Further examples of cocleft module coalgebras

Akira Nishikawa

§1. Cocleft module coalgebras

For a bialgebra, the concept of cocleft module coalgebras, that is, the dual one of cleft comodule algebras in [1], was introduced in [2], and for a subbialgebra A of a bialgebra C over a commutative ring, we gave two conditions for the A-module coalgera C to be cocleft ([2], Proposition [6]). Furthermore, we gave two examples of cocleft module coalgebras ([2, §4]).

Before presenting the proposition and the corollary which contain the contents of [2, Proposition 6], we give the definition of cocleft module coalgebras for a bialgebra:

Let A be a bialgebra over a commutative ring R, C a right A-module coalgebra. An R-linear mapping ϕ: $C \longrightarrow A$ is called to be a cointegral when ϕ is an A-module morphism, *i.e.*, $m_A(\phi \otimes I) = \phi \omega_C$, where m_A: $A \otimes A \longrightarrow A$ is the multiplication mapping of A, and ω_C: $C \otimes A \longrightarrow C$ is the A-module structure mapping of C. We call a right A-module coalgebra C to be cocleft when there exists a $*$-invertible cointegral $C \longrightarrow A$.

Proposition. *Let A be an algebra, C a coalgebra over a commutative ring R. An R-module morphism ϕ: $C \longrightarrow A$ is $*$-invertible in the situation* (*i*) *or* (*ii*):

(*i*) *C is a Hopf algebra, and ϕ is an algebra morphism.*

(*ii*) *A is a Hopf algebra, and ϕ is a coalgebra morphism.*

Proof. In the situation (i), by [3, Lemma 4.0.3(i)] (even if R is not necessarily a field), ϕS_C is the $*$-inverse of ϕ, where S_C is the antipode of C.

In the situation (ii), by [3, Lemma 4.0.3(ii)], $S_A \phi$ is the $*$-inverse of ϕ, where S_A is the antipode of A.

So in both cases, ϕ is $*$-invertible.

Now we have a result which covers the contents of [2, Proposition 6]:

Corollary. *Let A be a bialgebra, C an A-module coalgebra. If there exists a cointegral ϕ: $C \longrightarrow A$ with the property* (*i*) *or* (*ii*) *in the proposition, then C is a cocleft A-module coalgebra.*

2. Examples of cocleft module coalgebras

Example 1. *Suppose R is a commutative ring of prime characteristic p. We put $C = R[X,]$, $A = R[X^p]$, and define Δ_C: $C \longrightarrow C \otimes C$, ε_C: $C \longrightarrow R$ and S_C: $C \longrightarrow C$ as $\Delta_C(X) = X \otimes 1 + 1 \otimes X$, $\varepsilon_C(X) = 0$ and $S_C(X) = -X$. Then C is a polynomial Hopf algebra* *and A is a Hopf subalgebra of C, so C is naturally an A-module coalgebra.*

We define an R-module morphism ϕ: $C \longrightarrow A$ by $\phi(X^n)=X^n$ if $p|n$ and $\phi(X^n)=0$, otherwise. Then, as easily verified, the following diagram is commutative:

$$\begin{array}{ccc} C\otimes A & \xrightarrow{\phi\otimes I} & A\otimes A \\ {\scriptstyle \rho_c}\downarrow & & \downarrow{\scriptstyle m_A} \\ C & \xrightarrow{\phi} & A \end{array}$$

Where ρ_C is the natural A-module structure mapping of C. So, ϕ is a cointegral.

Next, as R is of characteristic p,

$(\phi\otimes\phi)\Delta_C(X^{mp})=(\phi\otimes\phi)\ (\Delta_C(X^p))^m=(X^p\otimes 1+1\otimes X^p)^m=(\Delta_A(X^p))^m=\Delta_A(\phi(X^p))^m$

$=\Delta_A(\phi(X^{pm}))$, $m\geqq 0$.

If $p|n$, then

$(\phi\otimes\phi)\Delta_C(X^n)=(\phi\otimes\phi)((X\otimes 1+1\otimes X)^n)=(\phi\otimes\phi)\ (\sum_{i=0}^{n}\binom{n}{i}\ X^i\otimes X^{n-i})$, both i and $n-i$

cannot be devided by p in the case $p\nmid n$, so either $\phi(X^i)$ or $\phi(X^{n-i})$ is 0, and so, $(\phi\otimes\phi)\Delta_C(X^n)=0=\Delta_A\phi(X^n)$.

On the other hand, $\phi\varepsilon_C(X^n)=0=\varepsilon_A(X^n)$, $n\geqq 1$, and $\phi\varepsilon_C(1)=1=\varepsilon_A(1)$.

Now we have: $(\phi\otimes\phi)\Delta_C=\Delta_A\phi$, $\phi\varepsilon_C=\varepsilon_A$. Hence ϕ is a coalgebra morphism, and we come to the situation (ii) of the proposition, and so, C is a cocleft A-module coalgebra by the corollary.

Example 2. *Let G be a finite group, H a subgroup of G. We set a coset decomposition $G=\bigcup_{i=1}^{n} g_iH$, $g_1=1$.*

The dual Hopf algebra $A=(RG)^*$ consists of the elements $\sum_{i=1}^{n}\sum_{h\in H} \lambda_{i,h}(g_ih)^*$ ($\lambda_{i,h}\in R$, $h\in H$), where the set $\{(g_ih)^* \mid 1\leqq i\leqq n,\ h\in H\}$ is the dual basis.

Algebra structure mappings m_A: $A\otimes A\longrightarrow A$ and u_A: $R\longrightarrow A$ are given by $m_A(g^*\otimes\tilde{g}^*)=\delta_{g,\tilde{g}}g^*$ and $u_A(1)=\sum_{g\in G}g^*$, and coalgebra structure mappings Δ_A: $A\longrightarrow A\otimes A$ and ε_A: $A\longrightarrow R$ are given by $\Delta_A(g^*)=\sum_{\tilde{g}\in G}\tilde{g}^*\otimes(\tilde{g}^{-1}g)^*$ and $\varepsilon_A(g^*)=\delta_{1,g}$, where δ is the Kronecker's delta. The antipode S_A of A is given by $S_A(g^*)=(g^{-1})^*$.

A Hopf algebra $C=(RH)^*$ is a right A-module through the natural resriction mapping $A\longrightarrow C$ defined by $(g_ih)^*\longrightarrow\delta_{1,i}h^*$.

Moreover, the module structure mapping $C\otimes A\longrightarrow C$ is a coalgebra morphism, i.e., the following diagram is commutative:

$$\begin{array}{ccc} C\otimes A & \longrightarrow & C \\ \Delta_{C\otimes A}\downarrow & & \Delta_C\downarrow \\ C\otimes A\otimes C\otimes A & \longrightarrow & C\otimes C, \end{array}$$

indeed, for any basis element $h^*\otimes(g_i\tilde{h})^*$ *in* $C\otimes A$,

$\Delta_C(h^*(g_i\tilde{h})^*)=\delta_{1,i}\delta_{h,\tilde{h}}\Delta_C(h^*)=\delta_{1,i}\delta_{h,\tilde{h}}\sum_{h_1\in H}h_1{}^*\otimes(h_1{}^{-1}h)^*$, *and* $\Delta_{C\otimes A}(h^*\otimes(g_i\tilde{h})^*)=\sum_{h_1\in H}\sum_{h_2\in H}$ $\sum_{j=1}^{n} h_1{}^*\otimes(g_jh_2)^*\otimes(h_1{}^{-1}h)^*\otimes(h_2{}^{-1}g_j{}^{-1}g_ih)^*\longrightarrow\sum_{h_1\in H}\sum_{h_2\in H}\sum_{j=1}^{n}\delta_{1,j}\ \delta_{h_1,h_2}\ \delta_{i,j}\ \delta_{h,\tilde{h}}h_1{}^*\otimes(h_1{}^{-1}h)^*$ $=\delta_{1,i}\ \delta_{h,\tilde{h}}\sum_{h_1\in H}h_1{}^*\otimes(h_1{}^{-1}h)^*$. *Therefore* C *is a right* A*-module coalgebra.*

Let ϕ: $C\longrightarrow A$ be a morphism of R-modules with $\phi(h^*)=h^*$ $(h\in H)$, where h^* in C satisfies $\langle h^*,\tilde{h}\rangle=\delta_{h,\tilde{h}}$, and h^* in A satisfies $\langle h^*,g_i\tilde{h}\rangle=\delta_{1,i}\delta_{h,\tilde{h}}$. Then, for $h,\tilde{h}\in H$ and $1\leqq i\leqq n$,

$m_A(\phi\otimes I)\ (h^*\otimes(g_ih)^*)=h^*(g_ih)^*=\delta_{1,i}\ \delta_{h,\tilde{h}}h^*$
$=\phi(h^*(g_i\tilde{h})^*)=\phi(\rho_C(h^*\otimes(g_ih)^*))$,

so ϕ is a cointegral.

Furthermore, for any basis element $h^*\otimes\tilde{h}^*$ in $C\otimes C$,

$\phi m_C(h^*\otimes\tilde{h}^*)=\phi(\delta_{h,\tilde{h}}h^*)=\delta_{h,\tilde{h}}h^*$
$=m_A(h^*\otimes\tilde{h}^*)=m_A(\phi\otimes\phi)\ (h^*\otimes\tilde{h}^*)$,

and $u_A\phi(h^*)=u_A(h^*)=\delta_{1,h}=u_C(h^*)$,

so $\phi m_C=m_A\ (\phi\otimes\phi)$ and $u_A\phi=u_C$, *i.e.*, ϕ is an algebra morphism.

Hence, we come to the situation (i) of the proposition, and so, by the corollary, C is a cocleft A-module coalgebra.

Example 3. *Suppose* R *is a commutative ring of characteristic* p, *and* $G=\langle g\rangle$ *is a cyclic group of order* p.

Let $A=RG$ *be a group Hopf algebra, and* $C=\sum_{i,j=0}^{p-1}\oplus Re_{ij}$ *a coalgebra with* $\Delta_C(e_{ij})=\sum_{k=0}^{p-1}e_{ik}\otimes e_{kj}$, *and* $\varepsilon_C(e_{ij})=\delta_{ij}$.

We define the action of A *on* C *by* $e_{ij}g^n=e_{i+n\ j+n}$ (*where each suffix number is read as modulus* p).

Then, $\Delta_C(e_{ij})\ (g^n\otimes g^n)=\sum_{k=0}^{p-1}e_{ik}g^n\otimes e_{kj}g^n=\sum_{k=0}^{p-1}e_{k+n\ i+n}\otimes e_{k+n\ j+n}$
$=\Delta_C(e_{i+n\ j+n})=\Delta_C(e_{ij}g^n)$,
and $\varepsilon_C(e_{i+n\ j+n})=\varepsilon_C(e_{ij}g^n)$,

so C *is a right* A*-module coalgebra.*

Let ϕ: $C \longrightarrow A$ be an R-module morphism with $\phi(e_{ij})=\delta_{ij}g^i$.
Then, $\phi(e_{ij}g^n)=\phi(e_{i+n\ j+n})=\delta_{ij}g^{i+n}$
$=\delta_{ij}g^i g^n=\phi(e_{ij})g^n$,
so ϕ is a cointegral.
Furthermore, we see,

$$\Delta_A\phi(e_{ij})=\delta_{ij}\Delta_A(g^i)=\delta_{ij}g^i\otimes g^i$$
$$=\sum_{k=0}^{p-1}\phi(e_{ik}))\otimes\phi(e_{kj})=(\phi\otimes\phi)\Delta_C(e_{ij}),$$

and $\varepsilon_A\phi(e_{ij})=\delta_{ij}\varepsilon_A(g^i)=\delta_{ij}=\varepsilon_C(e_{ij})$, $0\leqq i, j\leqq p-1$.

Hence $\Delta_A\phi=(\phi\otimes\phi)\Delta_C$ and $\varepsilon_A\phi=\varepsilon_C$, *i.e.*, ϕ is a coalgebra morphism, and we come to the case (ii) of the proposition, so by the corollary, C is a cocleft A-module coalgebra.

REFERENCES

[1] Y. Doi and M. Takeuchi: Cleft comodule algebras for a bialgebra, Com. in Algebra. (1986), 801–807.

[2] A. Nishikawa: Cocleft module coalgebras for a bialgebra, J. of the Fac. of Liberal Arts, SHINSHU UNIVERSITY 21 (1987).

[3] M. E. Sweedler: Hopf Algebras, Benjamin (1969).

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