On Differential Geometry of a Closed Space Curve at the Points Deviding into 2n Parts of Equal Length

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Let us taka a fixed closed space curve C having the constant length l, and we assume it is of class C^3. If we take an original point O on it, as any point P on it can be expressed by the arc-length S measured from O to P, every point P on C can be denoted by the symbol $P(s)$, where $O \leqq s \leqq l$.

We shall consider the system $\mathfrak{S}$ of the points $P_i(s_i)$, $i=1,2$, ordered along the curve C, dividing it into two parts of equal length. Let s_1 be the arc-length of OP_1. We shall take a fixed rectangular coordinate system $O-x^1x^2x^3$, and if we denote the coordinates of the points P_i by (x_i^1, x_i^2, x_i^3), all these coordinates are one-valued continuous functions of s. Therefore, the system $\mathfrak{S}=\{P_i(s_i),\ i=1,2\}$ is a function of s_1.

The curvatures κ_i of the curve C at the two points $P_i(s_i)$ are represented by

$$\kappa_i=\sqrt{(\boldsymbol{x}_i''\ \boldsymbol{x}_i'')}\ , \qquad i=1,2,$$

where κ_1, κ_2 are the curvatures of the curve C and $\boldsymbol{x}_1''$, $\boldsymbol{x}_2''$ are principal normal vectors at the points P_1, P_2 respectively ; the symbol $\boldsymbol{x}_i''$ are denoted by

$$\boldsymbol{x}_i''=\frac{d^2\boldsymbol{x}_i}{ds^2}.$$

Now we put

$$f(s_1)=\sqrt{(\boldsymbol{x}_1''\ \boldsymbol{x}_1'')}-\sqrt{(\boldsymbol{x}_2''\ \boldsymbol{x}_2'')},$$

then $f(s_1)$ is a one-valued continnous function of s_1 (this function can be considered as a function of the system $\mathfrak{S}$). If we replace s_1 by s_2 (i. e., take another system $\mathfrak{S}'=\{P_i(s_i')\}$, with $s_1'=s_2$),
we have easily

$$f(s_2)=\sqrt{(\boldsymbol{x}_2''\ \boldsymbol{x}_2'')}-\sqrt{(\boldsymbol{x}_1''\ \boldsymbol{x}_1'')},$$

and

$$f(s_1)=-f(s_2).$$

Therefore, we have at least one such ξ as satisfies both $f(\xi)=O$ and $s_1<\xi<s_2$. This ξ renders the curvatures of the two points $s=\xi$, $s=\xi+\frac{l}{2}$ equal.

Hence, we have the following theorem.

Theorem 1. *Among the systems* $\mathfrak{S}=\{P_i,\ i=1,2\}$, *ordered along the curve* C, *dividing* *a rectifiable fixed closed curve C which is of class C^3, into two parts of equal length, there exists at least one system that makes the two curvatures equal. Moreover, there exists at least one system $\mathfrak{S}$ making the torsions equal.*

The proof of the latter half of the theorem is as follows:
The torsions τ_i of the curve C at the points $P_i(s_i)$ are represented by

$$\tau_i = \frac{1}{\kappa_i^2} \mid \boldsymbol{x}_i' \, \boldsymbol{x}_i'' \, \boldsymbol{x}_i''' \mid, \qquad i=1, 2,$$

where derivatives with respect to arc-length s will be characterized by primes,

$$\boldsymbol{x}_i' \equiv \frac{d\boldsymbol{x}_i}{ds}, \quad \boldsymbol{x}_i'' \equiv \frac{d^2\boldsymbol{x}_i}{ds^2}, \quad \boldsymbol{x}_i''' \equiv \frac{d^3\boldsymbol{x}_i}{ds^3}.$$

If we put

$$g(s_1) = \frac{1}{\kappa_1^2} \mid \boldsymbol{x}_1' \, \boldsymbol{x}_1'' \, \boldsymbol{x}_1''' \mid - \frac{1}{\kappa_2^2} \mid \boldsymbol{x}_2' \, \boldsymbol{x}_2'' \, \boldsymbol{x}_2''' \mid,$$

(s_1) is a one-valued continuous function. We replace s_1 by s_2 then

$$g(s_2) = \frac{1}{\kappa_2^2} \mid \boldsymbol{x}_2' \, \boldsymbol{x}_2'' \, \boldsymbol{x}_2''' \mid - \frac{1}{\kappa_1^2} \mid \boldsymbol{x}_1' \, \boldsymbol{x}_1'' \, \boldsymbol{x}_1''' \mid.$$

Obviously, we have $g(s_1) = -g(s_2)$.
Therefore, we can find at least one auch η as satisfies both $g(\eta)=O$ and $s_1<\eta<s_2$.

Hence, the torsions of the curve at the two points $s=\eta$ and $s=\eta+\frac{l}{2}$ are equal. Thus, the latter half of the above theorem has been demonstrated.

Now we shall consider on the circle of curvature and osculating sphere of the curve C. We denote the radius of the circle of curvature and osculating sphere at the point of the curve C by ρ and R respectively. Then, these are given as

$$\rho = \frac{1}{\kappa}, \qquad R^2 = \rho^2 + \left(\frac{d\rho}{ds}\,\frac{1}{\tau}\right)^2.$$

The radius of the circle of curvature at the points which have equal curvatures is equal.

If now we put

$$h(s_1) = \left[\rho_1^2 + \left(\frac{d\rho_1}{ds}\,\frac{1}{\tau_1}\right)^2\right]^{\frac{1}{2}} - \left[\rho_2^2 + \left(\frac{d\rho_2}{ds}\,\frac{1}{\tau_2}\right)^2\right]^{\frac{1}{2}},$$

where ρ_1, ρ_2 are the radius of curvature at the two points P_1, P_2 in the curve respectively, and then replace s_1 by s_2, we have the following theorem, as the result of the same processes as above.

Theorem 2. (A). *Among the systems of points $\mathfrak{S}=\{P_i(s_i),\ i=1,2\}$, dividing C into two parts of equal length, there exists at least one system making the areas of the two circles of curvature equal.*

(B). *Among the systems of points $\mathfrak{S}=\{P_i(s_i),\ i=1,2\}$, dividing C into two parts of equal length, there exists at least one system making the volumes of the two osculating spheres equal.*

We have hitherto stated on the case of the system $\mathfrak{S}=\{P_i(s_i),\ i=1,2\}$, we shall consider on $\mathfrak{T}=\{P_j(s_j), j=1, 2, 3, 4\}$.

The curvatures κ_j, $(j=1, 2, 3, 4)$ of the curve C at the four points $P_j(s_j)$ are represented by

$$\kappa_j=\sqrt{(x_j''\ x_j'')}, \qquad (j=1, 2, 3, 4).$$

The function

$$\sqrt{(x_1''\ x_1'')}-\sqrt{(x_2''\ x_2'')}+\sqrt{(x_3''\ x_3'')}-\sqrt{(x_4''\ x_4'')}$$

can be considered as a function of the system $\mathfrak{T}$ which is a one-valued continuous function of s_1.

We put

$$\varphi(s_1)=\sqrt{(x_1''\ x_1'')}-\sqrt{(x_2''\ x_2'')}+\sqrt{(x_3''\ x_3'')}-\sqrt{(x_4''\ x_4'')},$$

and if we replace s_1 by s_2, we have easily $\varphi(s_1)=-\varphi(s_2)$.
Therefore, we have at least one such ξ as satisfies both $\varphi(\xi)=0$ and $s_1<\xi<s_2$. This ζ renders the sum of the curvatures at the opposite two points P_1, P_3 and that at the other opposite two points P_2, P_4 equal.

As for the torsion, we put

$$\phi(s_1)=\frac{1}{\kappa_1^2}\mid x_1'\ x_1''\ x_1'''\mid-\frac{1}{\kappa_2^2}\mid x_2'\ x_2''\ x_2'''\mid+\frac{1}{\kappa_3^2}\mid x_3'\ x_3''\ x_3'''\mid-\frac{1}{\kappa_4^2}\mid x_4'\ x_4''\ x_4'''\mid,$$

by the same considerations on the function $\phi(s_1)$, we may have the similar result.

In the case of the division of the space curve C with all the properties abovesaid into 2n parts of equal length, we can generally follow the same processes and have the same results.

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