COCLEFT MODULE COALGEBRAS FOR A BIALGEBRA

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In memory of Professor Akira HATTORI

§ 1. Introduction

For a Hopf algebra over a field, Sweedler's theorem [5, Theorem 4.1.1] on the structure of Hopf modules implies that the category of Hopf modules is equivalent to that of vector spaces. This theorem was generalized successively in [1, Theorem 3], [2, Theorem 3.3], [3, Theorem 2.8] and [4, Theorem 9] by Y. Doi and M. Takeuchi. In the process of generalizations, the structure theory of module coalgebras for a bialgebra over a commutative ring has been developed. Furthemore, in [4, Theorem 9], it was proved cleftness of a comodule algebra is a sufficient condition of the equivalence of the category of Hopf modules with that of right modules over the invariant subcomodule algebra.

In this paper we define the dual concepts of "(generalized) integral", "cleft" and "anti-cleft", and prove the following dual theorem of [4, Theorem 9] in the case that the A-module coalgebra C is flat as an R-module:

Main Theorem. *Let A be a bialgebra over a commutative ring R, C a right A-module coalgebra. When C is flat as an R-module, the following are equivalent:*

(i) *C is cocleft.*

(ii) *There exists a left $\bar{C}$-comodule and right A-module isomorphism $\bar{C}\otimes A \longrightarrow C$, and Ψ_N: $N \longrightarrow \bar{N}\square_{\bar{C}}C$ is bijective for every N in the category of $[C, A]$-Hopf modules.*

(iii) *There exists a left $\bar{C}$-comodule and right A-module isomorphism $\bar{C}\otimes A \longrightarrow C$, and $\Psi_{C\otimes A}: C\otimes A \longrightarrow C\square_{\bar{C}}C$ is bijective.*

In this case, the category of $[C, A]$-Hopf modules is equivalent to that of right $\bar{C}$-comodules.

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§ 2. Auxiliary results

Let R be a commutative ring with identity. All our algebras, coalgebras and modules are over R, and $\otimes$, Hom are also over R. Throughout this paper A is a bialgebra.

For a right A-module coalgebra C, *i.e.*, coalgebra which is a right A-module with the module structure map ω_C: $C\otimes A\longrightarrow C$ being a coalgebra morphism. We call an R-module map $C\longrightarrow A$ a "cointegral" if it is an A-module morphism. We say a coalgebra C to be "cocleft" (resp. "anti-cocleft") when there exists a $*$-invertible (resp. $\times$-invertible) cointegral, where $*$ and $\times$ are convolution product and twist convolution product (see [4], for example).

As the dual of [4, Lemma 4], we have:

Lemma 1. *Let C be a right A-module coalgebra. The following are equivalent for an R-module morphism ϕ: $C\longrightarrow A$.*

(i) *ϕ is a cointegral.*

(ii) $\phi\omega_C = (\phi p_1)*p_2$

(iii) $\phi\omega_C = (\phi p_1)\times p_2$,

Where p_1: $C\otimes A\longrightarrow C$, $c\otimes a\longmapsto c\varepsilon(a)$, p_2: $C\otimes A\longrightarrow A$, $c\otimes a\longrightarrow \varepsilon(c)a$.

Proof. $(\phi p_1)*p_2 = m_A(\phi\otimes I_A)\ (p_1\otimes p_2)\Delta_{C\otimes A} = m_A(\phi\otimes I_A)$, so (i) and (ii) are equivalent.

$(\phi p_1)\times p_2 = m_A(\phi\otimes I_A)\ (p_1\otimes p_2)T\Delta_{C\otimes A} = m_A(\phi\otimes I_A)$, so (i) and (iii) are equivalent.

As the dual of [4, Proposition 5], we have:

Proposition 2. *Let C be a right A-module coalgebra. Assume that there exists a cointegral ϕ: $C\longrightarrow A$ which is $*$-invertible (resp. $\times$-invertible) in Hom (C, A). Then:*

(1) *The map p_2: $C\otimes A\longrightarrow A$ is $*$-invertible (resp. $\times$-invertible).*

(2) $\phi^{-1}\omega_C = p_2*(\phi^{-1}p_1)$ (*resp.* $\phi^{-}\omega_C = p_2^{-}\times(\phi^{-}p_1)$).

If, in addition, there exists a coalgebra morphism $R\longrightarrow C$, then;

(3) *The identity map I_A is $*$-invertible (resp. $\times$-invertible), i.e., A is a Hopf algebra (resp. an anti-Hopf algebra).*

(4) $p_2^{-1} = I_A^{-1}p_2$ (*resp.* $p_2 = I_A^{-}p_2$).

(5) $\phi^{-1}\omega_C = m_AT(\phi^{-1}\otimes I_A^{-1})$ (*rep.* $\phi^{-}\omega_C = m_AT(\phi^{-}\otimes I_A^{-})$).

Proof. Since ω_C and p_1 are coalgebra morphisms, $\phi\omega_C$ and p_1 are $*$-invertible (resp. $\times$-invertible). As ϕ is $*$-invertible (resp. $\times$-invertible), (1) and (2) follow from the preceding lemma.

(3) Suppose β: $R\longrightarrow C$ is a coalgebra morphism. Then, $A\cong P\otimes A\xrightarrow{\beta\otimes I}C\otimes A$ is also a coalgebra morphism, and $p_2(\beta\otimes I)\ (a)=p_2(1\otimes a)=I(a)\ (a\in A)$, *i.e.*, $p_2(\beta\otimes I)=I_A$, so by [4, Lemma 1], $p_2^{-1}(\beta\otimes I)=(p_2(\beta\otimes I))^{-1}=I_A^{-1}$ (resp. $p_2^{-}(\beta\otimes I)=(p_2(\beta\otimes I))^{-}=I_A^{-}$).

(4) As p_2 is a coalgebra morphism, so by [4, Lemma 1], $I_A^{-1}p_2=(I_Ap_2)^{-1}=p_2^{-1}$ (resp. $I_A^{-}p_2=(I_Ap_2)^{-}=p_2^{-}$).

(5) By (2), (4) and [4, Lemma 2],

$\phi^{-1}\omega_C = p_2^{-1}*(\phi^{-1}p_1) = m_A T(\phi^{-1}\otimes I_A^{-1})$
(resp. $\phi^{-}\omega_C = p_2^{-}\times(\phi^{-}p_1) = m_A T(\phi^{-}\otimes I_A^{-})$).

As the dual of [4, Corollary 6], we have:

Corollary 3. *Any bialgebra A is cocleft (resp. anticocleft) as an A-module coalgebra if and only if there exists I_A^{-1} (resp. I_A^{-}) i.e., A is a Hopf algebra (resp. an anti-Hopf algebra).*

As the dual of Proposition 8 of [4], we have:

Proposition 4. *Let A be a bialgebra, C a right A-module coalgebra. Then:*

(1) *If A is an anti-Hopf algebra and C is cocleft, then C is anti-cocleft.*

(2) *If A is a Hopf algebra and C is anti-cocleft, then C is cocleft.*

(3) *When A is a Hopf algebra with the bijective antipode, then C is cocleft if and only if it is anti-cocleft.*

Proof. (1) Let ϕ: $C \longrightarrow A$ be a $*$-invertible cointegral.

By Proposition 2 (1), p_2:$C\otimes A \longrightarrow A$ is $*$-invertible. We write $\bar{S}$ as the pode I_A^{-} of A, then $\bar{S}$ is an algebra anti-morphism and p_2 is an algebra morphism, so

$$\bar{S}p_2^{-1} = (\bar{S}p_2)^{-} = (\bar{S})^{-}p_2 = p_2.$$

With this formula, [4, Lemma 1 (3)] and Proposition 2 (2),

$$(\bar{S}\phi)^{-}\omega_C = \bar{S}\phi^{-1}\omega_C = (\bar{S}\phi^{-1}p_1) \times (\bar{S}p_2^{-1}) = (\bar{S}\phi)^{-}\times p_2.$$

Then by Lemma 1, $(\bar{S}\phi)^{-}$ is a cointegral, and is $\times$-invertible because $(\bar{S}\phi)^{--} = \bar{S}\phi$. Hence C is anti-cocleft.

(2) Let ϕ': $C \longrightarrow A$ be a $\times$-invertible cointegral. By Proposition 2 (1), p_2: $C\otimes A \longrightarrow A$ is $\times$-invertible. If we write S as the antipode I_A^{-1}, then we have $Sp_2^{-} = (Sp_2)^{-1} = S^{-1}p_2 = p_2$. With this formula, [4, Lemma 1 (3)] and Proposition 2 (2),

$$(S\phi')^{-1}\omega_C = S\phi'^{-}\omega_C = S(p_2^{-}\times(\phi'^{-}p_1))$$
$$= (S\phi'^{-}p_1)*(Sp_2^{-}) = ((S\phi')^{-1}p_1)*p_2.$$

Then, by Lemma 1, $S\phi'^{-}$ is a cointegral and $*$-invertible because $((S\phi')^{-1})^{-1} = S\phi'$. Hence C is cocleft.

(3) is clear by (1) and (2).

Now we go on to the dualization of [4, Theorem 9] which is a generalization of [3, Theorem 2.8].

Let N be a right A-module and a right C-comodule with the structure mappings ω_N: $N\otimes A \longrightarrow N$, and ρ_N: $N \longrightarrow N\otimes C$.

N is a right $[C, A]$-Hopf module when the diagram

$$\begin{array}{ccc} N\otimes A & \xrightarrow{\omega_N} N \xrightarrow{\rho_N} & N\otimes C \\ {\scriptstyle \rho_N\otimes\Delta_A}\downarrow & & \downarrow{\scriptstyle \omega_N\otimes\omega_C} \\ N\otimes C\otimes A\otimes A & \xrightarrow{I\otimes T\otimes I} & N\otimes A\otimes C\otimes A \end{array}$$

is commutative.

If we put $A^+=\mathrm{Ker}\ \varepsilon_A$, $\bar{C}=C/CA^+$ and $\bar{N}=N/NA^+$, then $\bar{C}$ is a coalgebra and $\bar{N}$ is a right C-comodule. We write the natural mappings $C\longrightarrow\bar{C}$ and $N\longrightarrow\bar{N}$ as $p=p_C$ and $\pi=\pi_N$.

Let W be a right C-comodule. When C is flat as an R-module, an R-submodule $W\square_{\bar{C}}C$ of $W\otimes C$ is a [C, A]-Hopf submodule ([5; 1.3 Proposition]). Hence we assume that C is a flat R-module.

If $\mathbf{M}_A^C$ is the category of right [C, A]-Hopf modules and $\mathbf{M}^{\bar{C}}$ is that of right $\bar{C}$-comodules, then $N\longmapsto\bar{N}$ is a covariant functor

$\Phi:\mathbf{M}_A^C\longrightarrow\mathbf{M}^{\bar{C}}$, which is the left adjoint of the functor

$\Psi:\mathbf{M}^{\bar{C}}\longrightarrow\mathbf{M}_A^C$ defined by $W\longmapsto W\square_{\bar{C}}C$.

Correspondences of morphisms are following:

(1) $\mathbf{M}_A^C(N, W\square_{\bar{C}}C)\longrightarrow\mathbf{M}^{\bar{C}}(\bar{N}, W)$, $f\longmapsto[\pi(n)\longmapsto(I\otimes\varepsilon_C)f(n)]$.

The well-definedness of this mapping is verified with the commutative diagram

$$\begin{array}{ccc} C\otimes A & \xrightarrow{\omega_C} & C \\ {\scriptstyle \varepsilon_{C\otimes A}}\downarrow & & \downarrow{\scriptstyle \varepsilon_C} \\ R\otimes R & \xrightarrow{\sim} & R. \end{array}$$

(2) $\mathbf{M}^{\bar{C}}(\bar{N}, W)\longrightarrow\mathbf{M}_A^C(N, W\square_{\bar{C}}C)$, $g\longmapsto(g\otimes I)(\pi\otimes I)\rho_N$.

This is well-defined since

$$N\xrightarrow{\rho_N}N\otimes C\xrightarrow{\pi\otimes I}\bar{N}\otimes C\xrightarrow{g\otimes I}W\otimes C\ \underset{I_W\otimes\rho_C}{\overset{\rho_W\otimes I_C}{\rightrightarrows}}\ W\otimes\bar{C}\otimes C$$

is a null sequence.

The adjunctions are as following:

$$\Psi_N: N\longrightarrow\bar{N}\square_{\bar{C}}C,\ n\longmapsto\sum\pi(n_{(0)})\otimes n_{(1)},$$

$$\Phi_W:\overline{W\square_{\bar{C}}C}\longrightarrow W,\ \pi_{W\square_{\bar{C}}C}(\sum_i w_i\otimes c_i)\longmapsto\sum_i w_i\varepsilon(c_i).$$

Consider $C\otimes A$ as a right [C, A]-Hopf module via $(c\otimes a)a'=c\otimes aa'$ and $\rho_{C\otimes A}(c\otimes a)=\sum c_{(1)}\otimes a_{(1)}\otimes c_{(2)}a_{(2)}$.

Then $\overline{C\otimes A}=(C\otimes A)/(C\otimes A)A^{+}=(C\otimes A)/(C\otimes A^{+})=C\otimes R\cong C$, so $\Psi_{C\otimes A}\colon C\otimes A\longrightarrow \overline{C\otimes A}\square_{\overline{C}}C=C\square_{\overline{C}}C, c\otimes a \longmapsto \sum c_{(1)}\otimes c_{(2)}a$.

Proposition 5. *Let A be a bialgebra, C a right A-module coalgebra. If C is cocleft, then for any* $[C, A]$-Hopf module N *there exists an isomorphism* $N\longrightarrow\overline{N}\otimes A$ *of right A-modules.*

Proof. Let $\phi\colon C\longrightarrow A$ be a $*$-invertible right cointegral. We define a mapping $Q=Q_N$ as:

$$N \xrightarrow{\rho_N} N\otimes C \xrightarrow{I\otimes\phi^{-1}} N\otimes A \xrightarrow{\omega_N} N,$$

that is, $Q(na)=\omega_N(I\otimes\phi^{-1})\rho_N\omega_N(n\otimes a)$

$$=\omega_N(\omega_N\otimes(p_2{}^{-1}*(\phi^{-1}p_1)))\left(\sum n_{(0)}\otimes a_{(1)}\otimes n_{(1)}\otimes a_{(2)}\right)$$

$$=\sum n_{(0)}\varepsilon(n_{(1)})a_{(1)}p_2{}^{-1}(n_{(2)}\otimes a_{(2)})\,\phi^{-1}p_1(n_{(3)}\otimes a_{(3)})$$

$$=\sum n_{(0)}\,\phi^{-1}(n_{(1)})\varepsilon_A(a)=Q(n\varepsilon_A(a)),$$

hence $Q(NA^{+})=0$.

So there exists the induced map $\overline{Q}:\overline{N}\longrightarrow N$ such that the diagram

$$\begin{array}{ccc} N & \xrightarrow{Q} & N \\ & {\scriptstyle\pi}\searrow \quad \nearrow{\scriptstyle\overline{Q}} & \\ & \overline{N} & \end{array}$$

is commutative.

Now we define two mappings:

$$f=f_{\overline{N}\otimes A}:\overline{N}\otimes A \xrightarrow{\overline{Q}\otimes I} N\otimes A \xrightarrow{\omega_N} N,$$

$$g=g_N\colon N \xrightarrow{\rho_N} N\otimes C \xrightarrow{\pi\otimes\phi} \overline{N}\otimes A.$$

Then, for $n\in N$ and $a\in A$,

$$gf(\pi(n)\otimes a)=\sum\pi(n_{(0)}\,\phi^{-1}(n_{(2)})_{(1)}a_{(1)})\otimes\phi(n_{(1)}\,\phi^{-1}(n_{(2)})_{(2)}a_{(2)})$$

$$=\sum\pi(n_{(0)})\otimes\phi(n_{(1)})\,\phi^{-1}(n_{(2)})a=\sum\pi(n_{(0)})\otimes\varepsilon_A(n_{(1)})a$$

$$=\pi(n)\otimes a,$$

and $fg(n)=\sum n_{(0)}\,\phi^{-1}(n_{(1)})\phi(n_{(2)})=\sum n_{(0)}\varepsilon(n_{(1)})=n$.

So we have $gf=I_{\overline{N}\otimes A}$ and $fg=I_N$, that is, $N\cong\overline{N}\otimes A$.

Furthermore, g is an isomorphism of A-modules, since ϕ is a cointegral.

§ 3. Proof of the main theorem

Proof of Main Theorem. (i)$\Rightarrow$(ii). Let N be a right $[C, A]$-Hopf module. By Proposition 5, g_N: $N \xrightarrow{\sim} \bar{N} \otimes A$ as right A-modules, especially, g_C: $C \xrightarrow{\sim} \bar{C} \otimes A$ as right A-modules, and in this case g_C is also an isomorphism of left $\bar{C}$-comodules, since

$$\rho_{\bar{C}\otimes A} g_C(c) = (\Delta_{\bar{C}} \otimes I)(\sum p(c_{(1)}) \otimes \phi(c_{(2)})) = (I \otimes g_C)(\sum p(c_{(1)}) \otimes c_{(2)}) = (I \otimes g_C)(\rho_C(c)),\ c \in C.$$

Now, as easily verified,

$$\operatorname{Im}(\rho_{\bar{N}} \otimes I : \bar{N} \otimes A \longrightarrow \bar{N} \otimes \bar{C} \otimes A) \subset \bar{N} \square_{\bar{C}} (\bar{C} \otimes A).$$

On the other hand, $(I_{\bar{N}} \otimes \varepsilon_{\bar{N}} \otimes I_A)(\rho_{\bar{N}} \otimes I_A) = I_{\bar{N}\otimes A}$ is clear, and for any $\sum_i \pi(n_i) \otimes p_C(c_i) \otimes a_i$ in $\bar{N} \square_{\bar{C}} (\bar{C} \otimes A)$, using the equality

$\sum_i \sum \pi(n_{i(0)}) \otimes p(n_{i(1)}) \otimes p(c_i) a_i = \sum_i \sum \pi(n_i) \otimes p(c_{i(1)}) \otimes p(c_{i(2)}) \otimes a_i$, we have $(\rho_{\bar{N}} \otimes I)$

$(I \otimes \varepsilon_{\bar{C}} \otimes I)(\sum_i \pi(n_i) \otimes p(c_i) \otimes a_i = \sum_i \pi(n_i) \otimes p(c_i) \otimes a_i$, so $\rho_{\bar{N}} \otimes I_A)(I_{\bar{N}} \otimes \varepsilon_{\bar{C}} \otimes I_A) = I_{\bar{N}\square_{\bar{C}}(\bar{C}\otimes A)}$, hence $\bar{N} \otimes A \cong \bar{N} \square_{\bar{C}} (\bar{C} \otimes A)$.

With this isomorphism and Proposition 5, we see

$$N \cong \bar{N} \otimes A \cong \bar{N} \square_{\bar{C}} (\bar{C} \otimes A) \cong \bar{N} \square_{\bar{C}} C.$$

This composite isomorphism maps n in N to :

$$n \longmapsto \sum \pi(n_{(0)}) \otimes \phi(n_{(1)}) \longmapsto \sum \pi(n_{(0)}) \otimes p(n_{(1)}) \otimes \phi(n_{(2)})$$
$$\longmapsto \sum \pi(n_{(0)}) \otimes n_{(1)} \phi^{-1}(n_{(2)}) \phi(n_{(3)}) = \sum \pi(n_{(0)}) \otimes n_{(1)},$$

so it coincides with the adjunction Ψ_N.

(ii) $\Rightarrow$ (iii) is clear since

$$\Psi_{C\otimes A} :\ C \otimes A \longrightarrow \overline{C \otimes A} \square_{\bar{C}} C \cong C \square_{\bar{C}} C.$$

(iii)$\Rightarrow$(i). Let α: $\operatorname{Hom}(C, A) \longrightarrow \operatorname{Com}_{\bar{C}\text{-}}(C, C)$ be $f \longmapsto (I \otimes \varepsilon) \Psi_{C\otimes A}(I \otimes f)\Delta_C$, that is, $\alpha(f)(c) = \sum c_{(1)} f(c_{(2)})$.

$\alpha(f)$ is a left $\bar{C}$-comodule endomorphism since

$\rho_C \alpha(f)(c) = \sum p(c_{(1)}) \otimes c_{(2)} f(c_{(3)}) = (I_{\bar{C}} \otimes \alpha(f)) \rho_{\bar{C}}(c)$, $c \in C$.

By the construction, the following diagram is commutative:

$$\begin{array}{ccc}
\operatorname{Com}\text{-}_C(C, C \otimes A) & \xrightarrow[\sim]{\operatorname{Hom}(I_C, \Psi_{C\otimes A})} & \operatorname{Com}\text{-}_C(C, C \square_{\bar{C}} C) \\
\wr\| & & \wr\| \\
\operatorname{Hom}(C, A) & \xrightarrow{\quad \alpha \quad} & \operatorname{End}_{\bar{C}\text{-}}(C)
\end{array}$$

So α is an isomorphism.

Let F: $C \longrightarrow \bar{C}\otimes A$ be a left $\bar{C}$-comodule and right A-module isomorphism. Define λ: $C \longrightarrow A$ by $\lambda = p_2 F$. Then, easily, $m_A(\lambda\otimes I)=\lambda\omega_C$; *i.e.*, λ is a cointegral.

Furthermore, since F is a morphism of left $\bar{C}$-comodules,

$$(I\otimes\lambda)\rho_C=(p_C\otimes\lambda)\Delta_C=(p_C\otimes(p_2F))\Delta_C=(I\otimes p_2)\ (I\otimes F)\rho_C=F.$$

Let G : $\bar{C}\otimes A \longrightarrow C$ be the composite inverse of F. Define h: $\bar{C} \longrightarrow C$ by $h=Gi_1$, where i_1 : $\bar{C} \longrightarrow \bar{C}\otimes A$ is $p_C(c) \longmapsto p_C(c)\otimes 1$.

Since G is a left $\bar{C}$-comodule morphism, so $hp \in \mathrm{End}_{\bar{C}}\text{-}(C)$.

On the other hand, since G is a right A-module isomorphism, we have, for any $p_C(c)$ in $\bar{C}$ and a in A,

$$\omega_C(h\otimes I)\ (p_C(c)\otimes a) = h(p_C(c))a = G(p_C(c)\otimes a),$$

hence $\omega_C(h\otimes I)=G$.

Now since α is bijective, there exists $\phi\in\mathrm{Hom}(C, A)$ with $\alpha(\phi)=hp$. Then $\lambda*\phi= m_A(\lambda\otimes I)\ (I\otimes\phi)\Delta_C = \lambda\omega_C(I\otimes\phi)\Delta_C = \lambda hp_C = p_2 i_1 p_C = u_A\varepsilon_C$.

Moreover,
$$\begin{aligned}\alpha(\phi*\lambda) &= \omega_C(I\otimes(m_A(\phi\otimes\lambda)\Delta_C))\Delta_C\\ &= \omega_C((\omega_C(I\otimes\phi)\Delta_C)\otimes\lambda)\Delta_C\\ &= \omega_C(G\otimes I)\ (i_1\otimes\varepsilon_{\bar{C}}\otimes I)\ (I\otimes F)\ (p_C\otimes I)\Delta_C\\ &= G(I\otimes m_A)\ (i_1\otimes I)\ F = I_C\\ &= \omega_C(I\otimes u_A)\ (I\otimes\varepsilon_C)\Delta_C = \alpha(u_A\varepsilon_C),\end{aligned}$$

which shows $\phi*\lambda = u_A\varepsilon_C$, since α is bijective, and so, (iii)$\Rightarrow$(i) is proved.

To prove the last assertion, it is sufficient to show that the adjunction Φ_W: $\overline{W\square_{\bar{C}}C} \longrightarrow W$ is bijective for any right $\bar{C}$-comodule W. By using [6; 1.3 Proposition],

$$\overline{W\square_{\bar{C}}C}=(W\square_{\bar{C}}C)/(W\square_{\bar{C}}(CA^+))\cong W\square_{\bar{C}}\bar{C}\cong W,$$

$$\sum_i w_i\otimes c_i \longmapsto \sum_i w_i\varepsilon_{\bar{C}}(p_{\bar{C}}(c_i)) = \sum_i w_i\varepsilon_C(c_i).$$

So this composite bijection coincides with Φ_W.

This completes the proof.

§ 4. Examples of cocleft module coalgebras

Let C be a bialgebra, A its subbialgebra. Then C is naturally an A-module coalgebra. We assume that there exists an R-module morphism ϕ:$C \longrightarrow A$ which is a cointegral.

Furthermore, we set the following two cases:

(i) C is a Hopf algebra, and ϕ is an algebra morphism.

(ii) A is a Hopf algebra, and ϕ is a coalgebra morphism.

In the case (i), using $m_A(\phi\otimes\phi)=\phi m_C$ and $\phi u_C=u_A$, we can see that ϕS_C is the $*$-inverse of ϕ. In the case (ii), using $(\phi\otimes\phi)\Delta_C=\Delta_A\phi$ and $\varepsilon_A\phi=\varepsilon_C$, we can see that $S_A\phi$ is the $*$-inverse of ϕ. Thus we have:

Proposition 6. *Let C be a bialgebra, A a subbialgebra. If there exists a cointegral ϕ: $C\longrightarrow A$ with the property of* (i) *or* (ii), *then C is a cocleft A-module coalgebra.*

Example 1. Let G be a semigroup, H its subgroup. We set a coset decomposition $G=\bigcup_i g_iH$.

If we put $C=RG$ and $A=RH$, *a* semigroup bialgebra and its Hopf subalgebra, then C is a cocleft A-module coalgebra.

Indeed, defining $\phi:C\longrightarrow A$ by $\phi(g_ih)=h$, $h\in H$, and R-linearly, we can see that ϕ is a $*$-invertible integral with the property of Proposition 6 (ii).

Eiample 2. Suppose R is a ring of prime charasteristic p and $q=p^n$ for some positive integer n. Let C be the Hopf algebra $R[X]/(X^q)=R[x]$, where x is the residue class of X and $\Delta(x)=x\otimes 1+1\otimes x$, $\varepsilon(x)=0$ and $S(x)=-x$. If we put

$A=R[X^p]/(X^q)=R[x^p]$, then A is a Hopf subalgebra of C.

Let ϕ: $C\longrightarrow A$ be an R-linear map with $x^n\longmapsto x^n$ if $p|n$ and $x^n\longmapsto 0$ if $p\nmid n$. Then ϕ is a cointegral, as easily verified. Furthermore, we see, with direct computation, that ϕ is a coalgebra morphism. Thus C is a cocleft A-module coalgebra, because of Proposition 6 (ii).

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Supplement. The contents from §2 to § 3 of this paper coincide with the most part of the equally entitled lecture in the symposium of "*Algebraic groups and Lie groups, and their representations. Hopf algebras and Galois theory*" at Osaka University, January 1986. The lecture was put into practice at the recomendation of Professor M. Takeuchi.