Note on ϕ-notions

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In [2], EILENBERG and CARTAN give a condition for a $R\Pi$-module Q to be ϕ-injective (or ϕ-projective) by means of norm homomorphism. In this note, we give such a condition by means of transfer homomorphism instead of norm homomorphism.

Throughout this note, let Π be a group and π be its subgroup of finite index, say n. Let R be a ring with the unit element 1. Define a ring homomorphism $\phi: R\pi \longrightarrow R\Pi$ of group rings by the inclusion $\pi \subset \Pi$. Then any $R\Pi$-module can be considered as an $R\pi$-module by means of ϕ (Cf. [2], Chap. II, §6). Now, for any R-homomorphism f of an $R\Pi$-module A into an $R\Pi$-module B, define its transfer $t(f) \in Hom_R(A, B)$ by

$$t(f)a = \sum_{i=1}^{n} \xi_i f(\xi_i^{-1} a) \qquad \text{for all } a \in A, \tag{1}$$

where $\xi_1 = \varepsilon$, $\xi_2, \cdots\cdots, \xi_n$ are the arbitrarily chosen left coset representatives of Π by π. Then, for any π-map f, we can prove that (i) $t(f)$ is independent on the choice of the representatives, and (ii) $t(f)$ is a Π-map (Cf. [2], Chap. XII, §8).

Lemma 1. *If f is a Π-map, then $t(f) = nf$.*

Lemma 2. *Let $f \in Hom_\Pi(A, B)$, $g \in Hom_\pi(B, C)$ and $h \in Hom_\Pi(C, D)$, then we have $t(hgf) = ht(g)f$.*

Definition 1. An $R\Pi$-module Q is said to be *ϕ-injective,* if one of the following three equivalent conditions is satisfied:

(I_1) For any $R\Pi$-module A and its $R\Pi$-submodule B, if a Π-map f: $B \longrightarrow Q$ extends to A as a π-map, then it extends to A as a Π-map.

(I_2) Let A be any $R\Pi$-module and B be its $R\Pi$-submodule, which is an $R\pi$-direct summand of A. Then, every Π-map $f: B \longrightarrow Q$ extends to a Π-map of A into Q.

(I_3) The Π-exact (in fact, π-splitting) sequence

$$0 \longrightarrow Q \xrightarrow{i} Hom_\pi(R\Pi, Q) \longrightarrow Coker.\, i \longrightarrow 0, \tag{2}$$

where i is defined by $i(x)\xi = \xi x$ for all $x \in Q$ and $\xi \in \Pi$, is Π-splitting.

(As to the equivalence of these three conditions, refer to [1].)

As the dual notion of the ϕ-injectivity, we define:

Definition 2. An $R\Pi$-module P is said to be *ϕ-projective,* if one of the following three equivalent conditions is satisfied:

(P_1) Given a Π-map p of an $R\Pi$-module A onto an $R\Pi$-module B, and a Π-map $f:P\longrightarrow B$, to which we have a π-map $g:P\longrightarrow A$ such that $pg=f$, there exists a Π-map $h:P\longrightarrow A$ such that $ph=f$.

(P_2) Let A be an $R\Pi$-module and B be its $R\Pi$-submodule, being an $R\pi$-direct summand of A. Then, for any Π-map $f:P\longrightarrow B$, there exists a Π-map $h:P\longrightarrow A$, satisfying $ph=f$, where $p:A\longrightarrow B$ is the projection.

(P_3) Let $\varphi:R\Pi\otimes_\pi P\longrightarrow P$ be a Π-map defined by $\varphi(\xi\otimes x)=\xi x$ for $\xi\epsilon\Pi$ and $x\epsilon P$, then the following Π-exact (in fact, π-splitting) sequence

$$0\longrightarrow Ker.\,\varphi\longrightarrow R\Pi\otimes_\pi P\overset{\varphi}{\longrightarrow}P\longrightarrow 0 \tag{3}$$

is Π-splitting.

Lemma 3. *The above three conditione* (P_1) $-$(P_3) *are equivalent.*

Proof. (P_1)$\Longrightarrow$(P_2) and (P_2)$\Longrightarrow$(P_3) are obvious.

(P_3)$\Longrightarrow$(P_1) : Since the sequence (3) is Π-splitting, there exists a Π-map $\psi:P\longrightarrow R\Pi\otimes_\pi P$, such that $\varphi\psi=1_P$(=the identity of P). Define a Π-map F: $R\Pi\otimes_\pi P\longrightarrow A$ by $F(\xi\otimes x)=\xi g(x)$ for all $\xi\epsilon\Pi$ and $x\epsilon P$. Put $h=F\psi$. Then, h is a required Π-map of P into A. In fact, if $\psi(x)=\sum\xi_\lambda\otimes x_\lambda$, $\xi_\lambda\epsilon R\Pi$, $x_\lambda\epsilon P$,

$$\begin{aligned} ph(x)&=pF\psi(x)=pF(\sum\xi_\lambda\otimes x_\lambda)=p(\sum\xi_\lambda g(x_\lambda))\\ &=\sum\xi_\lambda pg(x_\lambda)=\sum\xi_\lambda f(x_\lambda), \end{aligned}$$

on the other hand,

$$f(x)=f\varphi\psi(x)=f(\sum\xi_\lambda x_\lambda)=\sum\xi_\lambda f(x_\lambda),$$

therefore $ph=f$, as required.

Definition 3. An $R\Pi$-module Q is *weakly ϕ-injective relative to A,* provided that the condition (I_2) holds for a specified $R\Pi$-module A.

Dually, an $R\Pi$-module P is *weakly ϕ-projective relative to A,* provided that the condition (P_2) holds for a specified $R\Pi$-module A.

Theorem 1. *An $R\Pi$-module Q is ϕ-injecitve, if and only if there exists a π-endomorphism $\vartheta:Q\longrightarrow Q$ such that $t(\vartheta)=1_Q$.*

Proof. *Sufficiency.* Consider the Π-exact and π-splitting sequence

$$0\longrightarrow Q\underset{j}{\overset{i}{\rightleftarrows}}Hom_\pi(R\Pi,Q)\longrightarrow Coker.\,i\longrightarrow 0,$$

where j is the π-map satisfying $ji=1_Q$. Then $j'=t(\vartheta j)$: $Hom_\pi(R\Pi,Q)\longrightarrow Q$ is a Π-map, and $j'i=t(\vartheta ji)=t(\vartheta)=1_Q$.

Necessity. Since Q is ϕ-injective, we have the Π-exact and Π-splitting sequence

$$0\longrightarrow Q\underset{j}{\overset{i}{\rightleftarrows}}Hom_\pi(R\Pi,Q)\longrightarrow Coker.\,i\longrightarrow 0$$

where i and j are both Π-maps and $ji=1_Q$.

For any $x \epsilon Q$, define a π-map $f_x : R\Pi \longrightarrow Q$ by

$$f_x(\xi) = \begin{cases} \xi x & \text{if } \xi \epsilon \pi, \\ 0 & \text{if } \xi \notin \pi. \end{cases}$$

Let $\rho : Q \longrightarrow Hom_\pi(R\Pi, Q)$ be a π-map defined by $\rho(x) = f_x$ for all $x \epsilon Q$. Then, we have $t(\rho) = i$. In fact, for any $\xi \epsilon \Pi$, let $\xi = \varpi \xi_{i_0}^{-1}$, where $\varpi \epsilon \pi$ and ξ_{i_0} is one of the above chosen representatives of Π by π, then

$$\begin{aligned}[t(\rho)(x)](\xi) &= [\sum \xi_i \rho(\xi_i^{-1} x)](\xi) \\ &= \sum (\xi_i f_{\xi_i^{-1} x})(\xi) \\ &= \sum f_{\xi_i^{-1} x}(\xi \xi_i) \\ &= \varpi \xi_{i_0}^{-1} x = \xi x \\ &= i(x)(\xi).\end{aligned}$$

Putting $\vartheta = j\rho$, we have a π-endomorphism $\vartheta : Q \longrightarrow Q$, and $t(\vartheta) = t(j\rho) = j(t(\rho)) = ji = 1_Q$.

Theorem 2. *An $R\Pi$-module P is ϕ-projective, if and only if there exists a π-endomorphism $\vartheta : P \longrightarrow P$ such that $t(\vartheta) = 1_P$.*

Proof. *Sufficiency.* Consider the Π-exact and π-splitting exact sequence (3), i.e.,

$$0 \longrightarrow Ker.\ \varphi \longrightarrow R\Pi \otimes_\pi P \underset{\psi}{\overset{\varphi}{\rightleftarrows}} P \longrightarrow 0,$$

where ψ is a π-map defined by $\psi(x) = 1 \otimes x$ for all $x \epsilon P$. Then $\psi' = t(\psi\vartheta) : P \longrightarrow R\Pi \otimes_\pi P$ is a Π-map, and $\varphi\psi' = t(\varphi\psi\vartheta) = t(\vartheta) = 1_P$.

Necessity. Let P be a ϕ-projective module, then we have the Π-exact and Π-splitting sequence

$$0 \longrightarrow Ker.\ \varphi \longrightarrow R\Pi \otimes_\pi P \underset{\varphi}{\overset{\psi}{\rightleftarrows}} P \longrightarrow 0,$$

where both φ and ψ are Π-maps and $\varphi\psi = 1_P$. For any $x \epsilon P$, we can denote $\psi(x) = \sum_{i=1}^{n} \xi_i \otimes \vartheta_i(x)$, where $\xi_1 = \varepsilon, \xi_2, \cdots\cdots, \xi_n$ are the left cosets representatives of Π by π chosen in the definition of transfer. Then $\vartheta = \vartheta_1$ is a π-endomorphism of P, which satisfies $t(\vartheta) = 1_P$. In fact,

$$\psi(\xi_i^{-1} x) = \varepsilon \otimes \vartheta_1(\xi_i^{-1} x) + \cdots\cdots,$$

$$\xi_i^{-1} \psi(x) = \xi_i^{-1}(\sum \xi_j \otimes \vartheta_j(x)) = \cdots\cdots + \varepsilon \otimes \vartheta_i(x) + \cdots\cdots,$$

imply $t(\vartheta)x = \sum \xi_i \vartheta_i(x)$, because ψ is a Π-map.

On the other hand, $\varphi\psi(x) = x$ implies $\sum \xi_i \vartheta_i(x) = x$. Consequently, we have $t(\vartheta)x = x$ for all $x \epsilon P$.

Combining Theorem 1 and Theorem 2, we have

Theorem 3. *Given a group Π and its subgroup π of finite index, the* *notions of ϕ-injectivity and ϕ-projectivity coincide.*

Theorem 1 and SWAN's method provide the following

Proposition 1. *Let Q be an R-free $R\Pi$-module whose base is permuted up to sign by Π. If all the elements acting non-freely are contained in π, Q is ϕ-injective.*

If the group Π is finite solvable, then Π is represented as a product of its p-Sylow subgroups (Cf. [4]), or more generally if a subgroup π of Π has its complementary subgroup, e. g., $\Pi=\mathfrak{S}_n$ and its subgroup $\pi=\mathfrak{S}_{n-1}$, $\pi=$a cyclic group generated by a cycle (12n), then SWAN's Lemma 7. 2 can be restated with slight modifications. For a c. s. s. complex K, $\otimes^n C(K)$, where $C(K)$ is the normalized chain complex of K, is an $\mathfrak{S}_n$-complex, therefore our theorems will be applicable for the study of symmetric products (*Cf.* [1] and [3]).

Now, for any map $f\epsilon Hom_\pi(A, B)$, we can define $t'(f) \epsilon Hom_\Pi(A, B)$ by

$$t'(f)a=\sum\eta_i^{-1}f(\eta_i a) \qquad \text{for } a\epsilon A,$$

where $\eta_1=\varepsilon$, $\eta_2, \cdots, \eta_n$ are the right cosets representatives of Π by π. Obviously, this definition does not depend on the choice of representatives. Thus, we can say that $t'(f)$ is a transfer homomorphism of f, and indeed we have

Proposition 2. *For any $f\epsilon Hom_\pi(A, B)$, we have $t(f)=t'(f)$.*

To prove this, we need the following

Lemma 4. *Let Π be a group and π be its subgroup of finite index n. Then, there exist n elements $\xi_1=\varepsilon$, $\xi_2, \cdots\cdots, \xi_n$ in Π, such that*

$$\Pi=\xi_1\pi+\xi_2\pi+\cdots\cdots+\xi_n\pi,$$

$$=\pi\xi_1+\pi\xi_2+\cdots\cdots+\pi\xi_n.$$

Proof of the Proposition 2. Suppose that $\xi_1=\varepsilon$, $\xi_2, \cdots\cdots$, ξ_n be n elements of Π chosen in Lemma 4. If we consider ξ_i's as a system of right coset representatives, then ξ_i^{-1}'s are a system of left coset representatives, therefore, for every ξ_i, there exists a $\xi_{j(i)}$ and $\varpi_i\epsilon\pi$ such that (i) $\xi_i^{-1}=\xi_{j(i)}\varpi_i$, and (ii) $\{\xi_{j(i)}\}$ is a permutation of $\{\xi_i\}$, because ξ_i's can be considered as left cosets representatives. Consequently,

$$t'(f)a=\sum\xi_i^{-1}f(\xi_i a)=\sum\xi_{j(i)}\varpi_i f(\varpi_i^{-1}\xi_{j(i)}^{-1}a)=\sum\xi_j f(\xi_j^{-1}a)=t(f)a.$$

It remains the proof of Lemma 4, and for this purpose we prove the next two lemmas.

Lemma 5. *Let Π be a group and π be its arbitrary subgroup of finite index. Then, for any element ξ of Π, the numbers of left cosets and the right cosets in the double coset $\pi\xi\pi$ are the same.*

Proof. As is well-known, the numbers of left cosets and right cosets in the double coset $\pi\xi\pi$ are $[\xi^{-1}\pi\xi : \pi\cap\xi^{-1}\pi\xi]$ and $[\pi : \pi\cap\xi^{-1}\pi\xi]$, respectively. Now,

$$[\pi:\pi\cap\xi^{-1}\pi\xi]=[\xi\pi\xi^{-1}:\pi\cap\xi\pi\xi^{-1}]\leqq[\pi\cup\xi\pi\xi^{-1}:\pi]^{1)}\leqq[\Pi:\pi],$$

and

$$[\xi^{-1}\pi\xi:\pi\cap\xi^{-1}\pi\xi]\leqq[\pi\cup\xi^{-1}\pi\xi:\pi]\leqq[\Pi:\pi].$$

Therefore, $[\pi:\pi\cap\xi^{-1}\pi\xi]$ and $[\xi^{-1}\pi\xi:\pi\cap\xi^{-1}\pi\xi]$ are both finite, because of the finiteness of $[\Pi:\pi]$.
Then, the equalities

$$[\Pi:\xi^{-1}\pi\xi\cap\pi]=[\Pi:\pi]\,[\pi:\pi\cap\xi^{-1}\pi\xi],$$
$$=[\Pi:\xi^{-1}\pi\xi]\,[\xi^{-1}\pi\xi:\pi\cap\xi^{-1}\pi\xi],$$

and

$$[\Pi:\pi]=[\Pi:\xi^{-1}\pi\xi]$$

imply that

$$[\pi:\pi\cap\xi^{-1}\pi\xi]=[\xi^{-1}\pi\xi:\pi\cap\xi^{-1}\pi\xi].$$

Lemma 6. *Under the same assumption as in Lemma 5, if the left coset $\xi\pi$ meets with s right cosets, then the right coset $\pi\xi$ meets with s left cosets. Moreover, $s=[\pi:\pi\cap\xi^{-1}\pi\xi]$.*

Proof. Assume that $\xi\pi\cap\pi\eta\neq\phi$. Then, as easily seen, $\xi\pi$ and $\pi\eta$ are contained in the double coset $\pi\xi\pi$. Conversely, every right cosets in $\pi\xi\pi$ meets with the left coset $\xi\pi$. Therefore, the number of the right cosets which intersect with the left coset $\xi\pi$ is equal to $[\pi:\pi\cap\xi^{-1}\pi\xi]$. Similarly, we have that the number of the left cosets which intersect with the right coset $\pi\eta$ is equal to $[\xi^{-1}\pi\xi:\pi\cap\xi^{-1}\pi\xi]$. Then, our conclusion follows from Lemma 5.

Proof of Lemma 4. At first, we notice that the coset does not intersect with any other left and right cosets, and we can choose $\varepsilon=\xi_1$ as a common representative. Moreover, for any element ξ of Π, the left coset $\xi\pi$ and the right coset $\pi\xi$ have the non-void intersection.

Now, we proceed to prove by induction, and we assume that we have chosen k elements $\xi_1, \xi_2, \cdots\cdots, \xi_k$ $(2\leqq k<n)$, satisfying $\xi_i\pi\cap\pi\xi_i\neq\phi$ for $i=1, \cdots\cdots, k$.

Consider the left cosets $\xi\pi$'s contained in $\Pi\setminus\bigcup_{i=1}^{k}\xi_i\pi$[2] and the right cosets $\pi\eta$'s contained in $\Pi\setminus\bigcup_{i=1}^{k}\pi\xi_i$. Then, we have the following two possible cases:

(i) There exist a left coset $\xi\pi$ in $\Pi\setminus\bigcup_{i=1}^{k}\xi_i\pi$ and a right coset $\pi\eta$ such that

1) If π_1 and π_2 are subgroups of a group Π, then the symbol $\pi_1\cup\pi_2$ denotes the subgroup of Π generated by π_1 and π_2.

2) Here, the symbol $\cup$ implies the set-theoretic union and the symbol $\setminus$ denotes the set-theoretic difference.

$\xi\pi \cap \pi\eta \neq \phi$.

(ii) For any left coset $\xi\pi$ in $\Pi \setminus \bigcap_{i=1}^{k} \xi_i\pi$ and any right coset $\pi\eta$ in $\Pi \setminus \bigcup_{i=1}^{k} \pi\xi_i$, we have $\xi\pi \cap \pi\eta = \phi$.

If we are in the case (i), it suffices to choose an element ξ_{k+1} in $\xi\pi \cap \pi\eta$. Therefore, if we show that we are never in the case (ii), the proof completes. To this end, assume we are in the case (ii), and take a left coset $\xi\pi$ from $\Pi \setminus \bigcup_{i=1}^{k} \xi_i\pi$ and a right coset $\pi\eta$ from $\Pi \setminus \bigcup_{i=1}^{k} \pi\xi_i$. Considering the double cosets $\pi\xi\pi$ and $\pi\eta\pi$, we have the following two formal possibilities:

(a) $\pi\xi\pi \cap \pi\eta\pi \neq \phi$;

(b) $\pi\xi\pi \cap \pi\eta\pi = \phi$.

If we are in (a), it follows $\pi\xi\pi = \pi\eta\pi$, and Lemma 6 implies $\pi\xi \cap \eta\pi \neq \phi$. On the other hand, if we are in (b), our inductive assumption, the assumption (ii) and Lemma 6 tell us that there exists a right coset $\pi\eta'$ in $\Pi \setminus \bigcup_{i=1}^{k} \pi\xi_i$ for which $\pi\xi\pi \cap \pi\eta'\pi \neq \phi$. Thus, in either case, we are guided to the contradiction.

References

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