EQUIVARIANT BORDISM AND SEMI-FREE S^1 -ACTION

Masato NAKAMURA

1. Introduction

Let X be a topological space with $A\subset X$ a subspace, and let τ: $S^1 \times (X, A) \longrightarrow (X, A)$ be an S^1-action such that $\tau(z, A)\subset A$ for $z\in S^1=\{z\in C$: complex number $| |z|=1\}$.

We consider the semi-free (free) bordism group $\Omega_*(X, A, \tau)$ of S^1-action (X, A, τ) by the analogue of R. E. Stong.

A semi-free (free) equivariant bordism class of (X, A, τ) is an equivariant class of triples (M, μ, f) with M a compact differentiable manifold with boundary, μ : $S^1\times M \longrightarrow M$ a differentiable semi-free (free) S^1-action on M and f : $(M, \partial M) \longrightarrow (X, A)$ continuous equivariant map $[\tau f = f\mu]$ sending ∂M into A. Two triples (M, μ, f) and (M', μ', f') are equivariant, or bordant, if there is a 4 tuple (W, V, ν, g) such that W and V are compact differentiable manifolds with boundary, $\partial V = \partial M \cup \partial M'$ and $\partial W = M \cup M' \cup V / \partial M \cup \partial M' \equiv \partial V$, ν: $(W, V) \longrightarrow (W, V)$ is a differentiable semi-free (free) S^1-action restricting to μ on M and μ' on M', and g : $(W, V) \longrightarrow (X, A)$ is a continuous equivariant map $[\tau g = g\nu]$ restricting to f on M and f' on M'.

The disjoint union of triples induces an opration on the set of semi-free (free) equivariant bordism classes of (X, A, τ) making this set into an abelian graded group, where the grading is given by the dimension of the manifold M and lets $\Omega_*(X, A, \tau)$ be the group of the semi-free equivariant bordism classes of (X, A, τ). And we let $\hat{\Omega}_*(X, A, \tau)$ be the group of free equivariant bordism classes of (X, A, τ). If A is empty, we write $\Omega_*(X, \tau)$ and $\hat{\Omega}_*(X, \tau)$ for these groups. The purpose of this paper is to compute the groups $\Omega_*(X, \tau)$.

2. Calculation of free bordism

THEOREM. 1. $\hat{\Omega}_*(X, A, \tau) \cong \Omega_*(X \times S^\infty/\tau \times \alpha,\ A \times S^\infty/\tau \times \alpha)$ where α is the S^1-action on the infinite sphere: direct limit of α: $S^1 \times S^{2n+1} \longrightarrow S^{2n+1}$, $\alpha(z, (z_0, z_1, \cdots\cdots z_n)) = (zz_0, zz_1, \cdots\cdots zz_n)$.

PROOF Let $\alpha\in\hat{\Omega}_n(X, A, \tau)$ be represented by (M, μ, f). Then the principal S^1-

bundle $M \longrightarrow M/\mu$ is induced by a map $\bar{\varphi} : M/\mu \longrightarrow CP^{(\infty)}$ with equivariant covering map $\varphi : M \longrightarrow S^{\infty}$, S^{∞} being given the above S^1-action. We then have an equivariant map $f \times \varphi : (M, \partial M) \longrightarrow (X \times S^{\infty}, A \times S^{\infty})$ and $\overline{f \times \varphi} : (M/\mu, \partial(M/\mu)) \longrightarrow (X \times S^{\infty}/\tau \times a, A \times S^{\infty}/\tau \times a)$.

The assignment

$$(M, \mu, f) \longrightarrow [M/\mu, \overline{f \times \varphi}] \in \Omega_n(X \times S^{\infty}/\tau \times a, A \times S^{\infty}/\tau \times a)$$

defines a homomorphism

$$\rho : \hat{\Omega}_n(X, A, \tau) \longrightarrow \Omega_n(X \times S^{\infty}/\tau \times a, A \times S^{\infty}/\tau \times a).$$

Being given $\bar{g} : (N, \partial N) \longrightarrow (X \times S^{\infty}/\tau \times a, A \times S^{\infty}/\tau \times a)$ there is an induced

cover

$$\begin{array}{ccc} \tilde{N} = \bar{g}^*(\pi) & \xrightarrow{g'} & X \times S^{\infty} \\ \downarrow & & \downarrow \\ N & \xrightarrow{\bar{g}} & X \times S^{\infty}/\tau \times a, \end{array}$$

and letting $g = \pi_1 \circ g' : \tilde{N} \longrightarrow X$ and $\tilde{\nu} : S^1 \times \tilde{N} \longrightarrow \tilde{N}$ being the S^1-action : $\tilde{\nu}(z(x, z')) = (\tau(z, x), zz')$, $(\tilde{N}, \tilde{\nu}, g)$ is a free bordism element of (X, A, τ). The assignment $(N, \bar{g}) \longrightarrow (\tilde{N}, \tilde{\nu}, g) \in \hat{\Omega}_n(X, A, \tau)$ induces a homomorphism inverse to σ.

Notes. (1) If X is a point, $A = \phi$, this gives $\Omega_*(S^1) \cong \Omega_*(CP(\infty))$. For $\tau = 1$, this is $\hat{\Omega}_*(X, A, 1) \cong \Omega_*(X \times CP(\infty), A \times CP(\infty))$.

3. **Calculation of semi-free bordism**

Threre are the exact sequence of S^1-action$\cdots\cdots \longrightarrow \Omega_n(A, \tau) \xrightarrow{\Omega_n(i)} \Omega_n(X, \tau) \xrightarrow{\Omega_n(j)} \Omega_n(X, A, \tau) \xrightarrow{\partial_n} \Omega_n(A, \tau) \longrightarrow$ with $(A, \phi, \tau) \xrightarrow{i} (X, \phi, \tau) \xrightarrow{j} (X, A, \tau)$ the inclusion. (see Refference [1], [2])

THEOREM 2. The (semi-free) equivariant bordism exact sequence of the S^1-action (X, F, τ) is split exact.

Proof. We have the homomorphisms

$$\Omega_n(X, \tau) \xrightarrow{\Omega_n(j)} \Omega_n(X, F_\tau, \tau) \underset{\cong}{\xleftarrow{k_*}} \hat{\Omega}_n(X, F_\tau, \tau)$$ and it suffices to define a homomorphism $q : \hat{\Omega}_n(X, F, \tau) \longrightarrow \Omega_n(X, \tau)$ with $\Omega_n(j) \circ q(\alpha) = k_*(\alpha)$ for all α. Being given $\alpha \in \Omega_n(X, F, \tau)$ represented by (M, μ, f), we have a closed manifold $\bar{M}$ obtained from M by identifying each $m \in \partial M$ with $\mu(S^1 \times m) \subset \partial M$. (This is the manifold obtained from M by attaching the disc bundle $D(\xi)$ of the line bundle ξ associated to the S^1-principal fibration $\sigma : M \longrightarrow M/\mu$ along their comon boundary.) Since f is equivariant and $f(\partial M) \subset F_\tau$, $f(m) = f(\mu m)$ for $m \in \partial M$, and f factors through $\bar{f} : \bar{M} \longrightarrow X$, this being equivariant if $\bar{M}$ is given the S^1-action

induced by μ. Letting q (α) be the class of $(\overline{M}, \overline{\mu}, \overline{f})$ defines the homomorphism q : $\widehat{\Omega}_n(X, F_\tau, \tau) \longrightarrow \Omega_n(X, \tau)$.

Now $\kappa_*(\alpha)$ and $\Omega_n(j) \circ$q (α) are represented by (M, μ, f) and $(\overline{M}, \overline{\mu}, \overline{f})$ respectively, in $\Omega_n(X, F_\tau, \tau)$. Let $\overline{H} : M \times I \longrightarrow X$ be a homotopy of the map $\overline{f} = H(\ ,0)$ to a map$g = H(\ ,1)$ with $g|_V = \overline{f}|F_{\overline{\mu}} \circ \pi$ where $V \cong D(\nu)$ is a tubular neighborhood of $F_{\overline{\mu}}$, constructed by the standard radial deformation. Then $F_{\overline{\mu}} = \partial M/\mu$ with $\nu \cong \xi$, and we may find a map $h : M \longrightarrow \overline{M} \times 1$ identifying M with $\overline{M} - V^\circ$ and such that $gh = f$. Then $(\overline{M} \times 1, V \times I, \overline{\mu} \times I, H)$ is a bordism of $(\overline{M}, \overline{\mu}, \overline{f})$ and (M, μ, f), so $\kappa_*(\alpha) = \Omega_n(j) \circ$q (α).

COROLLARY. $\Omega_*(X, \tau) \cong \Omega_*(F_\tau, 1) \oplus \widehat{\Omega}_*(X, F_\tau, \tau)$.

Refterence

1 R. E. Stong. Bordism and involution, Ann. of Math. 90 (1969).