ON THE REPRESENTATIONS AT FIXED POINTS OF SMOOTH ACTION OF Z_4 AND Z_{p^2} FOR ODD PRIME P

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Let Ⓗ be a smooth action of a finite group G on a differentiable manifold M. If $x \in M$ is a stationary point of this action, there is an induced representation Ⓗ$_x$ of G on the tangent space to M at x.

In this paper we shall obtain some result which compare the represantation Ⓗ$_x$ and Ⓗ$_y$ for different stationary points by the method of. G. E. Bredon.

In this paper we consider the case $G=Z_4$, Z_{p^2} for odd prime P. Let E_G be a universal space for G and $B_G=E_G/G$ the corresponding classifing space. We may assume that E_G is a cw-complex with finite skeltons and that G acts cellulary. R(G) donotes the complex representation ring of G. The map $E_G \longrightarrow *$ of E_G to a point induces the homomorphism $\alpha : R(G) \approx K_G(*) \longrightarrow K_G(E_G) \approx K(B_G)$ in the equivariant K-theory. Restricting to the r-skelton $B_G^{(r)}$ of B_G, we obtain the homomorphism $\alpha^{(r)} : R(G) \longrightarrow K(B_G^{(r)})$.

Theorem (G. E. Bredon). Let Ⓗ be a smooth action of a finite group G on a simply connected manifold M. Asume that

$$H^i(G\ ;\ \pi_i(M))=O \qquad for\ 1 \leqq i \leqq r.$$

If x and y are stationary points of Ⓗ, then

$$\alpha^{(r)}(Ⓗ_x - Ⓗ_y)=O$$

Using the above theorem, we obtain the next theorem.

Theoreml. Let Ⓗ be a smooth action of Z_4 on a simply connected manifold M. Assume that

$$H^i(Z_4\ ;\ \pi_i(M))=O \qquad for\ 1 \leqq i \leqq 2n+1,$$

then Ⓗ$_x$ − Ⓗ$_y$ is divisible by 2^{n+1}.

Proof. For Z_4 the complex representation ring is

$$Z[\eta]/(1-\eta^4)$$

where η is the representation $Z_p \longrightarrow U(1)$ taking the generator g into $e^{2\pi i/4}$. $B_{Z_4}^{2n+1}$ can be taken to be the lens space $L^n(4)$. By the result of T Kobayashi and M Sugawara

$$\tilde{K}(L^n(4)) \cong Z_{2^{n+1}} + Z_{2^{[n/2]}} + Z_{2^{[(n-1)/2]}},$$

and the direct summands are generated by the three elements

$$\sigma,\ \sigma^2+2\sigma,\ \sigma^3+2\sigma^2+2^{n/2+1}\sigma\ (if\ n\ is\ even),$$

$$\sigma,\ \sigma^2+2\sigma+2^{[n/2]+1}\sigma,\ \sigma^3+2\sigma^2 \quad (if\ n\ is\ even),$$

respectively, where $\sigma=\alpha^{2n+1}(\eta-1)$.
Let $I(Z)$ be the augmentation ideal $(1-\eta)R(Z_4)$. Since $I(Z_4)\cong Z$ additively, this implies that $ker\ \alpha^{(2n+1)}\subset 2^{n+1}\ I(Z_4)$.
from which the theorem follow. Next we consider Z_{p^2} action for odd prime p. BZ_{p^2} can be taken to be the lens space $L^n(p^2)$. By the result of T. Kawaguchi and M. Sugawara, the structure of $\widetilde{K}(L^n(p^2))$ is following. Let

$n-p^i+1=a_i(p^{i+1}-p^i)+b_i\ (O\leqq b_i<p^{i+1}-p^i)$ for $i=O,\ 1$, and consider the following elements of $\widetilde{K}\ (L^n(p^2))$: $\sigma=\eta-1$, $\sigma(1)=\eta^p-1=(1+\sigma)^p-1$,

$$\sigma(1,k)=\begin{cases}\sigma(1)\sigma^k+p^{[(n-k)/p]}\sigma^{p+k} & \\ \qquad if\ b_1\leqq k<b_1+p-1\ or\ k<b_1-(p-1)^2 \\ \sigma(1)\sigma^k & \text{(otherwise)},\end{cases}$$

for $O\leqq k\leqq \min\ (p^2-p-1,\ n-p)$.
Let p be a prime.
Then

$$\widetilde{K}\ (L^n(P^2))\cong\sum_{k=1}^{m}Z_{t_k},\ m=min\ (p^2-1,n)\ \ (direct\ sum)\ \text{and}$$

$$t_k=\begin{cases}p^{2-i+a_i}(if\ p^i\leqq k<p^i+b_i\ (i=0,1)\\ p^{1-i+a_i}(if\ p^i+b_i\leqq k<p^{i+1}\ (i=0,1).\end{cases}$$

Also, the k-th direct summand Z_{t_k} is genenated by the element

$$\sigma^k\ \ (if\ 1\leqq k<p),\ \sigma\ (1,\ k-p)\ \ (if p\leqq k<p^2).$$

Let I (Z_{p2}) be the augmentation ideal $(1-\eta)$R(Z_{p2}). Since $1\ (Z_{p2})\cong Z$ additively, this implies that

$$ker\ \alpha^{(2n+1)}\subset p^{2+a_0}\ 1\ (Z_{p^2})$$

from which the theorem follows,

Theorem II Let Ⓗ be a smooth action of Z_{p^2} on a simply connected manifold M, where p is a odd prime. Let $n=a(p-1)+b\ (O\leqq b<p-1)$ Asume that

$$H^i(Z_{p^2}\ ;\ \pi_i(M))=O\qquad for\ I\leqq i\leqq 2n+1$$

then Ⓗ$_x$ $-$ Ⓗ$_y$ is divisible by p^{2+a}.

References

1 G. E. Bredon, Representations at fixed points of smooth action of compact groups, Ann. of Math. 89 (1969).

2 T. Kawaguchi and M. Sugawara, K-and KO-Rings of the Lens Space L^n (p^2) for Odd Prime p, Hiroshima Math. j. (1971) 273-286.