A Study of Hyperboloidic Position

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(Received Nov. 11, 1959)

Weiss considered in how many ways two triangles in a plane can be mutually in perspective if their relative position be put properly, and proved the greatest number of the possible way to be 6.

His problem, if similarly considered in space, may well be expressed as follows: in how many ways two tetrahedra, if their relative position be properly put, can be mutually in hyperboloidic position.

After a study of this problem, I have arrived at the following theorem.

Theorem. *Two tetrahedra in general positiom in space, if their relative position be put properly, can be in hyperboloidic position in 16 ways. Provided in this theorem that 'two tetrahedra in general position' implies such relation between the two solids that on any one of the four planes which make up either of the two tetrahedra no vertex of the other comes.*

In plane, if we try to get two triangles which are mutually in perspective in 6 ways, one of them proves to be an imaginary triangle (the one the co-ordinates of whose vertices include imaginary number). In elementary geometrical meaning, therefore, they can not be in perspective in 6 ways. But, in space, we can find the greatest number of the possible way of hyperboloidic position, i. e. 16, with each tetrahedron being real one. This may be worth noticing, and we will demonstrate the above theorem in the following.

Suppose one of the given tetrahedra to be AB CD and let it be the fundamental tetrahedron, which is A $(1,0,0,0)$, B $(0,1,0,0)$, C $(0,0,1,0)$, D $(0,0,0,1)$. Suppose the other to be $A'B'C'D'$ and the co-ordinates of its vertices A', B', C', D', with one of them put on the unit point, to be $(a_1, a_2, a_3, 1)$, $(b_1, b_2, b_3, 1)$, $(c_1, c_2, c_3, 1)$ and $(1,1,1,1)$, respectively.

Since, by hypothesis, these tetrahedra are in general positions, any of a_1, a_2, a_3; b_1, b_2, b_3; c_1, c_2, c_3 is not zero. Symbols $A\to A'$, $B\to B'$, $C\to C'$, $D\to D'$ denoting that the vertices A', B', C', D' of the tetrahedron $A'B'C'D'$ are supposed to correspond to the vertices A, B, C, D of the tetrahedron AB CD, respectively, Plücker's line co-ordinates of four straight lines AA', BB', CC', DD′ are

$$\begin{matrix} a_2, & 0, & -a_3, & 0, & 1, & 0; \\ -b_1, & b_3, & 0, & 0, & 0, & 1; \\ 0, & -c_2, & c_1, & 1, & 0, & 0; \\ 0, & 0, & 0, & -1, & -1, & -1, \end{matrix}$$

respectively.

Supposing Plücker's line co-ordinates of a straight line which intersects these four lines to be

$$p_{12}^{(1)},\ p_{23}^{(1)},\ p_{31}^{(1)},\ p_{34}^{(1)},\ p_{14}^{(1)},\ p_{24}^{(1)},$$

we find the following Plücker's relations:

$$\begin{aligned} p_{23}^{(1)} + a_2 p_{34}^{(1)} - a_3 p_{24}^{(1)} &= 0 \\ p_{31}^{(1)} - b_1 p_{34}^{(1)} + b_3 p_{14}^{(1)} &= 0 \\ p_{12}^{(1)} - c_2 p_{14}^{(1)} + c_1 p_{24}^{(1)} &= 0 \\ p_{12}^{(1)} + p_{23}^{(1)} + p_{31}^{(1)} &= 0. \end{aligned}$$

When we consider a matrix consisting of the coefficients of

$$p_{12}^{(1)},\ p_{23}^{(1)},\ p_{31}^{(1)},\ p_{34}^{(1)},\ p_{14}^{(1)},\ p_{24}^{(1)},$$

i. e.,

$$\begin{pmatrix} 0 & 1 & 0 & a_2 & 0 & -a_3 \\ 0 & 0 & 1 & -b_1 & b_3 & 0 \\ 1 & 0 & 0 & 0 & -c_2 & c_1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix},$$

then, if the rank of this matrix is equal to 3, the four straight lines AA′, BB′, CC′, DD′ can be in hyperboloidic position.

Then let each of the following fifteen determinants of the 4-th order be equal to zero,

$|1234| = 0 \Rightarrow a_2 = b_1,$ $|1235| = 0 \Rightarrow b_3 = c_2,$
$|1236| = 0 \Rightarrow a_3 = c_1,$ $|1345| = 0 \Rightarrow a_2(b_3 - c_2) = 0,$
$|1346| = 0 \Rightarrow a_2c_1 = a_3b_1,$ $|1356| = 0 \Rightarrow a_3(b_3 - c_2) = 0,$
$|1452| = 0 \Rightarrow a_2b_3 = b_1c_2,$ $|1456| = 0 \Rightarrow a_2b_3c_1 = a_3b_1c_2,$
$|1562| = 0 \Rightarrow a_3b_3 = b_3c_1,$ $|1624| = 0 \Rightarrow a_3b_1 = b_1c_1,$
$|2345| = 0 \Rightarrow c_2(a_2 - b_1) = 0,$ $|2346| = 0 \Rightarrow c_1(a_2 - b_1) = 0,$
$|2356| = 0 \Rightarrow a_3c_2 - b_3c_1 = 0,$ $|2456| = 0 \Rightarrow a_2b_3c_1 - a_3b_1c_2 = 0,$
$|3456| = 0 \Rightarrow a_2b_3c_1 - a_3b_1c_2 = 0.$

Provided here that every numeral indicates the order of column in the matrix and also stands for the elements involved in the column.

From the above fifteen equations, the relations

$$a_2 = b_1, \quad b_3 = c_2, \quad a_3 = c_1 \tag{1}$$

are obtained. From (1), therefore, it becomes clear that in order that the four straight lines AA′, BB′, CC′, DD′ may be in hyperboloidic position, the co-ordinates of the vertices of the tetrahedron A′B′C′D′ should be A′ $(a_1, a_2, a_3, 1)$ B′ $(a_2, b_2, b_3, 1)$, C′ $(a_3, b_3, c_3, 1)$ and D′ $(1, 1, 1, 1)$.

Let us advance to the next stage. The corresponding vertices A→B′, B→C′, C→D′, D→A′ being taken up, Plücker's line co-ordinates of four straight lines AB′, BC′, CD′, DA′ are

$$\begin{array}{cccccc} b_2, & 0, & -b_3, & 0, & 1, & 0\ ; \\ -a_3, & c_3, & 0, & 0, & 0, & 1\ ; \\ 0, & -1, & 1, & 1, & 0, & 1\ ; \\ 0, & 0, & 0, & -a_3, & -a_1, & -a_2, \end{array}$$

respectively.

If Plücker's line co-ordinates of a straight line intersecting these four lines be supposed to be

$$p_{12}^{(2)},\ p_{23}^{(2)},\ p_{31}^{(2)},\ p_{34}^{(2)},\ p_{14}^{(2)},\ p_{24}^{(2)},$$

then,

$$\begin{aligned} p_{23}^{(2)} + b_2 p_{34}^{(2)} - b_3 p_{24}^{(2)} &= 0 \\ p_{31}^{(2)} - a_3 p_{34}^{(2)} + c_3 p_{14}^{(2)} &= 0 \\ p_{12}^{(2)} - p_{14}^{(2)} + p_{24}^{(2)} &= 0 \\ a_3 p_{12}^{(2)} + a_1 p_{23}^{(2)} + a_2 p_{31}^{(2)} &= 0. \end{aligned}$$

And likewise, a matrix consisting of the coefficients of

$$p_{12}^{(2)},\ p_{23}^{(2)},\ p_{31}^{(2)},\ p_{34}^{(2)},\ p_{14}^{(2)},\ p_{24}^{(2)},$$

i. e.,

$$\begin{pmatrix} 0 & 1 & 0 & b_2 & 0 & -b_3 \\ 0 & 0 & 1 & -a_3 & c_3 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 \\ a_3 & a_1 & a_2 & 0 & 0 & 0 \end{pmatrix}$$

will be found. In order for the four straight lines AB′, BC′, CA′, DA′ to be in hyperboloidic position, the rank of this matrix must be equal to 3. Then, let the fifteen determinants of the 4-th order be equal to zero,

$|1234| = 0 \Rightarrow a_1b_2 - a_2a_3 = 0,$ $|1235| = 0 \Rightarrow a_2c_3 - a_3 = 0,$
$|1236| = 0 \Rightarrow a_1b_3 - a_3 = 0,$ $|1345| = 0 \Rightarrow b_2(a_2c_3 - a_3) = 0,$
$|1346| = 0 \Rightarrow a_3(a_2b_3 - b_2) = 0,$ $|1356| = 0 \Rightarrow b_3(a_2c_3 - a_3) = 0,$
$|1452| = 0 \Rightarrow a_1b_2c_3 - a_3^2 = 0,$ $|1456| = 0 \Rightarrow a_3(a_3b_3 - b_2c_3) = 0,$
$|1562| = 0 \Rightarrow c_3(a_1b_3 - a_3) = 0,$ $|1624| = 0 \Rightarrow a_3(a_1b_3 - a_3) = 0,$
$|2345| = 0 \Rightarrow a_1b_2 - a_2a_3 = 0,$ $|2346| = 0 \Rightarrow a_1b_2 - a_2a_3 = 0,$
$|2356| = 0 \Rightarrow a_1b_3 - a_2c_3 = 0,$ $|2456| = 0 \Rightarrow a_1(a_3b_3 - b_2c_3) = 0,$
$|3456| = 0 \Rightarrow a_2(a_3b_3 - b_2c_3) = 0.$

And these equations will resolve themselves into the following relations:

$$\left.\begin{aligned} a_1 b_2 &= a_2 a_3, \\ a_2 c_3 &= a_1 b_3 = a_3. \end{aligned}\right\} \qquad (2)$$

From (2) the co-ordinates of the vertices of the tetrahedron A′B′C′D′ are known to be

$$A' \ (a_1, a_2, a_3, 1), \quad B' \ \left(a_2, \frac{a_2 a_3}{a_1}, \frac{a_3}{a_1}, 1\right)$$

$$C' \ \left(a_3, \frac{a_3}{a_1}, \frac{a_3}{a_2}, 1\right), \quad D' \ (1, 1, 1, 1).$$

And next, taking up the case of the corresponding vertices A→ C′, B→D′, C→A′, D→B′ and supposing Plücker's line co-ordinates of a straight line intersecting the four lines AC′, BD′, CA′, DB′ to be

$$p_{12}^{(3)}, \ p_{23}^{(3)}, \ p_{31}^{(3)}, \ p_{34}^{(3)}, \ p_{14}^{(3)}, \ p_{24}^{(3)},$$

we can come to the following matrix

$$\begin{pmatrix} 0 & a_1a_2 & 0 & a_2a_3 & 0 & -a_1a_3 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & -a_2 & a_1 \\ a_3 & a_1a_2 & a_2a_3 & 0 & 0 & 0 \end{pmatrix}.$$

In the same way if we suppose the fifteen determinants of the 4-th order to be equal to zero, the relation

$$a_1 = a_2 \qquad (3)$$

will be arrived at. Then from (3) the co-ordinates of the vertices of the tetrahedron A′B′C′D′ are known to be

$$A' \ (a_1, a_1, a_3, 1), \quad B' \ \left(a_1, a_3, \frac{a_3}{a_1}, 1\right),$$

$$C' \ \left(a_3, \frac{a_3}{a_1}, \frac{a_3}{a_1}, 1\right), \quad D' \ (1, 1, 1, 1).$$

Again, considering the corresponding vertices A→D′, B→A′, C→B′, D→C′, we similarly have the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -a_1 & a_3 & 0 \\ 1 & 0 & 0 & 0 & -a_3 & a_1 \\ a_3 & a_1a_3 & a_3 & 0 & 0 & 0 \end{pmatrix}$$

and since any of the fifteen determinants of the 4-th order got out of this matrix proves to be zero, it is obvious that its rank is equal to 3. Therefore the four straight lines AD′, BA′, CB′, DC′, are in hyperboloidic position.

And then, advancing to the corresponding vertices A→B′, B→C′, C→A′, D→D′, we similarly have the relation

$$a_3 = a_1^2 \qquad (4)$$

Then from (4) the co-ordinates of the vertices of the tetrahedron $A'B'C'D'$ are clearly known to be

$$A'\ (a_1, a_1, a_1^2, 1), \qquad B'\ (a_1, a_1^2, a_1, 1),$$
$$C'\ (a_1^2, a_1, a_1, 1), \qquad D'\ (1, 1, 1, 1).$$

Finally, the consideration of the corresponding vertices $A \to D'$, $B \to A'$, $C \to C'$, $D \to B'$ leads to the relation

$$a_1^2 = 1.$$

And since $a_1 \neq 1$ is evident,
then

$$a_1 = -1 \tag{5}$$

becomes fixed; therefore the vertices of the tetrahedron $A'B'C'D'$ are found definitely to be

$$A\ (-1,\ -1,\ 1,\ 1), \qquad B\ (-1,\ 1,\ -1,\ 1),$$
$$C\ (1,\ -1,\ -1,\ 1), \qquad D\ (1,\ 1,\ 1,\ 1).$$

And it is easily made sure that on any plane of this tetrahedron none of the vertices A, B, C, D comes. Hence, these two tetrahedra are in general position. Now if we adopt rightly the above co-ordinates, it is known that among 24 combinations of four straight lines which are drawn between the vertices of the tetrahedron AB CD and those of $A'B'C'D'$, the following 16 combinations are in hyperboloidic position.

$$\begin{cases} A \longrightarrow A' \\ B \longrightarrow B' \\ C \longrightarrow C' \\ D \longrightarrow D' \end{cases} \quad \begin{cases} A \longrightarrow B' \\ B \longrightarrow C' \\ C \longrightarrow D' \\ D \longrightarrow A' \end{cases} \quad \begin{cases} A \longrightarrow C' \\ B \longrightarrow D' \\ C \longrightarrow A' \\ D \longrightarrow B' \end{cases} \quad \begin{cases} A \longrightarrow D' \\ B \longrightarrow A' \\ C \longrightarrow B' \\ D \longrightarrow C' \end{cases}$$

$$\begin{cases} A \longrightarrow A' \\ B \longrightarrow C' \\ C \longrightarrow D' \\ D \longrightarrow B' \end{cases} \quad \begin{cases} A \longrightarrow B' \\ B \longrightarrow C' \\ C \longrightarrow A' \\ D \longrightarrow D' \end{cases} \quad \begin{cases} A \longrightarrow C' \\ B \longrightarrow B' \\ C \longrightarrow A' \\ D \longrightarrow D' \end{cases} \quad \begin{cases} A \longrightarrow D' \\ B \longrightarrow C' \\ C \longrightarrow B' \\ D \longrightarrow A' \end{cases}$$

$$\begin{cases} A \longrightarrow A' \\ B \longrightarrow D' \\ C \longrightarrow B' \\ D \longrightarrow C' \end{cases} \quad \begin{cases} A \longrightarrow B' \\ B \longrightarrow A' \\ C \longrightarrow D' \\ D \longrightarrow C' \end{cases} \quad \begin{cases} A \longrightarrow C' \\ B \longrightarrow A' \\ C \longrightarrow B' \\ D \longrightarrow D' \end{cases} \quad \begin{cases} A \longrightarrow D' \\ B \longrightarrow A' \\ C \longrightarrow C' \\ D \longrightarrow B' \end{cases}$$

$$\begin{cases} A \longrightarrow A' \\ B \longrightarrow D' \\ C \longrightarrow C' \\ D \longrightarrow B' \end{cases} \quad \begin{cases} A \longrightarrow B' \\ B \longrightarrow D' \\ C \longrightarrow C' \\ D \longrightarrow A' \end{cases} \quad \begin{cases} A \longrightarrow C' \\ B \longrightarrow B' \\ C \longrightarrow D' \\ D \longrightarrow A' \end{cases} \quad \begin{cases} A \longrightarrow D' \\ B \longrightarrow B' \\ C \longrightarrow A' \\ D \longrightarrow C' \end{cases}$$

And the following 8 ones are known to be in no hyperboloidic position.

$$\begin{cases} A \longrightarrow A' \\ B \longrightarrow B' \\ C \longrightarrow D' \\ D \longrightarrow C' \end{cases} \quad \begin{cases} A \longrightarrow B' \\ B \longrightarrow A' \\ C \longrightarrow C' \\ D \longrightarrow D' \end{cases} \quad \begin{cases} A \longrightarrow C' \\ B \longrightarrow A' \\ C \longrightarrow D' \\ D \longrightarrow B' \end{cases} \quad \begin{cases} A \longrightarrow D' \\ B \longrightarrow C' \\ C \longrightarrow A' \\ D \longrightarrow B' \end{cases}$$

$$\begin{cases} A \longrightarrow A' \\ B \longrightarrow C' \\ C \longrightarrow B' \\ D \longrightarrow D' \end{cases} \quad \begin{cases} A \longrightarrow B' \\ B \longrightarrow D' \\ C \longrightarrow A' \\ D \longrightarrow C' \end{cases} \quad \begin{cases} A \longrightarrow C' \\ B \longrightarrow D' \\ C \longrightarrow B' \\ D \longrightarrow A' \end{cases} \quad \begin{cases} A \longrightarrow D' \\ B \longrightarrow B' \\ C \longrightarrow C' \\ D \longrightarrow A' \end{cases}$$

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