On Some Properties of a Closed Space Curve Devided into Special Numbers of Parts of Equal Length

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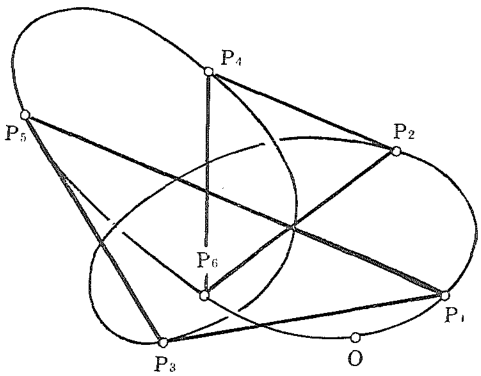
Let us take a rectifiable closed space curve C having the length l, and the definite oriention. If we take an original point O on it, as any point P on it can be expressed by the curve length s from O to P, every point P on C can be denoted by the symbol $P(s)$, where $O \leqq s \leqq l$.



We shall consider the system $\mathfrak{S}$ of the points $P_i(s_i)$, $i=1, 2, \cdots, 6$, ordered along the curve C, dividing it into six parts of equal length. Let s_1 be the curve length of OP_1. We shall take a fixed rectangular coordinate system $O-x_1x_2x_3$, and if we denote the coordinates of the points P_i by $(x_1^i,\ x_2^i,\ x_3^i)$, all these coordinates are one-valued continuous functions of s. Therefore, the system $\mathfrak{S}=\{P_i(s_i),\ i=1, 2, \cdots\cdots, 6\}$ is a function of s_1.

Let $\mathfrak{S}=\{P_i(s_i)\}$ be a system, and C_{135}, C_{246} be respectively the circumcenters of the circles $P_1P_3P_5$, $P_2P_4P_6$; x_i^{135}, x_i^{246}, $(i=1, 2, 3)$ being their respective coordinates. Then, these coordinates are defined by the following formulae :

$$x_1^{135}=\frac{1}{4\Delta_1}\begin{vmatrix} D_{123}^{135} & D_{31}^{135} & D_{12}^{135} \\ \sum(x_i^1)^2-\sum(x_i^3)^2 & 2\,(x_2^1-x_2^3) & 2\,(x_3^1-x_3^3) \\ \sum(x_i^3)^2-\sum(x_i^5)^2 & 2\,(x_2^3-x_2^5) & 2\,(x_3^3-x_3^5) \end{vmatrix}$$

$$x_2^{135}=\frac{1}{4\varDelta_1}\begin{vmatrix} D_{23}^{135} & D_{123}^{135} & D_{12}^{135} \\ 2\,(x_1^1-x_1^3) & \sum(x_i^1)-\sum(x_i^3)^2 & 2\,(x_3^1-x_3^3) \\ 2\,(x_1^3-x_1^5) & \sum(x_i^3)-\sum(x_i^5)^2 & 2\,(x_3^3-x_3^5) \end{vmatrix}$$

and

$$x_3^{135}=\frac{1}{4\varDelta_1}\begin{vmatrix} D_{23}^{135} & D_{31}^{135} & D_{123}^{135} \\ 2\,(x_1^1-x_1^3) & 2\,(x_2^1-x_2^3) & \sum(x_i^1)^2-\sum(x_i^3)^2 \\ 2\,(x_1^3-x_1^5) & 2\,(x_2^3-x_2^5) & \sum(x_i^3)^2-\sum(x_i^5)^2 \end{vmatrix}$$

where

$$D_{123}^{135}=\begin{vmatrix} x_1^1 & x_2^1 & x_3^1 \\ x_1^3 & x_2^3 & x_3^3 \\ x_1^5 & x_2^5 & x_3^5 \end{vmatrix}, \qquad D_{12}^{135}=\begin{vmatrix} x_1^1 & x_2^1 & 1 \\ x_1^3 & x_2^3 & 1 \\ x_1^5 & x_2^5 & 1 \end{vmatrix}$$

$$D_{23}^{135}=\begin{vmatrix} x_2^1 & x_3^1 & 1 \\ x_2^3 & x_3^3 & 1 \\ x_2^5 & x_3^5 & 1 \end{vmatrix}, \qquad D_{31}^{135}=\begin{vmatrix} x_3^1 & x_1^1 & 1 \\ x_3^3 & x_1^3 & 1 \\ x_3^5 & x_1^5 & 1 \end{vmatrix}$$

and

$$\varDelta_1=\begin{vmatrix} D_{23}^{135} & D_{31}^{135} & D_{12}^{135} \\ x_1^1-x_1^3 & x_2^1-x_2^3 & x_3^1-x_3^3 \\ x_1^3-x_1^5 & x_2^3-x_2^5 & x_3^3-x_3^5 \end{vmatrix}$$

and x_i^{246} are similarly defined using corresponding D_{123}^{246}, D_{12}^{246}, D_{23}^{246}, D_{31}^{246} and $\varDelta_2$.

Now put

$$f(s_1)=\pi\{\sum(x_i^1-x_i^{135})^2\}-\pi\{\sum(x_i^2-x_i^{246})^2\},$$

then $f(s_1)$ is a one-valued continuous functon of s_1 (this function can be considered as a function of the system $\mathfrak{S}$). If we replace s_1 by s_2 (*i. e.*, take another system $\mathfrak{S}'=\{P_i'(s_i')\}$, with $s_1'=s_2$), we have easily

$$f(s_2)=\pi\{\sum(x_i^2-x_i^{246})^2\}-\pi\{\sum(x_i^1-x_i^{135})^2\},$$

and

$$f(s_1)=-f(s_2).$$

Therefore, we have at least one such ξ as satisfies both $f(\xi)=O$ and $s_1<\xi<s_2$. This ξ renders the areas of the two circles $P_1P_3P_5$ and $P_2P_4P_6$ equal.

Hence, we have the following theorem.

Theorem 1. *Among the systems $\mathfrak{S}=\{P_i,\ i=1, 2, \cdots\cdots, 6\}$, ordered along the curve C, dividiug a rectifiable closed space curve C into six parts of equal length, there exists at least one system that makes the two circles $P_1P_3P_5$ and $P_2P_4P_6$ have an equal area.*

Moreover, there exists at least one system $\mathfrak{S}$ makiug the areas of two triangles $P_1P_3P_5$ and $P_2P_4P_6$ are equal.

The proof of the latter half of the theorem is as follows :

Let us denote the areas of the two triangles $P_1P_3P_5$ and $P_2P_4P_6$ by S_{135} and S_{246} respectively, with the areas of their respective projections into the three coordinates planes x_1x_2, x_2x_3 and x_3x_1 denoted by S'_{135}, S''_{135}, S'''_{135} and S'_{246}, S''_{246}, S'''_{246}. Then

$$S_{135}=\frac{1}{2}\{(S'_{135})^2+(S''_{135})^2+(S'''_{135})^2\}^{\frac{1}{2}}$$

$$=\frac{1}{2}\left\{\begin{vmatrix} x_1^1 & x_2^1 & 1 \\ x_1^3 & x_2^3 & 1 \\ x_1^5 & x_2^5 & 1 \end{vmatrix}^2+\begin{vmatrix} x_2^1 & x_3^1 & 1 \\ x_2^3 & x_3^3 & 1 \\ x_2^5 & x_3^5 & 1 \end{vmatrix}^2+\begin{vmatrix} x_3^1 & x_1^1 & 1 \\ x_3^3 & x_1^3 & 1 \\ x_3^5 & x_1^5 & 1 \end{vmatrix}^2\right\}^{\frac{1}{2}}$$

With S_{246}, we have the similar formulae.

If we put

$$g(s_1)=S_{135}-S_{246}$$
$$=\frac{1}{2}\{(S'_{135})^2+(S''_{135})^2+(S'''_{135})^2\}^{\frac{1}{2}}-\frac{1}{2}\{(S'_{246})^2+(S''_{246})^2+(S'''_{246})^2\}^{\frac{1}{2}}$$

using this $g(s_1)$ instead of the above $f(s_1)$, we may have the desired result.

A similar theorem to this Theorem 1 holds also in the case of a rectifiable closed plane chrve.

From the results above, we can deduce the following :

For any systems of points $\mathfrak{S}=\{P_i(s_i),\ i=1, 2, \cdots\cdots, 6\}$ dividing the rectifiable closed space curve into six parts of equal length, circumcenters of the two triangles $P_1P_3P_5$ and $P_2P_4P_6$ can not be coincident. It is just the same with the incenters, or the centroids, or the orthocenters, or the excenters of the triangles concerned.

Subsequently, by similar considerations, we shall prove the two properties of the systems of points $\mathfrak{T}=\{P_1, P_2, P_3, P_4\}$, ordered along the curve, dividing C into four parts of equal length.

Let us denote the coordinates of points P_j by x_α^j, $(j=1,2,3,4;\ \ \alpha=1,2,3)$. Considering the direction ratios of the straight lines P_1P_3, P_2P_4, put

$$\varphi(s_1)=(x_1^3-x_1^1)(x_1^4-x_1^2)+(x_2^3-x_2^1)(x_2^4-x_2^2)+(x_3^3-x_3^1)(x_3^4-x_3^2).$$

Replacing s_1 by s_2, then in the above formula, affixes 1, 3, 5 change to 2, 4, 6, and we have

$$\varphi(s_1)=-\varphi(s_2).$$

Therefore, we have at least one such η as satisfies both $\varphi(\eta)=O$ and $s_1<\eta<s_2$. For this η, the two straight lines are perpendicular. Hence we have the following theorem.

Theorem 2. (A) *Among the systems of points* $\mathfrak{T}=\{P_j,\ j=1,2,3,4\}$, *dividing* C *into four parts of equal length, there exists at least one system making the two straight lines* P_1P_3 and P_2P_4 *perpendicular.*

(B) *Among the systems of points* $\mathfrak{T}=\{P_j,\ j=1,2,3,4\}$, *dividing* C *into four parts of equal length, there exists at least one system making two segments* P_1P_3 *and* P_2P_4 *equal.*

A similar theorem holds also with a rectifiable closed plane curve.

Lastly, by similar considerations, we shall prove the two properties of the systems of points $\mathfrak{U}=\{P_k,\ k=1,2,\cdots\cdots,8\}$, ordered along the curve C, dividing it into eight parts of equal length.

Denoting the coordinates of points P_k, $(k=1,2,\cdots\cdots,8)$ by x_β^k, $(\beta=1,2,3)$, we shall define a function $\Psi(s_1)$ as the difference of the volumes of the two tetrahedra $P_1P_3P_5P_7$ and $P_2P_4P_6P_8$, i.e.,

$$\Psi(s_1)=\frac{1}{6}\left\{\begin{vmatrix} x_1^1 & x_2^1 & x_3^1 & 1 \\ x_1^3 & x_2^3 & x_3^3 & 1 \\ x_1^5 & x_2^5 & x_3^5 & 1 \\ x_1^7 & x_2^7 & x_3^7 & 1 \end{vmatrix}-\begin{vmatrix} x_1^2 & x_2^2 & x_3^2 & 1 \\ x_1^4 & x_2^4 & x_3^4 & 1 \\ x_1^6 & x_2^6 & x_3^6 & 1 \\ x_1^8 & x_2^8 & x_3^8 & 1 \end{vmatrix}\right\}.$$

Replacing s_1 by s_2, we have

$$\Psi(s_1)=-\Psi(s_2).$$

Therefore, we can find at least one such ζ as satisfies both $\Psi(\zeta)=0$ and $s_1<\zeta<s_2$. For this ζ, the two tetrahedra $P_1P_3P_5P_7$ and $P_2P_4P_6P_8$ have the same volume. Thus we have proved the following.

Theorem 3. *Among the systems of points* $\mathfrak{U}=\{P_k,\ k=1,2,\cdots\cdots,8\}$, *dividing* C *into eight parts of eqnal length, there exists at least one system, in which the two tetrahedra* $P_1P_3P_5P_7$ *and* $P_2P_4P_6P_8$ *have the same volume.*

An analogous theorem with a rectifiable closed plane curve is

Theorem 4. *Among the systems of points* $\mathfrak{U}=\{P_k,\ k=1,2,\cdots\cdots,8\}$, *dividing* C *into eight parts of equal length, there exists one system, in which the two quadrilaterals* $P_1P_3\ P_5P_7$ *and* $P_2P_4P_6P_8$ *have the same area.*

Proof. We shall consider a function $h(s_1)$ of the areas of the two quadrilaterals $P_1P_3P_5P_7$ and $P_2P_4P_6P_8$ defined by

$$h(s_1)=\frac{1}{2}\begin{vmatrix} x_1^1 & x_2^1 & 1 \\ x_1^3 & x_2^3 & 1 \\ x_1^5 & x_2^5 & 1 \end{vmatrix}+\frac{1}{2}\begin{vmatrix} x_1^1 & x_2^1 & 1 \\ x_1^5 & x_2^5 & 1 \\ x_1^7 & x_2^7 & 1 \end{vmatrix}-\frac{1}{2}\begin{vmatrix} x_1^2 & x_2^2 & 1 \\ x_1^4 & x_2^4 & 1 \\ x_1^6 & x_2^6 & 1 \end{vmatrix}-\frac{1}{2}\begin{vmatrix} x_1^2 & x_2^2 & 1 \\ x_1^6 & x_2^6 & 1 \\ x_1^8 & x_2^8 & 1 \end{vmatrix}.$$

Here, replacing s_1 by s_2, we have easily $h(s_1)=-h(s_2)$. (An area is taken as positive when its perimeter is described in the counterclockwise sense, and otherwise as negative.) The rest is similar to the proofs of the above theorems, and will be omitted.

Remark. It was proved by ZINDLER[(1)] in 1918 that, by similar considereations, among the systems of points dividing a rectifiable closde space curve into four parts of equal length, there exists, at least, one system of coplanar points.

It is known that at least one system can be of concircular points. But it still remains an unsolved question whether there are any systems of points dividing a rectifiable closed space curve into five equal parts and lying on the same sphere.

It was proved by Professor SHINICHI TAKAHASHI[(2)] of Nagoya Technical College in 1956 that among the systems of points dividing a rectifiable closed space curve into six parts of equal length, there is, at least, one system, for which the three straight lines P_1P_4, P_2P_5 and P_3P_6 are concurrent.

References :

(1) ZINDLER, K. ; (1918) Über die viertelnder Ebenen der geschlossenen Raumkurven. Wien Ber. 127, 1723-1728.

(2) TAKAHASHI, S. ; (1956) Arabesque of Elementary Mathematics Kyoritu Syuppan, Tokyo, (in Japanese).