Free S^1-Action

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1. Introduction

If $T: S^1 \times M^{2n+1} \rightarrow M^{2n+1}$ is a free S^1-action on a closed U-manifold, then there is an S^1-action $S: S^1 \times V^{2n+2} \rightarrow V^{2n+2}$ on a compact U-manifold with boundary, so that the boundary of (V^{2n+2}, S) is precisely (M, T). Of course, the S^1-action S may have fixed points. One such S^1-action is given by

$$V^{2n+2} = M^{2n} \times D^2/(m, z) \sim (T(\lambda, m), \lambda z),\ z \in D^2,\ \lambda \in S^1,$$

with S^1-action S induced by $(m, z) \rightarrow (m, \lambda z)$, $\lambda \in S^1$, for which the fixed point is M^{2n+1}/S.

Following Conner, Floyd and Uchida [1, 2] we know that the class of the free S^1-action (M^{2n}, T) in the bordism group $U_{2n-1}(BS^1)$ of free S^1-action which comute with the U structure is determined by characteristic numbers

$$s_\omega(\tau(M/S^1))c^{n-|\omega|}[M/S^1],$$

where for $\omega = (i_1, \cdots\cdots, i_r)$, $|\omega| = i_1 + \cdots\cdots + i_r$, and c is the Euler class of the principal fibration.

Let ρ denote the standard representation of S^1 on the complex numbers C, and $\bar{\rho}$ the conjugate representation. If the isotopy representation do not involve $\bar{\rho}$, the S^1-action is said to be a regular S^1-action.

Henceforward, we assume that an S^1-action (V^{2n+2}, S) is reguler and commutes with U-structure. Then we have a formula for these characteristic numbers in terms of the fixed point set of the S^1-action S. This is analogous to involution [3].

Proposition 1. For $|\omega| \leqq n$,

$$s_\omega c^{n-|\omega|}[M/S^1] = (-1)^{n+|\omega|} \sum_{j=0}^{n} \frac{s_\omega(1+y_1, \cdots, 1+y_{n+1-j}, z_1, \cdots, z_j)}{\prod_{1}^{n+1-j}(1+y_i)} [F^{2j}]$$

where F^{2j} is the union of the $2j$-dimensional components of the fixed set of S, and the elementary symmetric function $\sigma_i(y)$ and $\sigma_i(z)$ are replaced by the i-th Chern classes $v_i(\nu)$ of the normal bundle of F^{2j} in V^{2n+2} and $v_i(\tau)$ of the tangent bundle of F^{2j}, giving a cohomology class which is evaluated on the fundamental homology class $[F^{2j}]$ of F^{2j}.

Note. F^{2n+2} makes no contribution to the formula.

Proof. Being given a regular U-S^1-action (V^{2n+2}, S) with $\partial(V^{2n+2}, S) = (M^{2n+1}, T)$

being free, we may delate from V an invariant tubular neighborhood of the fixed point set of S to obtain a free bordism from (M^{2n+1}, T) to the disjoint union of the S^1-actions

$$(S(\nu^{n+1-j}), A)$$

given by the scalar multiplication on the sphere bundles $S(\nu^{n+1-j})$, where ν^{n+1-j} is the norml bundle of F^{2j} in V^{2n+2}. We then have

$$s_\omega c^{n-|\omega|}[M/S^1]=\sum_{j=0}^{n} s_\omega c^{n-|\omega|}[CP(\nu^{n+1-j})]$$

evaluating the characteristic number on each S^1-action of scaler multiplication.
Note. F^{2n+2} was removed completely.

To proof the equality

$$s_\omega c^{n-|\omega|}[CP(\nu)]=(-1)^{n+|\omega|}\frac{s_\omega(1+y,z)}{\Pi(1+y)}[F^{2j}]$$

we recall the standard facts about the cohomology $H^*(CP(\nu); Z)$ where ν is a complex $(n+1-j)$ plane bundle over a closed U-manifold F of dimension $2j$. The cohomology $H(CP(\nu); Z)$ is the free $H^*(F; Z)$ module on $1, c, \cdots\cdots, c^{n-j}$ where c is the characteristic class of the canonical bundle. The ring structure is then determined by the relation

$$\sum_{i=0}^{n+1-j}(-1)^{n+1-j-i}v_i c^{n+1-j-i}=0,$$

where $v_0=1, v_1, \cdots\cdots, v_{n+1-j}$ are the the Chern classes of ν. The Chern class of $CP(\nu)$ is

$$c(CP(\nu))=c(F)\cdot\{(1-c)^{n+1-j}+(1-c)^{n-j}v_1+\cdots\cdots\cdots+v_{n+1-j}\}$$

and if $\alpha\in H^*(F; Z)$, we have from [3] that

$$c^i\alpha[CP(\nu)]=\begin{cases}0 & \text{if} \quad i<n-j\\ (-1)^{j-i}\bar{v}_{j-i}\alpha[F] & \text{if} \quad i\geq n-j\end{cases}$$

where $\bar{v}$ denotes the dual Chern class defined by $v\bar{v}=1$.

It is convenient to formally write

$$c(F)=\prod_1^j(1+z_i)$$

and $\quad c(\nu)=\prod_1^{n+1-j}(1+y_i)$

using the splitting principle. We then have formally

$$c(CP(\nu))=\prod_1^j(1+z_i)\cdot\prod_1^{n+1-j}(1-c+y_i).$$

If we now expand the expression

$$s_\omega(1+y_1,\cdots\cdots\cdots,1+y_{n+1-j},z_1,\cdots\cdots,z_j)=\sum_{i=0}^{j}\gamma^i\in H^*(F; Z)$$

with $\gamma_i\in H^{2i}(F; Z)$, we have

$$\begin{aligned}s_\omega(-c+y_1,\cdots,-c+y_{n+1-j},z_1,\cdots,z_j)c^{n-|\omega|}[CP(\nu)]&=(\sum_{i=0}^{j}\gamma^i(-c)^{|\omega|-i})c^{n-|\omega|}[CP(\nu)]\\&=\sum_{i=0}^{j}(-1)^{|\omega|-i}\gamma^i c^{n-i}[CP(\nu)]\end{aligned}$$

$$=\sum_{i=0}^{j}(-1)^{|\omega|-i+j-i}\gamma^i \bar{v}_{j-i}c^{n-j}[CP(\nu)]$$
$$=(-1)^{n+|\omega|}\sum\gamma^i \bar{v}_{j-i}[F]$$

The proof is complete.

Proposition 1 provides direct implications converting hypotheses about the fixed set of S into conditions on (M^{2n+1}, T). Next proposition is analogous to [4, proposition 3].

Proposition 2. If (M^{2n+1}, T) is a free U-S^1-action represented as

$$\sum_{i=0}\beta_{2i}[S^{2(n+i)-1}, A]$$

where the S^1-action A is the scalar multiplication, and if $(M^{2n+1}, T)=\partial(V^{2n+2}, S)$ with the fixed point set of S having dimension less than or equal to $2k$, then the Euler characteristic of β_{2r} is zore for $r>2k$.

In order to prove this proposition, we shall make use of the characteristic number formula.

Being given a regular U-S^1-action (V^{2n+2}, S) with S free on ∂V, we let

$$\sigma(V^{2n+2},S)=\sum_{j=0}^{n+1}\left(\sum_{r=0}^{\infty}\frac{\sigma_r(1+y_1,\cdots\cdots,1+y_{n+1-j},z_1,\cdots\cdots,z_j)t^r}{\prod_1^{n+1-j}(1+y_i)}\left[F^{2j}\right]\right.$$

in $Z[t]$, where σ_i is the i-th elementary symmetric function.

By the formula, we have that

$$\sigma(V^{2n+2},S)=\sum_{r=0}^{n}(-1)^{n-r}v_r(\tau)c^{n-r}[\partial V/S^1]\,t^r+(\sum_{j=0}^{n+1}\chi(F^{2j})t^{n+1}$$

where χ is the Euler characteristic. Note: F^{2n+2} maters here.

We have the obvious additive rule for disjoint unions. Also we have a product rule.

Lemma. $\sigma(V_1^{2n_1+2}\times V_2^{2n_2+2},S_1\times S_2)=\sigma(V_1^{2n_1+2},S_1)\cdot\sigma(V_2^{2n_2+2},S_2)$.

Proof. For $i=1, 2$, we express the Chern class of the tangent bundle of the fixed point set in the form

$$c(F^{j_i})=(1+z_1^i)\cdots\cdots\cdots(1+z_{j_i}^i)$$

and Chern class of the normal bundle ν_i of F^{j_i} in the form

$$c(\nu_i)=(1+y_1^i)\cdots\cdots\cdots(1+y^i{}_{n_i+1-j_i}).$$

Then

$$\sigma(V_1^{2n_1+2}\times V_2^{2n_2+2},S_1\times S_2)$$
$$=\sum_{j_1+j_2=0}^{n_1+n_2+2}\left(\sum_{r=0}^{\infty}\frac{\sigma_r(1+y_1^1,\cdots,1+y_{n_1+1-j_1}^1,1+y_1^2\cdots,1+y_{n_2+1-j_2}^2,z_1^1\cdots.z_{j_1}^1,z_1^2,\cdots,z_{j_2}^2)t^r}{\prod_1^{n_1+1-j_1}(1+y_i^1)\cdot\prod_1^{n_2+1-j_2}(1+y_i^2)}\left[F^{2j_1}\times F^{2j_2}\right]\right)$$
$$=\sum_{j_1+j_2=0}^{n_1+n_2+2}\left(\sum_{r_1+r_2=r}^{\infty}\frac{\sigma_{r_1}(1+y_1^1,\cdots,1+y_{n_1+1-j_1}^1,z_1^1,\cdots,z_{j_1}^1)\sigma_{r_2}(1+y_1^2\cdots,1+y_{n_2+1-j_2}^2,z_1^2\cdots.z_{j_2}^2)t^r}{\prod_1^{n_1+1-j_1}(1+y_i^1)\cdot\prod_1^{n_2+1-j_2}(1+y_i^2)}\left[F^{2j_1}\times F^{2j_2}\right]\right)$$

$$=\left(\sum_{j_1=0}^{n_1+1}\left(\sum_{r_1=0}^{\infty}\frac{\sigma_{r_1}(1+y_1^1,\cdots,1+y_{n_1+1-j_1}^1,z_1^1\cdots.z_{j_1}^1)t^{r_1}}{\prod_1^{n_1+1-j_1}(1+y_i^1)}\left[F^{2j_1}\right]\right)\left(\sum_{j_2=0}^{n_2+1}\left(\sum_{r_2=0}^{\infty}\frac{\sigma_{r_2}(1+y_1^2\cdots,,1+y_{n_2+1-j_2}^2,z_1^2,\cdots,z_{j_2}^2)t^{r_2}}{\prod_1^{n_2+1-j_2}(1+y_i^2)}\left[F^{2j_2}\right]\right)$$

$=\sigma(V_1, S_1)\cdot\sigma(V_2, S_2)$.

By D^n, we denote the n-dimensional disk. We easily verify

Lemma. $([D^2], A)=1+t$, where the S^1-action A is the scaler multiplication.

Applying the product rule then gives

Lemma. $\sigma(\bigcup_{i=0}^{n} B^{2i}\times[D^{2(n+1-i)}, A])=\sum_{i=0}^{n}\chi(B^{2i})t^i(1+t)^{n+1-i}$.

Proof of Proposition 2. By hyposesis $\partial(V^{2n+2}, S)=\sum_{j=0}^{n}\beta_{2j}(S^{2(n-j)+1}, A)\in U_{2n}(BS^1)$, and $\sigma(V^{2n+2}, S)=\sigma(\bigcup_{j=0}^{n} B^{2j}\times[D^{2(n+1-j)}, A])$ plus a multiple of t^{n+1}, where B^{2j} is the representative of β_{2j}, for these have the same boundary, up to free bordism.

On the other hand, the fixed data of (V^{2n+2},S) is $\bigcup_{j=0}^{k}(F^{2j},\nu^{n+1-j})$, so by stability,

$\nu^{n+1-j}\simeq\xi^j\oplus(n+1-2j)$ and

$$\sigma(V^{2n+2}, S)=\sum_{j=0}^{k}\sigma(D(\nu^{n+1-j}), A)$$
$$=\sum_{j=0}^{k}\sigma(D(\xi^j), A)\cdot(1+t)^{n+1-2j}.$$

Hence $\sigma(V^{2n+2}, S)$ is divisible by $(1+t)^{n+1-2k}$, so $\chi(B^{2j})$ is zero for $j>2k$.

Proposition 2 is related to the bordism J homomorphism

$$J:\ \bigoplus_{j=0}^{k} U_{2j}(BU_{n+1-j})\longrightarrow U_{2n}(BS^1)$$

which assigns to $\xi^{n+1-j}\to N^{2j}$ the class of the free U-S^1-action on the sphere bundle of ξ^{n+1-j}.

In this notation, Proposition 2 induces

Proposition 3. $J:\ \bigoplus_{j=0}^{[n/2]-1} U_{2j}(BU_{n+1-j})\longrightarrow U_{2n}(BS^1)$ is not epic.

Proof. By Proposition 2, $[CP(2[n/2]-1)]\cdot[S^{2(n-2[n/2]+1)-1}, A]$ cannot belong to image..

We consider the epicity of the bordism J homomorphism.

Proposition 4. $J:\ \bigoplus_{j=0}^{[n/2]} U_{2j}(BU_{n+1-j})\longrightarrow U_{2n}(BS^1)$ is epic.

To prove Proposition 4. we have first

Lemma. Let $\alpha\in U_{2n}$. There is a class

$$x\in\bigoplus_{j=0}^{[n/2]} U_{2j}(BU_{n+1-j})\quad\text{with}\quad J(x)=\alpha[S^1, A].$$

Proof. The generators of the bordism ring U_* of U-manifolds are $[CP(n)]$ $(n\geq 1)$ and $[H_{m,n}]$ $(1<m\leq n)$. There is a semi-free U-S^1-action $(CP(n), S)$ defined by

$$S(z(z_1:\cdots\cdots:z_n))=(z_1:\cdots\cdots:z_{[\frac{n}{2}]}:zz_{[\frac{n}{2}]+1}:\cdots\cdots:zz_n),\ z\epsilon S^1,$$

and a semi-free U-S^1-action $(H_{m,n}, T)$ defined by

$$T(z,(z_1:\cdots\cdots:z_m)\times(z_1:\cdots\cdots:z_n))$$
$$=(z_1:\cdots\cdots:z_{[\frac{m}{2}]}:zz_{[\frac{m}{2}]+1}:\cdots\cdots:zz_m)\times(z_1:\cdots\cdots:z_{[\frac{n}{2}]}:\bar{z}z_{[\frac{n}{2}]+1}:\cdots\cdots:\bar{z}z_n).$$

Then, dim $F_S \leqq 2[n/2]$ and dim $F_T \leqq 2[(m+n-1)/2]$, $(m+n-1=1/2 \text{ dim } H_{m,n})$.

Hence, there is a U-manifold with the U-S^1-action (N^{2n}, T) with dim $F_T \leqq 2[n/2]$ and $[N]=\alpha$ and let

$$x=\sum_{j=0}^{[n/2]}(F_T^{2j},\ \nu^{n-j}\oplus 1),$$

where ν^{n-j} is the normal bundle of F_T^{2j} in N^{2n}.

We form a semi-free U-S^1-action $\Gamma(N^{2n}, T)$ from $(-D^2\times N^{2n}, A\times T)$, ($A$ is the scalar multiplication) and $(D^2\times N^{2n}, A\times T)$ by identifying the boundaries via the equivariant diffeomophism

$$\varphi:(S^1\times N^{2n},\ A\times 1)\longrightarrow(S^1\times N^{2n},\ A\times T)$$

which is defined by $\varphi(z, n)=(z, T(z, n))$. The fixed point data of $\Gamma(N^{2n}, T)$ is $x\oplus(N^{2n}, 1)$ and $JF=0$ so $J(-x)=J(N, 1)=[N][S^1, A]$, or $\alpha[S^1, A]=J(-x)$.

Proof of proposition 4. We proof proposition by induction on n. The case $n=0$ is simple, for $U_0(BS)\cong Z$ on the class $[S^1, A]=J(D^2, A)$ from $U_0(BU_1)$.

Now consider the commutative diagram

$$\begin{array}{ccc}
\bigoplus_{j=0}^{[n/2]} U_{2j}(BU_{n+1-j}) & \xrightarrow{\ J\ } & U_{2n}(BS^1) \\
\Big\downarrow I_* & & \Big\downarrow \Delta \\
\bigoplus_{j=0}^{[(n-1)/2]} U_{2j}(BU_{n-j}) & \xrightarrow{\ J'\ } & U_{2(n-1)}(BS^1)
\end{array}$$

where I_* adds a trivial line bundle and Δ is the Smith homomorphism. We assume J' is epic.

Letting $\beta \epsilon U_{2n}(BS^1)$, $\Delta\beta=J'(y)$ for some y, and $\Delta JI_*(y)=J(y)=\Delta\beta$ so $\beta-JI_*(y)\ \epsilon$ kernel Δ. Now kernel Δ cosists of the classes $\alpha\ [S^1,A]$, $\alpha\in U_{2n}$, so $\beta-JI_*(y)=\alpha[S^1,A]$ for some α. By the lemma, $\alpha[S,A]=J(x)$ for some x, and so $\beta=J(x+I_*(y))$. Thus J is epic.

Proposition. 3 and Proposition 4 are equivalent to

Proposition 3′. There exist a free U-S^1-action (M^{2n+1},T) which do not bound any U-S^1-action having fixed point set of dimension less than $2[n/2]$.

Proposition 4′. If (M^{2n+1}, T) is a free U-S^1-action, there is a semi-free U-S^1-action (V^{2n+2}, S) with the fixed point set of S having dimension less than or equal $2[n/2]$.

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