ON THE UNIT GROUPS OF THE BURNSIDE RINGS OF REAL-ELEMENTARY GROUPS

Tadao OBAYASHI

In memory of Professor Akira HATTORI

§ 1. Introduction

Let $A(G)$ be the Burnside ring of a finite group G and $A^*(G)$ the unit group of $A(G)$. Then there exists a homomorphism

$$u_G\ ;\ RO(G) \longrightarrow A^*(G)$$

defined by T. tom. Dieck (cf. [4]), where $RO(G)$ is the character ring of real representations of G. The map u_G is not necessarily surjective. However, it is known that u_G is surjective, if G is one of the following groups:

(1) G is a group of odd order (A. Dress, [1]),

(2) G is a 2-group (J. Tornehave, [5]),

(3) G is an abelian group (T. Matsuda, [2]), or

(4) G has a normal subgroup H of odd index such that

$$u_H\ ;\ RO(H) \longrightarrow A^*(H)$$

is surjective (T. Matsuda and T. Miyata, [3]).

Let $Cl(G)$ be the set of the conjugate classes of subgroups of a finite group G and $\{\pm 1\}^{Cl(G)}$ be the group consisting of the functions defined on $Cl(G)$ with values in the set $\{\pm 1\}$. Then, $A^*(G)$ is identified with a subgroup of $\{\pm 1\}^{Cl(G)}$ and the tom. Dieck homomorphism u_G is canonically extended to a homomorphism

$$\tilde{u}_G\ ;\ R(G) \longrightarrow \{\pm 1\}^{Cl(G)},$$

where $R(G)$ is the character ring of complex representations of G.

In this paper, we will study the map $\tilde{u}_G$ and prove

Theorem 1. *If G is a real-elementary group, then $A^*(G)$ is contained in the image of the map $\tilde{u}_G$.*

For any finite group G and its subgroup H, let

$$jnd_H\ ;\ A^*(H) \longrightarrow A^*(G)$$

be the multiplicative induction map. Then, as a corollary of this theorem, we have the following theorem.

Theorem 2. *For any finite group G, the image of the map $\bar{u}_G$ contains the subgroup $\prod_H jnd_H\,(A^*(H))$ of $A^*(G)$, where H runs over the set of all real-elementary subgroups of G.*

§ 2. Preliminaries

The Burnside ring $A(G)$ of a finite group G is defined as a Grothendieck ring on the finite G-sets, with addition induced by disjoint union and multiplication by cartesian product. Additively $A(G)$ is a free abelian group on the isomorphism classes $[G/H]$ of homogeneous G-sets G/H. It is known that $A(G)$ can be identified with a subring of the ring consisting of integer-valued functions which are defined on the set $Cl(G)$ of the conjugate classes (S) of subgroups S of G. Then we view frequently $A^*(G)$ as a subgroup of $\{\pm 1\}^{Cl(G)}$. Especially, the group $A^*(G)$ has exponent 2.

For any real representation V of G and any subgroup S of G, we denote by V^S the S-invariant subspace of V. Then the function $u_G([V])$ defined by the following formula

$$u_G([V])(S) = (-1)^{\dim V^S} \quad (S \leq G) \tag{1}$$

is an element of $A^*(G)$, and the assignment $[V] \longrightarrow u_G([V])$ defines the tom. Dieck homomorphism

$$u_G\,;\ RO(G) \longrightarrow A^*(G)$$

of the additive structure of the ring $RO(G)$ to $A^*(G)$ (for detail, see [4]).

In the formula (1), if we replace V by any complex representation V of G, then the resultant function $\bar{u}_G([V])$ belongs to the group $\{\pm 1\}^{Cl(G)}$. The assignment $[V] \longrightarrow \bar{u}_G([V])$ defines a homomorphism

$$\bar{u}_G\,;\ R(G) \longrightarrow \{\pm 1\}^{Cl(G)}$$

Let $c: RO(G) \longrightarrow R(G)$ be the complication map and i; $A^*(G) \longrightarrow \{\pm 1\}^{Cl(G)}$ be the inclusion map. Then we have the commutative diagram

$$\begin{array}{ccc} R(G) & \xrightarrow{\ \bar{u}_G\ } & \{\pm 1\}^{Cl(G)} \\ \uparrow c & & \uparrow i \\ RO(G) & \xrightarrow{\ u_G\ } & A^*(G) \end{array} \tag{2}$$

Let

$$\langle \chi, \theta \rangle_G = \frac{1}{|G|} \sum_{g \in G} \chi(g)\,\theta(g^{-1})$$

be the usual inner product of class functions χ and θ on G. If χ is the character afforded by a complex representation V of G, then for each subgroup S of G, the S-invariant space V^S is a representation of the subquotient group $N_G(S)/S$ ($N_G(S)$ is the normalizer of S). Let χ_{V^S} be the character afforded by V^S. Then by an easy representation theory, the values of χ_{V^S} are given by

$$\chi_{V^S}(gS) = \frac{1}{|S|}\sum_{s\in S}\chi(gs) \qquad (gS \in N_G(S)/S).$$

Hence we obtain

Lemma 1 (cf., [6]). *For each virtual character χ of $R(G)$,*

$$\tilde{u}_G(\chi)(S) = (-1)^{\langle \chi_S, 1_S\rangle_S} \qquad (S \leq G),$$

where χ_S is the restriction of χ to the subgroup S and 1_S is the trivial character of S.

Let H be a subgroup of G. Then we have the restriction map res_H; $A^*(G) \longrightarrow A^*(H)$ and the multiplicative induction map jnd_H; $A^*(H) \longrightarrow A^*(G)$. As a function, $jnd_H(\alpha)$ ($\alpha \in A^*(H)$) is defined by

$$jnd_H(\alpha)(S) = \Pi_{g\in H\backslash G/S}\,\alpha(H\cap S^g) \qquad (S \leq G),$$

where $S^g = gSg^{-1}$ and g runs over a complete set of representatives of the double coset $H\backslash G/S$. Moreover, each element g of G induces the conjugation map $conj_H{}^g$; $A^*(H) \longrightarrow A^*(H^g)$. Together with these maps *res*, *jnd* and *conj*, $A^*(G)$ is considered as a G-functor.

On the otherhand, together with the usual restiction map *res*, the induction map *ind* and the conjugation map *conj* of characters, $RO(G)$ is also a G-functor, and the tom. Dieck homomorphism u_G is a morphism between the G-functors $RO(G)$ and $A^*(G)$ (for detail, see [6]). Analogously, $\{\pm 1\}^{Cl(G)}$ and $R(G)$ are also G-functors, and the commutative diagram (2) is that of morphisms of G-functors.

In the nexst section, we need further maps, so called invaiant and inflation. Let N be a normal subgroup of G. Then the invariant map inv_N; $A^*(G) \longrightarrow A^*(G/N)$ and the inflation map inf_N; $A^*(G/N) \longrightarrow A^*(G)$ are defined by

$$inv_N(\alpha)(S/N) = \alpha(S) \qquad (\alpha \in A^*(G),\ N \leq S \leq G)$$

and

$$inf_N(\bar{\alpha})(S) = \bar{\alpha}(SN/N) \qquad (\bar{\alpha} \in A^*(G/N),\ S \leq G),$$

respectively. Similarly, we have the maps inv_N; $RO(G) \longrightarrow RO(G/N)$ and inf_N; $RO(G/N) \longrightarrow RO(G)$. Moreover, it is clear that u_G commutes with these maps.

§ 3. Proof of Theorem 1 and Theorem 2

Let p be a prime number. A group G is said to be real-p-elementary, if it is a semidirect product $C \cdot P$ of a p-group P and a cyclic group $C=\langle x\rangle$ of order prime to p such that

$$yxy^{-1} = x^{\pm 1} \qquad \text{for each} \qquad y \in P.$$

A group G is said to be real-elementary if it is real-p-elemetary for at least one prime number p.

We shall start the proof of Theorem 1 from the following trivial cases.

Lemma 2. *Let G be a real-p-elementary group. Then $A^*(G)$ is contained in the image of $\bar{u}_G$ if $p \neq 2$ or $p=2$ and P acts on C trivially.*

Proof. In each cases of the lemma, G becomes into a direct product $C \times P$. Then it follows from the results of J. Tornehave ([5]), T. Matsuda ([2]) and T. Matsuda and T. Miyata ([3]) that u_G is surjective (cf., §1). Thus, the lemma is obvious from the commutative diagram (2).

In the following lines, we assume that G is a real-2-elementary group, and we set

$$G = C \cdot T,$$

where T is a 2-group and $C=\langle x\rangle$ is a cyclic group of odd order n and T acts on C non trivially. Let T_0 be the kernel of the action. Then T_0 is a maximal normal subgroup of T of index 2.

Let $\mathbf{Z}_{(2)}$ be the ring of 2-local integers and let

$$A(G)_{(2)} = \mathbf{Z}_{(2)} \otimes A(G)$$

be the 2-local Burnside ring. Since the G-functor $A^*(G)$ (resp. $\{\pm 1\}^{Cl(G)}$) has exponent 2, it can be viewed as a module over the ring $A(G)_{(2)}$. Indeed, each generator $[G/H]$ of $A(G)_{(2)}$ acts on an element α of $A^*(G)$ (resp. $\{\pm 1\}^{Cl(G)}$) in the following way

$$\alpha \uparrow [G/H] = jnd_H \ res_H(\alpha).$$

For each subgroup $Q \leq C$, we denote by $e_{G,Q}$ the idempotent of $A(G)_{(2)}$ defined by the function

$$e_{G,Q}(S) = \begin{cases} 1 & \text{if } O^2(S) \text{ is conjugate to } Q \\ 0 & \text{otherwise,} \end{cases}$$

where $S \leq G$ and $O^2(S)$ means the subgroup generated by all 2'-elements of S.

Lemma 3. (1) *There is a direct decomposition*

$$A^*(G) = \Pi_Q A^*(G) \uparrow e_{G,Q}$$

into subgroups, where Q runs over the subgroups of C.
(2) *The restriction map res_{QT} induces an isomorphism*

$$r_Q ;\ A^*(G) \uparrow e_{G,Q} \xrightarrow{\sim} A^*(QT) \uparrow e_{QT,Q}.$$

Set $a = e_{G,Q}[G/QT]$. Then a is a unit of $A(G)_{(2)}$, and the map

$$j_Q = jnd_{QT} \uparrow a^{-1}$$

gives the inverse of r_Q.
(3) *Let $A^*(T)_Q$ be the subgroup of $A^*(T)$ consisting of the elements β such that for each $T' \leq T$, $\beta(T') = 1$ whenever QT' has a proper normal subgroup of odd index. Then the invariant map inv_Q induces an isomorphism*

$$v_Q ;\ A^*(QT) \uparrow e_{QT,Q} \xrightarrow{\sim} A^*(T)_Q$$

and the inverse map of v_Q is given by

$$v_Q^{-1}(\beta)(S) = \begin{cases} \beta(S/Q) & \text{if } Q \leq S \\ 1 & \textit{otherwise} \end{cases}$$

for any $\beta \in A^*(T)_Q$ and $S \leq QT$.

Proof. Since G is real-2-elementary, it is easily seen that a subgroup Q of G is 2-perfect if and only if $Q \leq C$, and so it is necessarily normal in G. Then the decomposition of (1) follows directly from the decomposition theorem by T. Yoshida ([6], Theorem B, (i)). Moreover, note that any conjugate of QT not identical with itself written as the form QT^c by some $c \in C\text{-}Q$ and that $QT \cap QT^C = QT_o$. Then, any element α of $A^*(QT)$ satisfies the equalities

$$res_{QT \cap QT^g}\ conj^g{}_{QT}(\alpha) = res_{QT \cap QT^g}(\alpha) \qquad \text{for all } g \in G.$$

Hence, the assertions (2) and (3) follow from the stable element theorem and the transfer theorem by T. Yoshida (*id.* Theorem B, (ii) and Theorem C, respectively).

Since $A^*(G)$ (resp. $\{\pm 1\}^{Cl(G)}$) has exponent 2, the map u_G (resp. $\bar{u}_G$) factors through the homomorphism u^*_G; $RO(G)/2RO(G) \longrightarrow A^*(G)$ (resp. $\bar{u}^*_G$; $R(G)/2R(G) \longrightarrow \{\pm 1\}^{Cl(G)}$). Together with the maps induced by *res*, *ind* and *conj*, $RO(G)/2RO(G)$ (resp. $R(G)/2R(G)$) comes into a G-functor. Then $A(G)_{(2)}$ acts on $RO(G)/2RO(G)$ (resp. $R(G)/2R(G)$), and the map u^*_G (resp. $\bar{u}^*_G$) is considered as a homomorphism of $A(G)_{(2)}$-modules.

From the decomposition (1) in the above lemma, to prove Theorem 1, it is sufficient to show that

$$A^*(G)\uparrow e_{G,Q} \leq \mathrm{Im}\,(\bar{u}_G) \tag{3}$$

for each $Q \leq C$. Let r_Q and v_Q be the isomorphisms in the preceding lemma. Then, via the isomorphism

$$v_Q r_Q \;;\; A^*(G)\uparrow e_{G,Q} \longrightarrow A^*(T)_Q$$

and by the surjectivity of the map u_T (cf., § 1), we can restate the above condition (3) in terms of characters.

Lemma 4. *Let Q be a subgroup of C. Then the inclusion* (3) *is verified, if for any virtual character ρ of $RO(T)$ such that $u_T(\rho) \in A^*(T)_Q$, there exsists a virtual character χ of $R(QT)$ with the following properties:for each $Q' \leq Q$ and $T' \leq T$*

$$(*) \quad \bar{u}_{QT}(\chi)(Q'T') = \begin{cases} u_T(\rho)(T') & \text{if } Q' = Q \\ 1 & \textit{otherwise.} \end{cases}$$

Proof. Let α be any element of $A^*(G)\uparrow e_{G,Q}$ and set

$$\beta = v_Q r_Q(\alpha).$$

Since u_T is surjective, there exsists a virtual character $\rho \in RO(T)$ such that $u_T(\rho) = \beta$. Then by assumption we can find a virtual character $\chi \in R(QT)$ with the above properties (*). But up to conjugate, any subgroup of QT is written as $Q'T'$ by some $Q' \leq Q$ and $T' \leq T$. Hence, by Lemma 2, (3) we see that

$$\bar{u}_{QT}(\chi) = v_Q^{-1}(\beta),$$

so by the same lemma, (2) we obtain

$$\alpha = r_Q^{-1} v_Q^{-1}(\beta) = r_Q^{-1}(\bar{u}_{QT}(\chi)) = jnd_{QT}(\bar{u}_{QT}(\chi))\uparrow a^{-1}.$$

Consider the following commutative diagram

$$\begin{array}{ccc} R(G)/2R(G) & \xrightarrow{\bar{u}^*_G} & \{\pm 1\}^{Cl(G)} \\ \uparrow ind^*_{QT} & & \uparrow jnd_{QT} \\ R(QT)/2R(QT) & \xrightarrow{\bar{u}^*_{QT}} & \{\pm 1\}^{Cl(QT)} \end{array}$$

of $A(G)_{(2)}$-homomorphisms. Set $\chi^* = \chi$ (mod. 2). Then

$$\begin{aligned} \alpha &= jnd_{QT}(\bar{u}_{QT}(\chi))\uparrow a^{-1} \\ &= jnd_{QT}(\bar{u}^*_{QT}(\chi^*))\uparrow a^{-1} \\ &= \bar{u}^*_G(ind^*_{QT}(\chi^*))\uparrow a^{-1} \\ &= \bar{u}^*_G(ind^*_{QT}(\chi^*)\uparrow a^{-1}). \end{aligned}$$

But the element $ind^*_{QT}(\chi^*)\uparrow a^{-1}$ belongs to $R(G)/2R(G)$. Then $\alpha \in \mathrm{Im}\,(\bar{u}^*_G) = \mathrm{Im}\,(\bar{u}_G)$, which proves the lemma.

Lemma 5. *Let $A^*(T)_0$ be the kernel of the restriction map $res_{T_0}; A^*(T) \longrightarrow A^*(T_0)$. Then for each $Q \leq C$*

$$A^*(T)_Q = \begin{cases} A^*(T) & \text{if } Q = 1 \\ A^*(T)_0 & \textit{otherwise.} \end{cases}$$

Proof. Recall that $A^*(T)_Q$ is consisted of the elements β of $A^*(T)$ with the property: for each $T' \leq T$, $\beta(T')=1$ whenever QT' has a proper normal subgroup of odd index. But we see easily that the condition on QT' is equivalent to that Q contains the commutator $[Q, T']$ as a proper subgroup. Since Q is an abelian group of odd order on which the 2-group T' acts, it is known in group theory that there is a direct decomposition

$$Q = C_Q(T') \times [Q, T'],$$

where $C_Q(T')$ means the centralizer of T' in Q. Hence, $Q \neq [Q, T']$ is equivalent to $C_Q(T') \neq 1$. When $Q = 1$, $C_Q(T') = 1$ for every $T' \leq T$, and when $Q \neq 1$, $C_Q(T') \neq 1$ if and only if $T' \leq T_0$. This proves the lemma.

Lemma 6. *Let β be any element of $A^*(T)_0$. Then there exists an induced character $\rho = ind_{T_0}(\rho_0)$ from some character ρ_0 of $RO(T_0)$ such that $\beta = u_T(\rho)$.*

Proof. Since u_T is surjective, there exists a virtual character ρ of $RO(T)$ such that $\beta = u_T(\rho)$. We may assume that ρ is a sum of absolutely irreducible characters. Let ϕ be an absolutely irreducible character of T. Since T_0 is a normal subgroup of index 2, the restriction to T_0 of ϕ has the form

$$\text{(i)} \quad res_{T_0}(\phi) = \phi_0 \quad \text{or}$$

$$\text{(ii)} \quad res_{T_0}(\phi) = \phi_0 + \phi_0{}^t,$$

where ϕ_0 is an absolutely irreducible character of T_0 and $\phi_0{}^t$ ($\neq \phi_0$) means the conjugate character by an element t of $T - T_0$.

In the case of (i), there exists an absolutely irreducible character ϕ' ($\neq \phi$) such that

$$res_{T_0}(\phi') = \phi_0 \quad \text{and} \quad ind_{T_0}(\phi_0) = \phi + \phi'.$$

Set $\psi = \phi + \phi'$. Since the restriction and u_T are commutable, we see that $u_T(\psi)$ belongs to $A^*(T)_0$. In the case of (ii), it is clear that $ind_{T_0}(\phi_0) = \phi$. If $\phi_0{}^t$ is equal to some algebraic conjugate of ϕ_0, then $u_T(\phi)$ also belongs to $A^*(T)_0$. By the result of J. Tornehave ([5]), the kernel of u_{T_0} is generated by the elements 2η and $\eta + \eta'$, where η is a virtual character and η' is it's algebraic conjugate. Thus, we see easily that $u_T(\rho)$ belongs to $A^*(T)_0$ if and only if ρ is a sum of characters ψ and ϕ of the above types. But, both ψ and ϕ are induced from characters ϕ_0 of T_0. Hence, ρ is also induced from some character ρ_0 of T_0, which proves the lemma.

Now we will show that for each $Q\leq C$ and any $\rho\in RO(T)$ such that $u_T(\rho)\in A^*(T)_Q$, we can find a virtual complex character $\chi\in R(QT)$ with the properties (*) in Lemma 4. If $Q=1$, then there is nothing to do. In other cases, by Lemma 5 $A^*(T)_Q=A^*(T)_0$ for an arbitrary Q, and the group QT is also real-2-elementary. Then, it suffices to do only in the case of $Q=C$. Moreover, by the preceding lemma, we can assume that $\rho=ind_{T_0}(\rho_0)$ for a certain character ρ_0 of $RO(T_0)$. Hence by Lemma 4, Theorem 1 is proved, if we show the following lemma.

Lemma 7. *Let ρ_0 be a character of $RO(T_0)$ and set $\rho=ind_{T_0}(\rho_0)$. Then we can find a virtual complex character χ of G such that for each $Q\leq C$ and $T'\leq T$,*

$$\bar{u}_G(\chi)(QT')=\begin{cases} u_T(\rho)(T') & \text{if } Q=C \\ 1 & \textit{otherwise.}\end{cases} \tag{4}$$

Proof. Let n $(\neq 1)$ be the order of the cyclic group $C=\langle x\rangle$. Then, there exsist the integers $a_1,\ a_2,\ \cdots\cdots,\ a_{n-1}$ which satisfy the following system of congruences:

$$a_{n/d}+a_{2n/d}+\cdots\cdots+a_{(d-1)n/d}\equiv 1\ (\text{mod. } 2) \tag{5}$$

corresponding to each divisor d $(\neq 1)$ of n. Indeed, this is an easy consequence from linear algebra. Let

$$1_C=\xi_0,\ \xi_1,\ \xi_2,\cdots\cdots,\ \xi_{n-1}$$

be the distinct irreducible characters of C which are defined by

$$\xi_k(x)=e^{2\pi ki/n}\qquad (k=0,\ 1,\ 2,\ \cdots\cdots,\ n-1),$$

and set

$$\xi=1_C+\sum\nolimits_{k=1}^{n-1}a_k\xi_k.$$

Then ξ is a virtual character of C, and for each $Q\leq C$, it satisfies the congruences (mod. 2):

$$\frac{1}{|Q|}\sum\nolimits_{q\in Q}\xi(q)=\langle\xi_Q,\ 1_Q\rangle_Q\equiv\begin{cases}1 & \text{if } Q=C\\ 0 & \text{otherwise.}\end{cases} \tag{6}$$

In fact, we see that

$$\langle\xi_Q,\ 1_Q\rangle=1+\sum\nolimits_{k=1}^{n-1}a_k\langle(\xi_k)_Q,\ 1_Q\rangle_Q.$$

Since ξ_k is linear, $\langle(\xi_k)_Q,\ 1_Q\rangle_Q$ is equal to 0 or 1. When $Q=C$, obviously it is equal to 0 for $k=1,\ 2,\cdots\cdots,\ n\text{-}1$. Let $Q\neq C$ and $n/d=|Q|$ $(d\neq 1)$ be the order of Q. Then, by the definition of the character ξ_k, we see that $\langle(\xi_k)_Q,\ 1_Q\rangle$ is equal to 1 whenever k is a multiple of n/d. Hence, the congruences (6) follow from (5).

Let $\tilde{\xi}$ and $\tilde{\rho}_0$ be the extensions of the characters ξ and ρ_0 to the group CT_0, respectively. Indeed, these are well defined, because CT_0 is a direct product. Then, the induced character

$$\chi = ind_{CT_0}(\tilde{\xi}\ \tilde{\rho}_0)$$

is the required virtual character of $R(G)$. To show this, we calculate the inner product $<\chi_{QT'}, 1_{QT'}>_{QT'}$. Since

$$\sum_{q\in Q} \xi(q^t) = \sum_{p\in Q} \xi(q) \qquad \text{for each} \quad t\in T,$$

we obtain that

$$\begin{aligned}
<\chi_{QT'}, 1_{QT'}>_{QT'} &= \frac{1}{|QT'|}\sum_{qt'\in QT'} \chi(qt') \\
&= \frac{1}{|QT'|}\sum_{qt'\in QT'}\frac{1}{|CT_0|}\sum_{\substack{g\in G \\ (qt')^g\in CT_0}} \tilde{\xi}\ \tilde{\rho}_0((qt')^g) \\
&= \frac{1}{|QT'|}\sum_{q\in Q}\sum_{t'\in T'\in T_0}\frac{1}{|T_0|}\sum_{t\in T}\xi(q^t)\,\rho_0(t'^t) \\
&= \frac{1}{|QT'|}\sum_{t'\in T'\cap T_0}\frac{1}{|T_0|}\sum_{t\in T}(\sum_{q\in Q}\xi(q^t)\,\rho_0(t'^t) \\
&= \frac{1}{|Q|}\sum_{q\in Q}\xi(q)\frac{1}{|T'|}\sum_{t'\in T'\cap T_0}\frac{1}{|T_0|}\sum_{t\in T}\rho_0(t'^t) \\
&= \frac{1}{|Q|}\sum_{q\in Q}\xi(q)\frac{1}{|T'|}\sum_{t'\in T'} ind_{T_0}(\rho_0)(t') \\
&= \frac{1}{|Q|}\sum_{q\in Q}\xi(q)\frac{1}{|T'|}\sum_{t'\in T'}\rho(t') \\
&= \frac{1}{|Q|}\sum_{q\in Q}\xi(q)\ <\rho_{T'}, 1_{T'}>_{T'}.
\end{aligned}$$

Consequently, by Lemma 1 we have

$$\begin{aligned}
\tilde{u}_G(\chi)(QT') &= (-1)^{<\chi_{QT'},\ 1_{QT'}>_{QT'}} \\
&= u_T(\rho)(T')^{\frac{1}{|Q|}\sum_{q\in Q}\xi(q)}.
\end{aligned}$$

So by the congruences (6), χ indeed satisfies the equality (4). Hence the lemma is proved, and we have completed the proof of Theorem 1.

Since $\tilde{u}_T$ is a morphism of G-functors, Theorem 2 follows directly form Theorem 1.

REFERENCES

[1] A. Dress: A characterization of solvable groups, Math. Z., 110 (1969), 213-217.

[2] T. Matsuda: On the unit groups of Burnside rings, Japanese J. Math., 8 (1982), 71-93.

[3] T. Matsuda and T. Miyata: On the unit groups of Burnside rings of finite groups,

J. Math. Soc. Japan, 35 (1983), 345-354.

[4] T. tom. Dieck: Transformation groups and representation theory: 766 (1979), Lecture Notes in Math., Springer.

[5] J. Tornehave: The unit theorem for the Burnside ring of a 2-group, Aarhus Universitet Preprint Series, 41 (1983/84).

[6] T. Yoshida: On the unit groups of Burnside rings, (preprint).