On the Concircularity of Kiepert's Points

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(Received Nov. 30, 1960)

In the plane geometry, the following is well known as Kiepert's theorem:

Let BCA', CAB' and ABC' be three outer isosceles triangles with the same base angle, described on the sides of a triangle ABC. Then the triangles ABC and $A'B'C'$ are in perspective.

We call the centre of the perspectivity the *Kiepert's point* of the triangle ABC.

Here, we generalize this theorem in the following form:

Lemma. *Let M_{23}, M_{31} and M_{12} be the respective mid-points of the sides A_2A_3, A_3A_1 and A_1A_2 of the triangle $A_1A_2A_3$, and P_{23}, P_{31} and P_{12} be the points on the outer perpendicuilar bisectors of the sides A_2A_3, A_3A_1 and A_1A_2, satisfying the relation*

$M_{23}P_{23} : M_{31}P_{31} : M_{12}P_{12} = A_2A_3 : A_3A_1 : A_1A_2$. Then A_1P_{23}, A_2P_{31} and A_3P_{13} meet in a point.

If we denote the value of the above ratio by λ, we call the point in which A_1P_{23}, A_2P_{31}, and A_3P_{12} meet the *λ-Kieperts point* of the triangle $A_1A_2A_3$, and denote it by $P_{123}(\lambda)$.

Proof of the Lemma.

Without loss of generality, we can assume that the circumscribed circle of $A_1A_2A_3$ is the unit circle. Then, if we denote the three points A_1, A_2 and A_3 by the complex numbers t_1, t_2 and t_3, respectively, we have $|t_i| = 1$, $i = 1, 2, 3$.

Let m_{ij} and p_{ij} be the complex numbers of the mid-points M_{ij} and P_{ij}, respectively ($i \neq j$, $i, j = 1, 2, 3$), then we have

$$m_{ij} = \frac{1}{2}(t_i + t_j),$$

and

$$p_{ij} = \frac{1}{2}(t_i + t_j) - \sqrt{-1}\,\lambda(t_j - t_i), \quad (\lambda \geqq 0).$$

Then, in the current coordinate z, the equation of A_1P_{23} is

$$\begin{vmatrix} z & \bar{z} & 1 \\ t_1 & \bar{t}_1 & 1 \\ p_{23} & \bar{p}_{23} & 1 \end{vmatrix} = 0,$$

or

(1) $$(\bar{t}_1 - \bar{p}_{23})z - (t_1 - p_{23})\bar{z} = -(t_1\bar{p}_{23} - \bar{t}_1 p_{23}),$$

where — denotes the complex conjugate.

Simiarily, the equations of A_2P_{31} and A_3P_{12} are

(2) $$(\bar{t}_2) - \bar{p}_{31})z - (t_2 - p_{31})\bar{z} = -(t_2\bar{p}_{31} - \bar{t}_2 p_{31}),$$

and

(3) $$(\bar{t}_3 - \bar{p}_{12})z - (t_3 - p_{12})\bar{z} = -(t_3\bar{p}_{12} - \bar{t}_3 p_{12}).$$

In these equations, t_i and p_{ij} satisfy

(4) $$t_i \cdot \bar{t}_i = 1, \quad \bar{p}_{ij} = p_{ij}/ t_i \cdot t_j.$$

We now denote by Δ the determinant obtained from the coefficients of the equations (1), (2)and (3), i. e.,

$$\Delta = \begin{vmatrix} \bar{t}_1 - \bar{p}_{23} & -(t_1 - p_{23}) & t_1\bar{p}_{23} - \bar{t}_1 p_{23} \\ \bar{t}_2 - \bar{p}_{31} & -(t_2 - p_{31}) & t_2\bar{p}_{31} - \bar{t}_2 p_{31} \\ \bar{t}_3 - \bar{p}_{12} & -(t_3 - p_{12}) & t_3\bar{p}_{12} - \bar{t}_3 p_{12} \end{vmatrix}$$

Multiplying $t_1t_2t_3$ to every row, and using the relations (4), we have

$$\Delta = -\frac{1}{(t_1 t_2 t_3)^2} \begin{vmatrix} t_2 t_3 - t_1 p_{23} & t_1 - p_{23} & t^2{}_1 p_{23} - t_2 t_3 p_{23} \\ t_3 t_1 - t_2 p_{31} & t_2 - p_{31} & t^2{}_2 p_{31} - t_3 t_1 p_{31} \\ t_1 t_2 - t_3 p_{12} & t_3 - p_{12} & t^2{}_3 p_{12} - t_1 t_2 p_{12} \end{vmatrix}$$

Then, adding the first and the second rows to the third row, every element of the third row vanishes. Therefore, we have

$$\Delta = 0, \text{ since } t_i \neq 0 \text{ for } i = 1, 2, 3.$$

This implies that the three straight lines A_1P_{23}, A_2P_{31} and A_3P_{12} meet in a point $P_{123}(\lambda)$. Thus the proof of the lemma is established.

Now, our main result is the following theorem :

Theorem 1. *Let $A_1A_2A_3A_4$ be a convex quadrilateral inscribed in a circle, and $P_{123}(\lambda)$, $P_{234}(\lambda)$, $P_{341}(\lambda)$ and $P_{412}(\lambda)$, be the λ-Kiepert's points of the triangles $A_1A_2A_3$, $A_2A_3A_4$, $A_3A_4A_1$ and $A_4A_1A_2$, respectively. Then, these four points are concircular.*

Proof. Let $z_{123}(\lambda)$, $z_{234}(\lambda)$, $z_{341}(\lambda)$ and $z_{412}(\lambda)$ be the complex numbers of the points $P_{123}(\lambda)$, $P_{234}(\lambda)$, $P_{341}(\lambda)$ and $P_{412}(\lambda)$, respectively.

Since $z_{123}(\lambda)$ is the solution z of the system of equations (1) and (3),

$$z = \frac{\begin{vmatrix} t_1\bar{p}_{23} - \bar{t}_1 p_{23} & t_1 - p_{23} \\ t_2\bar{p}_{31} - \bar{t}_2 p_{31} & t_2 - p_{31} \end{vmatrix}}{\begin{vmatrix} \bar{t}_1 - \bar{p}_{23} & -(t_1 - p_{23}) \\ \bar{t}_2 - \bar{p}_{31} & -(t_2 - p_{31}) \end{vmatrix}}$$

that is,

$$(5)\qquad z_{123}(\lambda) = \frac{(\bar{t}_1 p_{23} - t_1 \bar{p}_{23})(t_2 - p_{31}) - (t_1 - p_{23})(\bar{t}_2 p_{31} - t_2 \bar{p}_{31})}{(\bar{t}_1 - \bar{p}_{23})(t_2 - p_{31}) - (t_1 - p_{23})(\bar{t}_2 - \bar{p}_{31})},$$

Similarily, considering the λ-Kiepert's points of the other three triangles $A_2 A_3 A_4$, $A_3 A_4 A_1$ and $A_4 A_1 A_2$, we have

$$(6)\qquad z_{234}(\lambda) = \frac{(\bar{t}_2 p_{34} - t_2 \bar{p}_{34})(t_3 - p_{42}) - (t_2 - p_{34})(\bar{t}_3 p_{42} - t_3 \bar{p}_{42})}{(\bar{t}_2 - \bar{p}_{34})(t_3 - p_{42}) - (t_2 - p_{34})(\bar{t}_3 - \bar{p}_{42})},$$

$$(7)\qquad z_{341}(\lambda) = \frac{(\bar{t}_3 p_{41} - t_3 \bar{p}_{41})(t_4 - p_{13}) - (t_3 - p_{41})(\bar{t}_4 p_{13} - t_4 \bar{p}_{13})}{(\bar{t}_3 - \bar{p}_{41})(t_4 - p_{13}) - (t_3 - p_{41})(\bar{t}_4 - \bar{p}_{13})},$$

$$(8)\qquad z_{412}(\lambda) = \frac{(\bar{t}_4 p_{12} - t_4 \bar{p}_{12})(t_1 - p_{24}) - (t_4 - p_{12})(\bar{t}_1 p_{24} - t_1 \bar{p}_{24})}{(\bar{t}_4 - \bar{p}_{12})(t_1 - p_{24}) - (t_4 - p_{12})(\bar{t}_1 - \bar{p}_{24})},$$

To prove the concircularity of these four points $P_{123}(\lambda)$, $P_{234}(\lambda)$, $P_{341}(\lambda)$ and $P_{412}(\lambda)$, it suffices to show that

$$z = \frac{z_{123}(\lambda) - z_{341}(\lambda)}{z_{234}(\lambda) - z_{341}(\lambda)} \times \frac{z_{234}(\lambda) - z_{412}(\lambda)}{z_{123}(\lambda) - z_{412}(\lambda)}$$

is real, i.e., $z = \bar{z}$.

Since the denominator, $(\bar{t}_1 p_{23} - t_1 \bar{p}_{23})$ and $(\bar{t}_2 p_{31} - t_2 \bar{p}_{31})$ in the numerator of (5) are pure imaginary, we can represent $z_{123}(\lambda)$ as

$$z_{123}(\lambda) = \frac{\sqrt{-1}\,\alpha_1 (t_2 - p_{31}) - \sqrt{-1}\,\alpha_2 (t_1 - p_{23})}{\sqrt{-1}\,\alpha},$$

$$= \frac{\alpha_1 (t_2 - p_{31}) - \alpha_2 (t_1 - p_{23})}{\alpha},$$

where α, α_1 and α_2 are real.

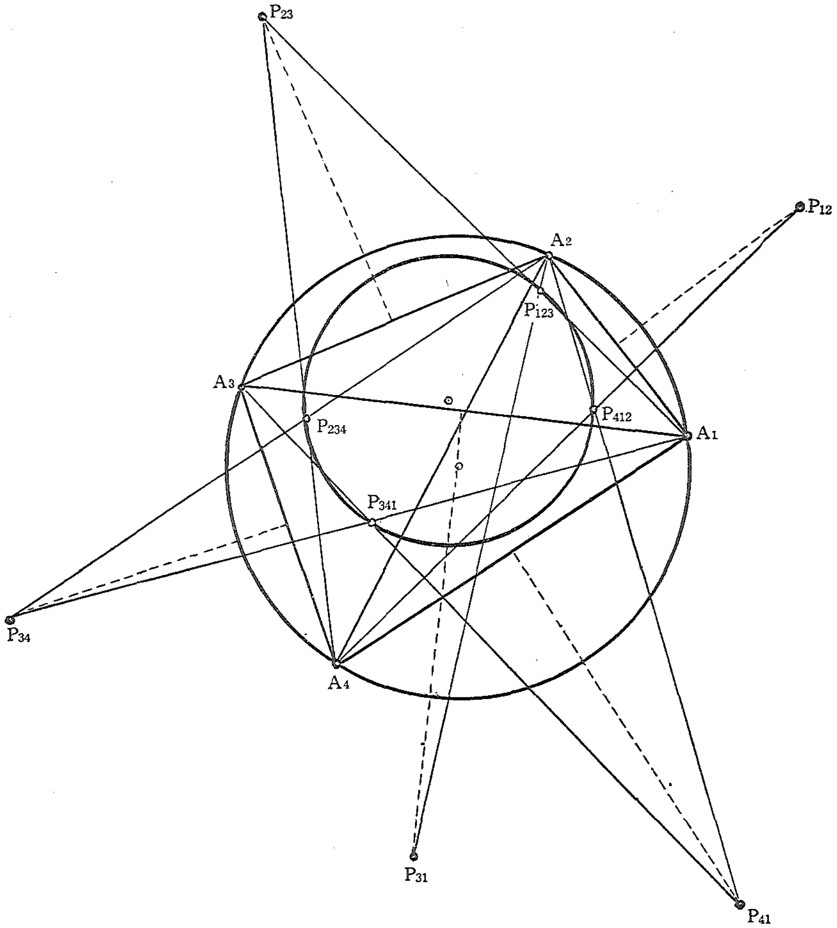
Similarily, we can represent (6), (7) and (8) as

$$z_{234}(\lambda) = \frac{\beta_1 (t_3 - p_{42}) - \beta_2 (t_2 - p_{34})}{\beta},$$

$$z_{341}(\lambda) = \frac{\gamma_1 (t_4 - p_{13}) - \gamma_2 (t_3 - p_{41})}{\gamma},$$

$$z_{412}(\lambda) = \frac{\delta_1 (t_1 - p_{24}) - \delta_2 (t_4 - p_{12})}{\delta},$$

where β's, γ's and δ's are all real.



$(\lambda = 1)$

Consequently, we have

$$z - \bar{z} = \frac{\dfrac{\alpha_1(t_2 - p_{34}) - \alpha_2(t_1 - p_{23})}{\alpha} - \dfrac{\gamma_1(t_4 - p_{13}) - \gamma_2(t_3 - p_{41})}{\gamma}}{\dfrac{\beta_1(t_3 - p_{42}) - \beta_2(t_2 - p_{34})}{\beta} - \dfrac{\gamma_1(t_4 - p_{13}) - \gamma_2(t_3 - p_{41})}{\gamma}}$$

$$\times \frac{\dfrac{\beta_1(t_3 - p_{42}) - \beta_2(t_2 - p_{34})}{\beta} - \dfrac{\delta_1(t_1 - p_{24}) - \delta_2(t_4 - p_{12})}{\delta}}{\dfrac{\alpha_1(t_2 - p_{34}) - \alpha_2(t_1 - p_{23})}{\alpha} - \dfrac{\delta_1(t_1 - p_{24}) - \delta_2(t_4 - p_{12})}{\delta}}$$

$$- \frac{\dfrac{\alpha_1(\bar{t}_2 - \bar{p}_{34}) - \alpha_2(\bar{t}_1 - \bar{p}_{23})}{\alpha} - \dfrac{\gamma_1(\bar{t}_4 - \bar{p}_{13}) - \gamma_2(\bar{t}_3 - \bar{p}_{41})}{\gamma}}{\dfrac{\beta_1(\bar{t}_3 - \bar{p}_{42}) - \beta_2(\bar{t}_2 - \bar{p}_{34})}{\beta} - \dfrac{\gamma_1(\bar{t}_4 - \bar{p}_{13}) - \gamma_2(\bar{t}_3 - \bar{p}_{41})}{\gamma}}$$

$$\times \frac{\dfrac{\beta_1(\bar{t}_3-\bar{p}_{42})-\beta_2(\bar{t}_2-\bar{p}_{34})}{\beta}-\dfrac{\delta_1(\bar{t}_1-\bar{p}_{24})-\delta_2(\bar{t}_4-\bar{p}_{12})}{\delta}}{\dfrac{\alpha_1(\bar{t}_2-\bar{p}_{34})-\alpha_2(\bar{t}_1-\bar{p}_{23})}{\alpha}-\dfrac{\delta_1(\bar{t}_1-\bar{p}_{24})-\delta_2(\bar{t}_4-\bar{p}_{12})}{\delta}}$$

$$= \frac{\alpha_1\beta\gamma(t_2-p_{31})-\alpha_2\beta\gamma(t_1-p_{23})-\alpha\beta\gamma_1(t_4-p_{13})+\alpha\beta\gamma_2(t_3-p_{41})}{\alpha\beta_1\gamma(t_3-p_{42})-\alpha\beta_2\gamma(t_2-p_{34})-\alpha\beta\gamma_1(t_4-p_{13})+\alpha\beta\gamma_2(t_3-p_{41})}$$

$$\times \frac{\alpha\beta_1\delta(t_3-p_{42})-\alpha\beta_2\delta(t_2-p_{34})-\alpha\beta\delta_1(t_1-p_{24})+\alpha\beta\delta_2(t_4-p_{12})}{\alpha_1\beta\delta(t_2-p_{31})-\alpha_2\beta\delta(t_1-p_{23})-\alpha\beta\delta_1(t_1-p_{24})+\alpha\beta\delta_2(t_4-p_{12})}$$

$$- \frac{\alpha_1\beta\gamma(\bar{t}_2-\bar{p}_{31})-\alpha_2\beta\gamma(\bar{t}_1-\bar{p}_{23})-\alpha\beta\gamma_1(\bar{t}_4-\bar{p}_{13})+\alpha\beta\gamma_2(\bar{t}_3-\bar{p}_{41})}{\alpha\beta_1\gamma(\bar{t}_3-\bar{p}_{42})-\alpha\beta_2\gamma(\bar{t}_2-\bar{p}_{34})-\alpha\beta\gamma_1(\bar{t}_4-\bar{p}_{13})+\alpha\beta\gamma_2(\bar{t}_3-\bar{p}_{41})}$$

$$\times \frac{\alpha\beta_1\delta(\bar{t}_3-\bar{p}_{42})-\alpha\beta_2\delta(\bar{t}_2-\bar{p}_{34})-\alpha\beta\delta_1(\bar{t}_1-\bar{p}_{24})+\alpha\beta\delta_2(\bar{t}_4-\bar{p}_{12})}{\alpha_1\beta\delta(\bar{t}_2-\bar{p}_{31})-\alpha_2\beta\delta(\bar{t}_1-\bar{p}_{23})-\alpha\beta\delta_1(\bar{t}_1-\bar{p}_{24})+\alpha\beta\delta_2(\bar{t}_4-\bar{p}_{12})}$$

Reducing the above fractions to a common denominator, we have

$$\begin{aligned}
\textit{the numerator} &= \{\alpha_1\beta\gamma(t_2-p_{31})-\alpha_2\beta\gamma(t_1-p_{23})-\alpha\beta\gamma_1(t_4-p_{13})+\alpha\beta\gamma_2(t_3-p_{41})\}\\
&\times\{\alpha\beta_1\delta(t_3-p_{42})-\alpha\beta_2\delta(t_2-p_{34})-\alpha\beta\delta_1(t_1-p_{24})+\alpha\beta\delta_2(t_4-p_{12})\}\\
&\times\{\alpha\beta_1\gamma(\bar{t}_3-\bar{p}_{42})-\alpha\beta_2\gamma(\bar{t}_2-\bar{p}_{34})-\alpha\beta\gamma_1(\bar{t}_4-\bar{p}_{13})+\alpha\beta\gamma_2(\bar{t}_3-\bar{p}_{41})\}\\
&\times\{\alpha_1\beta\delta(\bar{t}_2-\bar{p}_{31})-\alpha_2\beta\delta(\bar{t}_1-\bar{p}_{23})-\alpha\beta\delta_1(\bar{t}_1-\bar{p}_{24})+\alpha\beta\delta_2(\bar{t}_4-\bar{p}_{12})\}\\
&-\{\alpha\beta_1\gamma(t_3-p_{42})-\alpha\beta_2\gamma(t_2-p_{34})-\alpha\beta\gamma_1(t_4-p_{13})+\alpha\beta\gamma_2(t_3-p_{41})\}\\
&\times\{\alpha_1\beta\delta(t_2-p_{31})-\alpha_2\beta\delta(t_1-p_{23})-\alpha\beta\delta_1(t_1-p_{24})+\alpha\beta\delta_2(t_4-p_{12})\}\\
&\times\{\alpha_1\beta\gamma(\bar{t}_2-\bar{p}_{31})-\alpha_2\beta\gamma(\bar{t}_2-\bar{p}_{23})-\alpha\beta\gamma_1(\bar{t}_4-\bar{p}_{13})+\alpha\beta\gamma_2(\bar{t}_3-\bar{p}_{41})\}\\
&\times\{\alpha\beta_1\delta(\bar{t}_3-\bar{p}_{42})-\alpha\beta_2\delta(\bar{t}_2-\bar{p}_{34})-\alpha\beta\delta_1(\bar{t}_1-\bar{p}_{24})+\alpha\beta\delta_2(\bar{t}_4-\bar{p}_{12})\}\\
&= 0.
\end{aligned}$$

On the other hand, being $A_1A_2A_3A_4$ a convex quadrilateral, the common denominator cannot be zero.

Therefore, we have $z-\bar{z}=0$, i.e. $z=\bar{z}$. Thus, our theorem has been proved completely.

If $\lambda=0$, four points $P_{123}(0)$, $P_{234}(0)$, $P_{341}(0)$ and $P_{412}(0)$ are the respective centroides of the triangles $A_1A_2A_3$, $A_2A_3A_4$, $A_3A_4A_1$ and $A_4A_1A_2$; and if $\lambda=\infty$, $P_{123}(\infty)$, $P_{234}(\infty)$, $P_{341}(\infty)$ and $P_{412}(\infty)$ are the respective orthocentres of the triangles $A_1A_2A_3$, $A_2A_3A_4$, $A_3A_4A_1$ and $A_4A_1A_2$.

Accordingly, we have the following

Theorem 2. *Let $A_1A_2A_3A_4$ be a convex quadrilateral inscribed in a circle, then the centroides of the four triangles $A_1A_2A_3$, $A_2A_3A_4$, $A_3A_4A_1$ and $A_4A_1A_2$ are concircular. The same result holds for the orthocentres of these four triangles.*

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