NON-SUBMERSIONS OF THE PUNCTURED LENS SPACES $L^n(4)-x$ AND $L^n(p^2)-x$ FOR ODD PRIME p

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§1 ntroduction

In this note we consider the non-submersibility of the punctured lens space $L^n(k)-x$, which is the lens space with a point removed. The lens space is defined as follow ; Let k be an integer and γ ~~be~~ the rotation of $(2n+1)$-sphere

$$S^{2n+1}=\left[(z_0,\ z_1,\ \cdots\cdots,\ z_n)/\sum_{i=0}^{n}|z_i|^2=1\right]$$

of the complex $(n+1)$-space C^{n+1} given by

$$\gamma(z_0,\ z_1,\ \cdots\cdots,\ z_n)=(e^{2\pi i/k}z_0,\ e^{2\pi i/k}z_1,\ \cdots\cdots,e^{2\pi i/k}z_n).$$

Then γ generates the topological transformation group Γ of S^{2n+1} of order k, and the lens space $L^n(k)$ is defined to be the orbit space :

$$L^n(k)=S^{2n+1}/\Gamma$$

This is the compact differentiable $(2n+1)$-manifold wihout boundary. In this note we consider the case $k=4$, p^2 for odd prime p.

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§2 Non-submersing Theorem

As is well known, the lens space $L^n(k)$ has a cell structure given by

$$L^n(k)=s^1\cup e^2\cup\cdots\cdots e^{2n+1}\ .$$

Since the punctured lens space $L^n(k)-x$ has $L^n{}_0(k)=s^1\cup e^2\cup\cdots\cdots e^{2n}$ as deformation retract, the study of the tangent bundle $T(L^n(k)-x)$ reduces to that of $T(L^n(k)-x)|L^n{}_0(k)$. Let T represent $T(L^n(k)-x)|L^n{}_0(k)$. This is the $(2n+1)$-dimensional bundle, so $\tau_0=T-(2n+1)\epsilon\widetilde{KO}(L^n{}_0(k))$, where $\widetilde{KO}(L^n{}_0(k))$ is the reduced Grothendieck ring of real vector bundles over $L^n{}_0(k)$.

If $L^n(k)-x$ submerse in R^{2n+1-k}, i,e if T has a trivial summand of dimension $(2n+1-k)$, then τ_0 has geometric dimension $n\leqq k$. This implies for the ~~G rothendick~~ operators that $\gamma^i(\tau_0)=0$ for i$>k$. According of the result of T. Kobayashi and M. Sugawara the structure of $\widetilde{KO}(L^n(4))$ is following.

Let ρ be the canonical complex line bundle over $L^n(4)$, and set

$$\sigma=\eta-1\epsilon\widetilde{K}(L^n(4))$$

Let ρ be the non-trivial real line bundle over $L^n(4)$ and set $k=\rho-1\epsilon\widetilde{KO}(L^n(4))$. Let $r\sigma\epsilon\widetilde{KO}(L^n(4))$ denote the real restriction of σ.

Theorem (T. Kobayashi and M. Sugawara)

$$\widetilde{KO}(L^n(4))\cong\begin{cases} Z_{2^{n+1}} \oplus Z_{2^{n/2}} & \text{for even } n>0 \\ Z_{2^n} \oplus Z_{2[n/2]+1} & \text{for } n\equiv 1 \bmod 4, \\ Z_{2^n} \oplus Z_{2[n/2]} & \text{for } n\equiv 3 \bmod 4, \end{cases}$$

and the first summand is generated by $r\sigma$ and the secound by $k+2^{[n/2]}r\sigma$, where it is able to replace the last element by k if $n\equiv 1 \bmod 4$.

The multiplicative structure in $\widetilde{KO}(L^n(4))$ is given by

$$(r\sigma)^2=-4r\sigma+2k, \begin{cases} (r\sigma)^{[n/2]+1}=0 & \text{if } n\not\equiv 1 \bmod 4, \\ (r\sigma)^{[n/2]+2}=0 & \text{if } n\equiv 1 \bmod 4\,; \end{cases}$$

$$k^2=k\cdot r\sigma=-2k, \qquad k^{[n/2]+2}=0.$$

The action of the γ^i by $\gamma_t(r\sigma)=1+r\sigma\cdot t-r\sigma\cdot t^2$.

Since $T=(n+1)r\eta-1$, then $\tau_0(L^n{}_0(4))=(n+1)r\sigma$.

Thus

$$\gamma_t(\tau_0(L^n{}_0(4))=(\gamma_t(r\sigma))^{n+1}=(1+r\sigma(t-t^2))^{n+1}=\sum_{i=0}^{\infty}\binom{n+1}{i}(r\sigma)^i(t-t^2)^i.$$

According T. Kobayashi and M. Sugawara $(r\sigma)^i$ is of order 2^{n-2i+2} if $n\equiv 1 \bmod 2$ and 2^{n-2i+3} if $n\equiv 0 \bmod 2$.

If $n\not\equiv 0 \bmod 4$, $\widetilde{KO}((L^n{}_0(4))\cong\widetilde{KO}(L^n(4))$, so we consider only the the case $n\not\equiv 0 \bmod 4$ after this. The condition $\gamma^i(\tau_0)=0$ is equivaliant to $\binom{n+1}{i}\equiv 0 \bmod 2^{n-2i+2}$ if $n\equiv 1 \bmod 2$ and $\binom{n+1}{i}\equiv 0 \bmod 2^{n-2i+3}$ if $n\equiv 0 \bmod 2$, $n\not\equiv 0 \bmod 4$.

We define $L(n,4)$ to be the integer given by

$$L(n,4)=\begin{cases} \max\ \{i|1\leq i\leq\left[\frac{n}{2}\right],\ \binom{n+1}{i}\not\equiv 0 \bmod 2^{n-2i+2}\} & \text{if } n\not\equiv 1 \bmod 2, \\ \max\ \{i|1\leq i\leq\left[\frac{n}{2}\right],\ \binom{n+1}{i}\not\equiv 0 \bmod 2^{n-2i+3}\} & \text{if } n\equiv 0 \bmod 2, n\not\equiv 0 \bmod 4. \end{cases}$$

By the Theorem of Atiyah, we obtain the following theorem.

Theorem 1. If $n\not\equiv 0 \bmod 4$, The punctured lens space $L^n(4)-x$ cannot be submersed in $R^{2n+1-\{2L(n,4)-1\}}=R^{2\{n+1-L(n,4)\}}$.

Next we consider the submersion of the punctured lens space $L^n(p^2)-x$ where p is the odd prime integer.

We remark that $\widetilde{KO}(L^n{}_0(p^2)\cong\widetilde{KO}(L^n(p^2))$. The order $(r\sigma)^i\epsilon\widetilde{KO}(L^n(p^2))$ is equal to $p^{2+[(n-2i)/(p-1)]}$ and $(r\sigma)^{[n/2]+1}=0$ ([4]). So we define the integer

$$L(n,\ p^2)=\max\left\{i\mid i\leq[n/2],\binom{n+1}{i}\not\equiv 0 \bmod p^{2+[(n-2i)/(p-1)]}\right\}.$$

By the method of Atiyah, we obtain the following theorem.

Theorem 2. The punctured lens space $L^n(p^2)-x$ cannot be submersed in $R^{2n+1-\{2L(n,p^2)-1\}}=R^{2\{n+1-L(n,p^2)\}}$.

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Refference

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