ON THE CORRESPONDENCE OF GROUP EXTENSIONS WITH THE SECOND COHOMOLOGY CLASSES

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§ 1 Introduction

Let G and C be multiplicative groups, and in addition, C be abelian and a left G-group, which means that there exists a function $G \times C \longrightarrow C$ written $(g, c) \longrightarrow c^g$ such that $(cd)^g = c^g d^g$, $(c^g)^h = c^{hg}$, $c^1 = c$ $(c, d \in C, g, h \in G)$.

For any group extension of C by G, *i. e.*, $\mathscr{E} : 1 \longrightarrow C \xrightarrow{\alpha} W \xrightarrow{\beta} G \longrightarrow 1$, there exists a section $\pi : G \longrightarrow W$ $(g \longrightarrow \pi_g)$ of W, *i. e.*, a map π such that $\beta(\pi_g) = g$.

With this section, an operation of G on C is associated by $c \longrightarrow \alpha^{-1}(\pi_g \alpha(c) \pi_g{}^{-1})$ being independent to the choice of π. We are interested in the extensions which endow C with the prescribed G-group structure.

Now for $\mathscr{E}$ above, we have a 2-cocycle $f : G \times G \longrightarrow C$ determined by $\pi_g \pi_h = \alpha(f(g, h))\pi_{gh}$ $(g, h \in G)$.

As is well-known, the 2-cocycle which is obtained with any other section of W is cohomologous to f. Moreover, for any equivalent two extensions, we can see that, by means of the compatible section, there corresponds the same 2-cocycle. In addition, any two equivalent extensions endow C with the same G-group structure.

Now we have explained that there is a mapping Φ of $\Sigma(G, C)$ onto $H^2(G, C)$, the former is the set of all equivalence classes of group extensions of C by G that endow C with the prescribed G-group structure, the latter is the second cohomology group of G with the coefficient group C.

Conversely, let $f : G \times G \longrightarrow C$ be a 2-cocycle which represents a given cohomology class in $H^2(G, C)$. Then, we obtain a group $W = \{(c, g) \mid c \in C, g \in G\}$ with the multiplication $(c, g)(d, h) = (cd^g f(g, h), gh)$ $(c, d \in C, g, h \in G)$, and a group extension $1 \longrightarrow C \xrightarrow{\alpha} W \xrightarrow{\beta} G \longrightarrow 1$, where $\alpha(c) = (c, 1)$ and $\beta(c, g) = g$, such that the induced G-group structure of C is identical with the prescribed one. Besides, it is a matter of common knowledge that the group extensions which correspond to cohomologous cocycles are equivalent to each other. Now we have explained that there is a mapping Ψ of $H^2(G, C)$ into $\Sigma(G, C)$.

The fact that there exists a 1-1 correspondence between $\Sigma(G, C)$ and $H^2(G, C)$ is proved in [1]~[5], and in [3]~[5], the proofs of this fact are given by verifying that Φ and Ψ are mutually inverses. $\Sigma(G, C)$ has a group structure with the *Baer sum*, and it seems that there are no papers which explicitly point out that

Φ is a group isomorphism, *a fortiori*, a natural equivalence of bifunctors with the variables C and G. In this note, we shall give an explicit proof of this fact, *i. e.*,

Theorem. *Let C be an abelian group, G an arbitrary group, and suppose that C has a left G-group structure. Then the 1-1 correspondence $\Sigma(G, C) \leftrightarrow H^2(G, C)$, which is described in [1]~[5], is a natural equivalence of bifunctors covariant in the variable C and contravariant in the variable G.*

§ 2 Isomorphism

For two group extensions $\mathscr{E}_1 : 1 \longrightarrow C \xrightarrow{\alpha_1} W_1 \xrightarrow{\beta_1} G \longrightarrow 1$ and $\mathscr{E}_2 : 1 \longrightarrow C \xrightarrow{\alpha_2} W_2 \xrightarrow{\beta_2} G$——1, the *Baer sum* $\mathscr{E}_1 + \mathscr{E}_2$ is obtained as the lowest row of the following commutative diagram of row exact :

$$
\begin{array}{lcccccccc}
\mathrm{E}' : 1 & \longrightarrow & C \times C & \xrightarrow{\alpha_1 \times \alpha_2} & W_1 \times W_2 & \xrightarrow{\beta_1 \times \beta_2} & G \times G & \longrightarrow & 1 \\
 & & \Vert & & \uparrow \xi & & \uparrow \Delta & & \\
\mathrm{E}'' : 1 & \longrightarrow & C \times C & \xrightarrow{\sigma} & V & \xrightarrow{\tau} & G & \longrightarrow & 1 \\
 & & \downarrow \nabla & & \downarrow \eta & & \Vert & & \\
\mathrm{E} : 1 & \longrightarrow & C & \xrightarrow{\alpha} & W & \xrightarrow{\beta} & G & \longrightarrow & 1,
\end{array}
$$

where $V = \{(w_1, w_2, g) \in W_1 \times W_2 \times G \mid \beta_1(w_1) = \beta_2(w_2) = g\}$, $\Delta(g) = (g, g)$, $\sigma(c_1, c_2) = (\alpha_1(c_1), \alpha_2(c_2), 1)$, $\tau(w_1, w_2, g) = g(= \beta_1(w_1) = \beta_2(w_2))$,
$\nabla(c_1, c_2) = c_1 c_2$, $W = (C \times V)/N$, $N = \{\nabla(c_1, c_2)^{-1}, \sigma(c_1, c_2)) \mid c_1, c_2 \in C\} = \{(c_1^{-1} c_2^{-1}, (\alpha_1(c_1), \alpha_2(c_2), 1) \mid c_1, c_2 \in C\}$, $\alpha(c) = (c, 1)N$, $\beta((c, x)N) = \tau(x)$, $\xi(w_1, w_2, g) = (w_1, w_2)$, and $\eta(x) = (1, x)N$.

Let $\pi_i : G \longrightarrow W_i$ be sections of W_i, and be written $\pi_i(g) = \pi_{ig} (i = 1, 2)$. Then, the 2-cocycles $f_i : G \times G \longrightarrow C$, which are determined by the following equalities, represent the cohomology casses $\Phi((E_i))$:

$$\pi_{ig}\, \pi_{ih} = \alpha_i(f_i(g, h))\, \pi_{igh} (g, h \in G,\ i = 1, 2).$$

Now we can naturally construct sections π', π'', and π of $W_1 \times W_2$, V, and W, respectively, where the diagram

$$
\begin{array}{ccc}
W_1 \times W_2 & \xleftarrow{\pi'} & G \times G \\
\uparrow \xi & & \uparrow \Delta \\
V & \xleftarrow{\pi''} & G \\
\downarrow \eta & & \Vert \\
W & \xleftarrow{\pi} & G
\end{array}
$$

is commutative. Actually, $\pi' g = \pi_{1g} \times \pi_{2g}$, $\pi'' g = (\pi_{1g}, \pi_{2g}, g)$, and $\pi g = (1, (\pi_{1g}, \pi_{2g}, g))N$

$(g\epsilon G)$.

Then, the mapping $f: G\times G \longrightarrow C$, which is determined by the equality $\pi_g\pi_h = \alpha(f(g, h))\ \pi_{gh}\ (g, h\epsilon G)$, represents the cohomology class $\Phi((\mathscr{E}_1)+(\mathscr{E}_2))$. So we have $(1,(\pi_{1g}, \pi_{2g}, g))(1, (\pi_{1g}, \pi_{2g}, h))N=\alpha(f(g, h))(1, (\pi_{1gh}, \pi_{2gh}, gh))N$, hence $f(g, h)=f_1(g, h)\ f_2(g, h)=(f_1\cdot f_2)(g, h)\ \ (g, h\epsilon G)$.

Therefore $\Phi((\mathscr{E}_1)+(\mathscr{E}_2))= \Phi((\mathscr{E}_1))\Phi((\mathscr{E}_2))$, proving that Φ is a group isomorphism.

§ 3 Natural equivalence

Let $\mathscr{E}: 1 \longrightarrow C \overset{\alpha}{\longrightarrow} W \overset{\beta}{\longrightarrow} G \longrightarrow 1$ be any group extension such that the induced G-group structure of C is the same as the prescribed one.

For any homomorphism $\kappa: C \longrightarrow C'$ of abelian groups, $\Sigma(1_G. \kappa)(\mathscr{E})$ is the lower row of the following commutative diagram :

$$\begin{array}{ccccccccc} 1 & \longrightarrow & C & \overset{\alpha}{\longrightarrow} & W & \overset{\beta}{\longrightarrow} & G & \longrightarrow & 1 \\ & & \downarrow \kappa & & \downarrow \xi & & \| & & \\ 1 & \longrightarrow & C' & \overset{\alpha'}{\longrightarrow} & U & \overset{\beta'}{\longrightarrow} & G & \longrightarrow & 1 \end{array}$$

where $U=(C'\times W)/N$, $N= \{(\kappa(c)^{-1}, \alpha(c))\,|\,c\epsilon C\}$, $\xi(w)=(1, w)N$, $\alpha'(c')=(c', 1)N$, and $\beta'((c', w)N)= \beta(w)$. For any section π of W, the section π' of U which is naturally obtained as in §2 turned out to be $\xi\circ\pi$. The 2-cocycles $f: G\times G \longrightarrow C$ and $f': G\times G \longrightarrow C'$ determined by the equalities $\pi_g\pi_h= \alpha(f(g, h))\ \pi_{gh}$ and $\pi'_g\pi'_h= \alpha'(f'(g, h))\ \pi_{gh}(g, h\epsilon G)$ represent the cohomology classes $\Phi((\mathscr{E}))$ and $\Phi(\Sigma(1_G, \kappa)(\mathscr{E}))$, respectively. Now with those equalities just above and $\pi'=\xi\circ\pi$, we obtain $f'=\kappa\circ f$, *i.e.*, the commutative diagram

$$\begin{array}{ccc} \Sigma(G,C) & \overset{\Phi_{G,C}}{\longrightarrow} & H^2(G,C) \\ \downarrow \Sigma(1_G, \kappa) & & \downarrow H^2(1_G, \kappa) \\ \Sigma(G,C') & \overset{\Phi_{G,C'}}{\longrightarrow} & H^2(G,C'), \end{array}$$

proving the natural equivalence of Φ in the variable C.

For any group homomorphism $\nu: G_1 \longrightarrow G$, $\Sigma(\nu, 1_C)(\mathscr{E})$ is the lower row of the following commutative diagram :

$$\begin{array}{ccccccccc} 1 & \longrightarrow & C & \xrightarrow{\alpha} & W & \xrightarrow{\beta} & G & \longrightarrow & 1 \\ & & \| & & \uparrow{\scriptstyle \eta} & & \uparrow{\scriptstyle \nu} & & \\ 1 & \longrightarrow & C & \xrightarrow{\alpha''} & V & \xrightarrow{\beta''} & G' & \longrightarrow & 1, \end{array}$$

where $V=\{(w, g') \in W\times G' \mid \beta(w)=\nu(g')\}$, $\eta(w, g')=w$, $\alpha''(c)=(c, o)$, and $\beta'(w, g')=g'$.

For any section π of W, we can naturally construct the section π'' of V as in §2, satisfying $\pi\circ\nu=\eta\circ\pi''$. The 2-cocycle $f'' : G'\times G'\longrightarrow C$ determined by the equality $\pi_g''\pi_h''=(\alpha''(g, h))\pi''_{gh}$ represents the cohomology class $\Phi(\Sigma(\nu, 1_C)(\mathscr{E}))$. With this equality, $\pi_g\pi_h=\alpha(f(g, h))\pi_{gh}$, and $\pi\circ\nu=\eta\circ\pi''$, we obtain $f\circ(\nu\times\nu)=f''$, *i. e.*, the commutative diagram :

$$\begin{array}{ccc} \Sigma(C,G) & \xrightarrow{\Phi_{G,C}} & H^2(G,C) \\ \downarrow{\scriptstyle \Sigma(\nu,1_C)} & & \downarrow{\scriptstyle H^2(\nu,1_C)} \\ \Sigma(G',C) & \xrightarrow{\Phi_{G',C}} & H^2(G',C), \end{array}$$

proving the natural equivalence of Φ in the variable G.

References

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