ON THE GROUP ISOMORPHISM OF $E(C, A)$ WITH $H^1(G, \mathrm{HOM}_K(C, A))$

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§1. Introduction

In this paper, we assume that K is a field, G is a finite group, and A, C are KG-modules.

In Chapter 10 of [4], it is proved that there is a one-to-one correspondence between the set of all equivalence classes of extentions of A by C and the *1-st* cohomology group $H^1(G, \mathrm{Hom}_K(C, A))$.

The same fact appears in Chapter 8 of [2]. On the other hand, in Chapter 3 of [3], it is proved that the former set $E(C, A)$ has the group structure with the *Baer sum*. In this paper, we shall prove with the direct computation that the one-to-one correspondence in [4] is a group isomorphism, which can be proved homologically by connecting the result of Chapter 11, §8, and Chapter 14, §1 in [1].

§2. The one-to-one correspondence

For an extension

$$\mathscr{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{Hom}_K(C, A) \xrightarrow{f_*} \mathrm{Hom}_K(C, B) \xrightarrow{g_*} \mathrm{Hom}_K(C, C) \longrightarrow 0$$

is an exact sequence of KG-modules and KG-homomorphisms.

Thus we have the following exact cohomology sequence

$$0 \longrightarrow \mathrm{Hom}_{KG}(C, A) \longrightarrow \mathrm{Hom}_{KG}(C, B) \longrightarrow \mathrm{Hom}_{KG}(C, C) \xrightarrow{\triangle} H^1(G, \mathrm{Hom}_K(C, A))$$

$$\longrightarrow H^1(G, \mathrm{Hom}_K(C, B)) \longrightarrow \cdots$$

where $\triangle$ is the connecting homomorphism.

To the equivalence class $(\mathscr{E})$ of $\mathscr{E}$, associate $\triangle(1_C)$, then we have the one-to-one correspondence in [4]:

$$E(C, A) \longleftrightarrow H^1(G, \mathrm{Hom}_K(C, A)).$$

In the extension $\mathscr{E}$, let ϕ be a K-splitting of g. Then, ϕ is in $\mathrm{Hom}_K(C, B)$, and if $\delta : \mathrm{Hom}_K(C, B) = C^0(G, \mathrm{Hom}_K(C, B)) \longrightarrow C^1(G, \mathrm{Hom}_K(C, B))$ is the differential map, then $\delta\phi$ is actually in $Z^1(G, \mathrm{Hom}_K(C, A))$, and the cohomology class of $\delta\phi$ is $\triangle(1_C)$.

Conversely for any element of $H^1(G, \mathrm{Hom}_K(C, A))$, let ψ be the representative 1-cocycle.

Take for B the set of all pairs (a, c), where $a \in A$ and $c \in C$, and regard it as the direct sum of A and C considered simply as K-spaces. Next define $\sigma(a, c)$ by

$$\sigma(a, c) = (\sigma a + \psi_\sigma(\sigma c), \sigma c).$$

This makes B into a KG-module as is easily verified.

With the mappings $f: a \mapsto (a, 0)$ and $g: (a, c) \mapsto c$, the sequence

$$\mathscr{E}:\ 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is an exact sequence of KG-modules and KG-homomorphisms, *i. e.*, an extension of A by C.

Moreover K-homomorphism $\phi: C \longrightarrow B$ such that $\phi(c) = (0, c)$ is a K-splitting of g and if we construct $\triangle(1_C)$ through ϕ, then it coincides with ψ.

§ 3. The *Baer sum*

Let $(\mathscr{E}_1)$ and $(\mathscr{E}_2)$ are in $E(C, A)$, *i. e.*,

$$\mathscr{E}_1:\ 0 \longrightarrow A \xrightarrow{f_1} B_1 \xrightarrow{g_1} C \longrightarrow 0$$

$$\mathscr{E}_2:\ 0 \longrightarrow A \xrightarrow{f_2} B_2 \xrightarrow{g_2} C \longrightarrow 0,$$

and ϕ_1 and ϕ_2 be K-splittings of g_1 and g_2, respectively.

First we describe the *Baer sum* of $(\mathscr{E}_1)$ and $(\mathscr{E}_2)$.

Let $\Delta : C \longrightarrow C \oplus C$ be the diagonal map and let X be KG-submodule of $B_1 \oplus B_2 \oplus C$ which is the set $\{(b_1, b_2, c) \in B_1 \oplus B_2 \oplus C \mid f_1(b_1) = f_2(b_2) = c\}$, then we have the commutative diagram of row exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A \oplus A & \xrightarrow{f_1 \oplus f_2} & B_1 \oplus B_2 & \xrightarrow{g_1 \oplus g_2} & C \oplus C & \longrightarrow & 0 \\
 & & \| & & \uparrow \alpha & & \uparrow \Delta & & \\
0 & \longrightarrow & A \oplus A & \xrightarrow{\tilde{f}} & X & \xrightarrow{\tilde{g}} & C & \longrightarrow & 0
\end{array}$$

where $\tilde{f}(a_1, a_2) = (f_1(a_1), f_2(a_2),)0$, $\tilde{g}(b_1, b_2, c) = c$, $\alpha(b_1, b_2, c) = (b_1, b_2)$.

Corresponding to the K-space structure of

$$B_1 \oplus B_2 = (A \oplus C) \oplus (A \oplus C),$$

we have

$$(A \oplus C) \oplus (A \oplus C) \supset X = \{((a_1, c), (a_2, c), c) \mid a_1, a_2 \in \mathrm{A}, c \in C\}.$$

Next let $\nabla : A \oplus A \longrightarrow A$ be the map $(a_1, a_2) \mapsto a_1 + a_2$, and let B be the factor module of $A \oplus X$ by N where $N = \{(-\nabla(a_1, a_2), \tilde{f}(a_1, a_2)) \mid a_1, a_2 \in A\}$, then we have the commutative diagram of row exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A \oplus A & \xrightarrow{\tilde{f}} & X & \xrightarrow{\tilde{g}} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \beta} & & \| & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0, \end{array}$$

where $f(a) = (a, 0) + N$, $g((a, x) + N) = \tilde{g}(x)$, $\beta(x) = (0, x) + N$.

The lower row of this diagram is the *Baer sum* of $\mathscr{E}_1$ and $\mathscr{E}_2$.

Corresponding to the K-space structure of

$$X = \{((a_1, c), (a_2, c), c) \mid a_1, a_2 \in A, c \in C\},$$

we have

$$N = \{(-a_1 - a_2, (a_1, 0), (a_2, 0), 0)\} \subset A \oplus (A \oplus C) \oplus (A \oplus C) \oplus C,$$

so

$$B = \{(a, (0, c), (0, c), c) + N\}.$$

Now let $\phi : C \longrightarrow B$ be the K-homomorphism such that

$$\phi(c) = (0, (0, c), (0, c), c) + N = (0, \phi_1(c), \phi_2(c), c) + N,$$

then ϕ is a K-splitting of g.

On the other hand, let us observe the K-module-structure of G.

For any σ in G, and $(\mathrm{a}, (0, c), (0, c), c) + N$ in B,

$$\sigma((a, (0, c), (0, c), c) + N) = (\sigma a, \sigma(0, c), \sigma(0, c), \sigma c) + N$$

$$= (\sigma a, (\psi_{1_\sigma}(\sigma c), \sigma c), (\psi_{2_\sigma}(\sigma c), \sigma c), \sigma c) + N,$$

where ψ_1 and ψ_2 are 1-cocycles corresponding to the extensions $\mathscr{E}_1$ and $\mathscr{E}_2$, which are constructed through ϕ_1 and ϕ_2, respectively.

§ 4. A direct proof of the group isomorphism $E(C, A) \cong H^1(G, \mathrm{Hom}_K(C, A))$

Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be the following extensions:

$$\mathscr{E}_1 : 0 \longrightarrow A \xrightarrow{f_1} B_1 \xrightarrow{g_1} C \longrightarrow 0,$$

$$\mathscr{E}_2 : 0 \longrightarrow A \xrightarrow{f_2} B_1 \xrightarrow{g_2} C \longrightarrow 0.$$

Then $B_1 = A \oplus C$, $B_2 = A \oplus C$ as K-spaces, and K-splittings ϕ_1 and ϕ_2 of g_1 and g_2 are given as

$$\phi_1(c) = (0, c) \in B_1, \quad \phi_2(c) = (0, c) \in B_2.$$

Let ψ_1 and ψ_2 be 1-cocycles $G \longrightarrow \mathrm{Hom}_K(C, A)$ which are obtained through ϕ_1 and ϕ_2, respectively, then

$$\psi_{i_\sigma}(c) = (\sigma\phi_i - \phi_i)(c) = \sigma\phi(\sigma^{-1}c) - \phi_i(c) \ (\sigma \in G, \ \ c \in C, \ \ i = 1, 2).$$

Here $\mathscr{E}_1 + \mathscr{E}_2$ is the extension

$$\mathscr{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

described in § 3.

Our purpose is to prove that the 1-cocycle corresponding to $\mathscr{E}$ is cohomologous to $\psi_1 + \psi_2$.

If ψ is the 1-cocycle corresponding to $\mathscr{E}$ through ϕ which is described in § 3, then for any σ in G, and any c in C,

$$\begin{aligned}
\psi_\sigma(c) &= (\sigma\phi - \phi)(c) = \sigma\phi(\sigma^{-1}c) - \phi(c) \\
&= \sigma(0, \phi_1(\sigma^{-1}c), \phi_2(\sigma^{-1}c), \sigma^{-1}c) - (0, \phi_1(c), \phi_2(c), c) + N \\
&= \sigma(0, (0, \sigma^{-1}c), (0, \sigma^{-1}c), \sigma^{-1}c) - (0, (0, c), (0, c), c) + N \\
&= (0, (\psi_{1\sigma}(c), c), (\psi_{2\sigma}(c), c), c) - (0, (0, c), (0, c), c) + N \\
&= (0, (\psi_{1\sigma}(c), 0), (\psi_{2\sigma}(c), 0), 0) + N \\
&= ((\psi_{1\sigma} + \psi_{2\sigma})(c), (0, 0), (\psi, 0), 0) + N,
\end{aligned}$$

hence

$$\psi_\sigma(c) = (\psi_1 + \psi_2)_\sigma(c) \qquad (\sigma \in G, c \in C).$$

So we have proved that ψ is actually equal to $\psi_1 + \psi_2$, and the proof of the group isomorphism

$$E(C, A) \cong H^1(G, \mathrm{Hom}_K(C, A))$$

is completed.

Now $H^1(G, \mathrm{Hom}_K(C, A))$ is a bifunctor in variables C and A ([2]), and $E(C, A)$ is also a bifunctor in variables C and A ([1]).

We can easily verify that the isomorphism above is an equivalence of functors. Thus we have, with direct computations:

Theorem. *Let G be a finite group, K be a field, and A, C be left KG-modules. Moreover let $E(C, A)$ be the set of equivalence classes of extensions of A by C, and observe it as an abelian group with the Baer sum. Then we have the natural equivalance of bifunctors*

$$E(C, A) \cong H^1(G, \mathrm{Hom}_K(C, A)).$$

References

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