 7. *If $y \sim y'$, then we have $x \cdot y \sim x \cdot y'$.*

Our repeated method and Lemma 6 will give the proof.

Now, let $\pi_n(K, L; a^0)$ be the set of homotopy classes in $\Gamma_n(K, L; a^0)$. The homotopy class containing x will be denoted by $\bar{x}$. Then, if $\xi = \bar{x}$ and $\eta = \bar{y}$ are any two elements of $\pi_n(K, L; a^0)$, by virtue of Lemmas 4–7, we can define $\zeta = \xi \cdot \eta$ by $\zeta = \bar{z}$, $z = x \cdot y$. Moreover, Lemma 2 enables us to define a map ∂: $\pi_n(K, L; a^0) \longrightarrow \pi_{n-1}(L; a^0)$[1] by $\partial \xi = \overline{\partial_0 x}$, where $\xi = \bar{x}$ is an arbitrary element of $\pi_n(K, L; a^0)$.

Theorem 1. *$\pi_n(K, L; a^0)(n \geqq 2)$ is a group.*

Proof. *Associativity.* Let x, y and z be arbitrary elements in $\Gamma_n(K, L; a^0)$. Define w_i's, $0 \leqq i \leqq n+2$, $i \neq n$, by the following:

$$w_i = a^{n+1} \qquad \text{for } 0 < i < n-1,$$

$$w_{n-1} = \text{a solvent of the equation defining } xy,$$

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1) If L is the subcomplex having only one non-degenerate simplex a^0, $\pi_n(K, L; a^0)$ is denoted by $\pi_n(K; a^0)$, and this is the same defined by Moore [10].

w_{n+1}=a solvent of the equation defining $(xy)z$,

w_{n+2}=a solvent of the equation defining yz,

w_0=a solvent of the equation $[\partial_0 w_1, \cdots\cdots, \partial_0 w_{n-1}, *, \partial_0 w_{n+1}, \partial_0 w_{n+2}]$.

Then, the solution of the equation $[w_0, w_1, \cdots\cdots, w_{n-1}, *, w_{n+1}, w_{n+2}]$ implies that $(xy)z=x(yz)$.

Divisibility. Let x and y be arbitrary elements in $\Gamma_n(K, L; a^0)$. Then, we can easily find the elements z_1 and z_2 in $\Gamma_n(K, L; a^0)$ satisfying $xz_1=y$ and $z_2x=y$.

Definition 2. $\pi_n(K, L; a^0)$ is called the *n-dimensional relative homotopy groups* of the Kan-pair $(K, L; a^0)$.

Lemma 2 and the definition of ∂ lead us to

Theorem 2. *$\partial : \pi_n(K, L; a^0) \longrightarrow \pi_{n-1}(L; a^0)$ $(n \geqq 2)$ is a group homomorphism.* (We call ∂ the *boundary homomorphism* of the relative homotopy groups of the Kan-pair $(K, L; a^0)$.)

2. ***Commutativity of*** $\pi_n(K, L; a^0)$

Lemma 8. *Let x, y and z be any elements of $\Gamma_n(K, L; a^0)$ $(n \geqq 3)$. If there exists such an $(n+1)$-simplex w as*

$$\partial_0 w \in L,$$
$$\partial_i w = a^n \qquad \text{for } 0 < i < n-2 \text{ and } i=n+1,$$
$$\partial_{n-2} w = x,$$
$$\partial_{n-1} w = y,$$
$$\partial_n w = z,$$

then $\bar{z} \cdot \bar{x} = \bar{y}$.

Proof. Define $n+2$ $(n+1)$-simplices v_i's, $0 \leqq i \leqq n+2$, $i \neq n$, by the following :

$$v_i = a^{n+1} \qquad \text{for } 0 < i < n\text{-}2,$$
$$v_{n-2} = s_n x,$$

v_{n-1}=a solvent of the equation, which asserts the existence of u satisfying $ux=y$,

$$v_{n+1} = w,$$
$$v_{n+2} = s_{n-2} x,$$

v_0=a solvent of the equation $[\partial_0 v_1, \cdots\cdots, \partial_0 v_{n-1}, *, \partial_0 v_{n+1}, \partial_0 v_{n+2}]$,

Then, the solution of the equation $[v_0, \cdots\cdots, v_{n-1}, *, v_{n+1}, v_{n+2}]$ implies that $zx \sim y$.

Lemma 9. *Let x, y and z be any elements of $\Gamma_n(K, L; a^0)$, $(n \geqq 3)$. If there exists such an $(n+1)$-simplex w as*

$$\partial_0 w \in L,$$
$$\partial_i w = a^n \qquad \text{for } 0 < i < n-2 \text{ and } i=n-1,$$

$$\partial_{n-2}w=x,$$

$$\partial_n w=y,$$

$$\partial_{n+1}w=z,$$

then we have $\bar{z}=\bar{x}\,\bar{y}$.

Proof. Define $n+2$ $(n+1)$-simplices v_i's, $0\leqq i\leqq n+2$, $i\neq n+1$, by the following :

$$v_i=a^{n+1} \qquad \text{for } 0<i<n-2,$$

$$v_{n-2}=s_{n-2}x$$

$v_{n-1}=$a solvent of the equation $[u_0, \cdots\cdots, u_{n-1}, *, u_{n-1}]$, where $\partial_0 u_0 \epsilon L$ and $u_j=a^n$ for $0<j, j\neq n$,

$$v_n=w,$$

$$v_{n+2}=s_n z,$$

$v_0=$a solvent of the equation $[\partial_0 v_1, \cdots\cdots, \partial_0 v_n, *, \partial_0 v_{n+2}]$.

(*N. B.* $u=\partial_n v_{n-1}$ is a representative of $\bar{x}^{-1}$ by virtue of Lemma 8.) Then, the solution of the equation $[v_0, \cdots\cdots, v_n, *, v_{n+1}]$ implies $u\cdot z=y$, hence we have $\bar{x}\cdot\bar{y}=\bar{z}$.

Lemma 10. *Let x_1, x_2, x_3 and x_4 be arbitrary four elements in $\Gamma_n(K, L; a^0)$, $(n\geqq 3)$. Then, the existence of such an $(n+1)$-simplex y as $\partial_0 y\epsilon L$, $\partial_i y=a^n$ for $0<i<n-2$, $\partial_{n-2}y=x_1$, $\partial_{n-1}y=x_2$, $\partial_n y=x_3$ and $\partial_{n+1}y=x_4$ implies $x_1x_3=x_2x_4$.*

Proof. Lemma 9 and the same procedure as in preceding two lemmas lead us to the conclusion.

Theorem 3. *$\pi_n(K, L; a^0)$ is abelian for $n\geqq 3$.*

Proof. Let ξ and η be arbitrary two elements of $\pi_n(K, L; a^0)$, and x and y their representatives in $\Gamma_n(K, L; a^0)$, respectively. Define $n+2$ $(n+1)$-simplices w_i's, $0\leqq i\leqq n+2$, $i\neq n-1$ by the following :

$$w_i=a^{n+1} \qquad \text{for } 0<i<n-2,$$

$$w_{n-2}=s_n x,$$

$$w_{n-1}=w_n=s_{n+1}y,$$

$w_{n+2}=$a solvent of the equation $[u, a^n, \cdots\cdots, a^n, x, a^n, a^n, *]$, where $u\epsilon L$ is a solvent of the equation $[a^{n-1}, \cdots\cdots, a^{n-1}, \partial_0 x, a^{n-1}, a^{n-1}, *]$,

$w_0=$a solvent of the equation $[\partial_0 w_1, \cdots\cdots, \partial_0 w_n, *, \partial_0 w_{n+2}]$.

Then, $\partial_{n+1}w_{n+2}$ represents ξ by Lemma 9. Therefore, the solution of the equation $[w_0, \cdots\cdots, w_n *, w_{n+2}]$ and Lemma 10 yield $\xi\cdot\eta=\eta\cdot\xi$.

3. Homotopy addition theorem.

Theorem 4. *Let $\xi_1, \cdots\cdots, \xi_{n+1}$ be the elements of $\pi_{n+k}(K, L; a^0)$, $(n\geqq 2, k\geqq 0)$, and x_i's their representatives in $\Gamma_{n+k}(K, L; a^0)$. If there exists such an $(n+k+1)$-simplex y as $\partial_0 y\epsilon L$, $\partial_i y=a^{n+k}$ for $0<i\leqq k$ and $\partial_{k+j}y=x_j$ for $1\leqq j\leqq n+1$,*

then we have

$$\prod_{i=0}^{n} \xi_{n+1-i}^{(-1)^i} = \varepsilon \text{ (the unit element of } \pi_{n+k}).$$

Proof. We shall proceed on the induction on n. Obviously, our theorem is valid for $n=2$. Therefore, we can assume that the theorem is valid for $n<N$ $(N>2)$. But, we shall need the following

Lemma 11. *If* $\xi_{N+1}=\varepsilon$, *our theorem holds for* $n=N$.

Lemma 12. *If* $\xi_1=\varepsilon$, *our theorem holds for* $n=N$.

Lemma 13. *If* $\xi_{N-1}=\varepsilon$, *our theorem holds for* $n=N$.

These three lemmas will be proved by the consideration of appropriate equations.

Now, define $N+k+2$ $(N+k+1)$-simplices w_i's, $0\leqq i\leqq n+2$, $i\neq N+k+1$, by the following :

$w_i=a^{N+k+1}$ for $0<i\leqq k$.

w_{k+j}=a solvent of the equation $[v_j, a^{N+k}, \cdots\cdots, a^{N+k}, *, a^{N+k}, a^{N+k}, x_j]$, where $v_j \in L$ is a solvent of the equation $[a^{N+k-1}, \cdots\cdots, a^{N+k-1}, *, a^{N+k-1}, \partial_0 x_j]$, for $1\leqq j\leqq N-2$. Then, $u_j = \partial_{N+k+2} w_{k+j}$ represents ξ_j by Lemma 10.

w_{N+k-1}=a solvent of the equation $[v_{N-1}, a_{N+k}, \cdots\cdots\cdots, a_{N+k}, u_1, \cdots\cdots\cdots, u_{N-2}, a_{N+k}, *, x_{N-1}]$, where $v_{N-2} \in L$ is a solvent of the equation $[a^{N+k-1}, \cdots\cdots, a^{N+k-1}, \partial_0 u_1, \cdots\cdots, \partial_0 u_{N-2}, a^{N+k-1}, *, \partial_0 x_{N-1}]$. Then, $u_{N-1}=\partial_{N+k} w_{N+k-1}$ represents $\prod_{i=1}^{N-1} \xi_{N-i}^{(-1)^{i+1}}$ by Lemma 13. (*N. B.* $\pi_{N+k}(K, L; a^0)$ is abelian for $N>2$.)

$w_{N+k}=s_{N+k}x_N$.

$w_{N+k+2}=y$.

w_0=a solvent of the equation $[\partial_0 w_1, \cdots\cdots, \partial_0 w_{N+k}, *, \partial_0 w_{N+k+2}]$.

Then, the solution of the equation $[w_0, , w_{N+k}, *, w_{N+k+2}]$ implies $\xi_{N+1}\xi_{N-1}u_{N-1}=\varepsilon$, *i. e.*, $\prod_{i=0}^{n} \xi_{N+1-i}^{(-1)^i} = \varepsilon$.

4. *Further properties.*

Definition 3. Let $(K, L; a^0)$ and $(K', L'; b^0)$ be c. s. s. pairs. A function f: $(K, L; a^0) \longrightarrow (K', L'; b^0)$ is a *c. s. s. map* (or simply *map*), provided that it satisfies (i) $f(K_n)\subset K_n'$, (ii) $\partial_i' \circ f = f \circ \partial_i$ for all i, and (iii) $s_j' \circ f = f \circ s_j$ for all j.

Proposition 1. *Let* $f:(K, L; a^0) \longrightarrow (K', L'; b^0)$ *be a c. s. s. map of Kan-pairs. Then, we can define the homomorphisms* f_*: $\pi_n(K, L; a^0) \longrightarrow \pi_n(K', L'; b^0)$ *for all* $n\geqq 2$. (f_*'s are called the *induced homomorphisms.*) *Moreover, we have the following commutative diagram*:

$$\begin{array}{ccc} \pi_n(K, L; a^0) & \xrightarrow{f_*} & \pi_n(K', L'; b^0) \\ \partial \downarrow & & \downarrow \partial' \\ \pi_{n-1}(L; a^0) & \xrightarrow{f_*'} & \pi_{n-1}(L'; b^0) \end{array}$$

where f_*' *is the homomorphism induced by the restriction* $f \mid L : (L; a^0) \longrightarrow (L'; b^0)$.

Proposition 2. *Let* $f : (K, L; a^0) \longrightarrow (K', L'; b^0)$ *and* $g : (K', L'; b^0) \longrightarrow (K'', L''; c^0)$ *be the c. s. s. maps of Kan-pairs. Then, we have* $(g \circ f)_* = g_* \circ f_*$.

Proposition 3. *Let* $1 : (K, L; a^0) \longrightarrow (K, L; a^0)$ *be the identity map. Then* $1_* = 1$.

Definition 4. Let $i : (L; a^0) \longrightarrow (K; a^0)$ and $j : (K, a^0; a^0) \longrightarrow (K, L; a^0)$ be the inclusion maps. Then, the following sequence is called the *homotopy sequence* of the Kan-pair $(K, L; a^0)$:

$$\cdots\cdots \longrightarrow \pi_{n+1}(K, L; a^0) \xrightarrow{\partial} \pi_n(L; a^0) \xrightarrow{i_*} \pi_n(K; a^0) \xrightarrow{j_*}$$
$$\pi_n(K, L; a^0) \xrightarrow{\partial} \pi_{n-1}(L; a^0) \longrightarrow \cdots\cdots$$
$$\longrightarrow \pi_2(K, L; a^0) \xrightarrow{\partial} \pi_1(L; a^0) \xrightarrow{i_*} \pi_1(K; a^0).$$

Proposition 4. *The homotopy sequence of the Kan-pair is exact.*

Proposition 5. *Let K be the c. s. s. complex which has only one non-degenerate simplex* a^0. *Then,* $\pi_n(K; a^0) = \varepsilon$ *for all* $n \geqq 0$.

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