"A Study on the Points at a Rectifiable Fixed Closed Curve Divided into Special Number of Parts of Equal Length"

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Based on the same theory, we once wrote the two papers on the geometry of points dividing a fixed rectifiable closed curve in 3-dimensicnal euclidean space into special number of parts of equal length.

The one* deals with the analytical geometry in the space and the other** with differential geometry.

This paper deals with some problems of differential geometry and position vectors.

In the following, the space denotes 3-dimensional euclidean space. Let us take a rectifiable closed curve C having the constant lengh l and the definite direcstion, and we assume it is of class C^3.

If we take the original point O on it since any point P on it can be expressed by the arc length s measured from O to P, every point P on C can be denoted by the symbol $P(s)$, where $o \leqq s \leqq l$.

We shall consider the system $\mathfrak{S}$ of the points P_i, $i=1, 2$ ordered along the curve C, dividing it into two parts of equal length.

Let s_1 be the arc length of OP_1. We shall take a fixed rectangular coordinate system O-$x_1\ x_2\ x_3$, and if we denote the coordinates of the points P_i on a fixed closed curve C by (x^i_1, x^i_2, x^i_3), all these coordinates are continuous functions of s_1.

Therefore, the system $\mathfrak{S}=\{P_i(s_i), i=1, 2\}$ is a function of s_1.

The osculating planes at the points $P_i(s_i)$ on the curve C are represented by

$$(1) \qquad |\mathbf{x}-\mathbf{x}^i(s_i) \quad \mathbf{x}^{i\prime}(s_i) \quad \mathbf{x}^{i\prime\prime}(s_i)| = o, \quad (i=1, 2)$$

where

$$\mathbf{x}^{i\prime}(s_i) \equiv \frac{d\mathbf{x}^i(s_i)}{ds}, \qquad \mathbf{x}^{i\prime\prime}(s_i) \equiv \frac{d^2\mathbf{x}^i(s_i)}{ds^2}, \qquad (i=1, 2).$$

The lengths d_i of the perpendiculars from origin O to the two osculating planes (1) are given by

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* See References (7). ** See References (9).

$$(2)\quad d_i=\frac{\text{abs.}\left\{\begin{vmatrix} x_2^{i\,\prime}(s_i) & x_3^{i\,\prime}(s_i) \\ x_2^{i\,\prime\prime}(s_i) & x_3^{i\,\prime\prime}(s_i)\end{vmatrix} x_1^i(s_i)+\begin{vmatrix} x_3^{i\,\prime}(s_i) & x_1^{i\,\prime}(s_i) \\ x_3^{i\,\prime\prime}(s_i) & x_1^{i\,\prime\prime}(s_i)\end{vmatrix} x_2^i(s_i)+\begin{vmatrix} x_1^{i\,\prime}(s_i) & x_2^{i\,\prime}(s_i) \\ x_1^{i\,\prime\prime}(s_i) & x_2^{i\,\prime\prime}(s_i)\end{vmatrix} x_3^i(s_i)\right\}}{\left\{\begin{vmatrix} x_2^{i\,\prime}(s_i) & x_3^{i\,\prime}(s_i) \\ x_2^{i\,\prime}(s_i) & x_3^{i\,\prime\prime}(s_i)\end{vmatrix}^2+\begin{vmatrix} x_3^{i\,\prime}(s_i) & x_1^{i\,\prime\prime}(s_i) \\ x_3^{i}(s_i) & x_1^{i\,\prime\prime}(s_i)\end{vmatrix}^2+\begin{vmatrix} x_1^{i\,\prime}(s_i) & x_2^{i\,\prime}(s_i) \\ x_1^{i\,\prime\prime}(s_i) & x_2^{i\,\prime\prime}(s_i)\end{vmatrix}^2\right\}^{\frac{1}{2}}},$$

$(i=1,\ 2)$.

Now, if we put

$$f(s_1)=d_1-d_2,$$

then $f(s_1)$ is a continuous function of s_1 (this function can be considered as a function of the system $\mathfrak{S}$).

If we replace s_1 by s_2 (*i. e.*, take another system $\mathfrak{S}'=\{P_i(s_i')\}$, with $s_1'=s_2$), we have easily $f(s_1)=-f(s_2)$. Therefore, we have at least one such ξ as satisfies both $f(\xi)=o$ and $s_1<\xi_2<s_2$. This ξ renders the lengths of two perpendiculars from the origin O to the osculating planes at the two points P_1 and P_2 on C equal. Hence, we have the following.

Theorem 1. *Among the system $\mathfrak{S}=\{P_i,\ i=1,\ 2\}$ ordered along the curve C, dividing a rectifiable fixed closed curve C which is of class C^3, into two parts of equal length, there exists at least one system that makes the lengths of the two perpendiculars from the origin to the osculating planes at the two points P_1 and P_2 on C dividing it into two parts of equal length equal.*

This theorem holds also in the case of replacing the osculating plane by the normal plane, and by the rectifying plane.

Now, we deal with some problems of position vectors.
Here, we assume a closed curve C as above constructed.
Denoting the two position vectors OP_1, OP_2 by $\mathbf{x}^1(s_1)$, $\mathbf{x}^2(s_2)$ as above, the cross products of the two position vectors to the centres of the two osculating circles at the points P_1 and P_2 on the curve C are represented by

$$(3)\qquad (\mathbf{x}^1(s_1)+\sqrt{\mathbf{x}^{1\prime\prime}(s_1)\cdot\mathbf{x}^{1\prime\prime}(s_1)}\ \ \xi_2(s_1))\times(\mathbf{x}^2(s_2)+\sqrt{\mathbf{x}^{2\prime\prime}(s_2)\cdot\mathbf{x}^{2\prime\prime}(s_2)}\ \ \xi_2(s_2)),$$

where $\xi_2(s_1)$, $\xi_2(s_2)$ are the principal unit vectors at the points P_1 and P_2, respectively.

Put this expression $g(s_1)$, and $g(s_1)$ is a continuous function of s_1. If we replace s_1 by s_2, we have obviously $g(s_1)=-g(s_2)$. Therefore, we have at least one such η as satisfies both $g(\eta)=o$ and $s_1<\eta<s_2$. This η renders the position vectors to the centres of the two osculating circles at the points P_1 and P_2 equal.

Theorem 2. *Among the system $\mathfrak{S}=\{P_i,\ i=1,\ 2\}$, ordered along the curve C, dividing a rectifiable closed curve C into two parts of eqnal length, there exists at least one system that makes the origin O and the two centres of osculating* *circles at P_1 and P_2 on the curve C collinear.*

A similar theorem holds on the centres of the two osculating spheres at the points P_1 and P_2 on the curve C.

If we take by similar considerations the cross product of the two position vectors $\mathbf{x}^1(s_1)$ and $\mathbf{x}^2(s_2)$, the following fact holds:

Theorem 3. *Among the system $\mathfrak{S}=\{P_i, i=1, 2\}$, ordered along the curve C, dividing a rectifiable closed curve C into two parts of equal length, there exists at least one system that the three points origion O, P_1 and P_2 are collinear.*

Now, we shall consider the system $\mathfrak{S}$ of the points $P_i, i=1, 2, 3, \cdots\cdots, 2n$ ordered along the curve C, dividing it into $2n$ parts of equal length. Let s_1 be the arc length of OP_1.

We shall take $2n$ position vectors $\mathbf{x}^1, \mathbf{x}^2, \cdots\cdots\cdots, \mathbf{x}^{2n}$ to the $2n$ points of the curve C, respetively.

If we denote

$$\mathbf{x}^{13\cdots\cdots(2n-1)}=\frac{1}{n}(\mathbf{x}^1+\mathbf{x}^3+\cdots\cdots+\mathbf{x}^{2n-1})$$

and

$$\mathbf{x}^{24\cdots\cdots(2n)}=\frac{1}{n}(\mathbf{x}^2+\mathbf{x}^4+\cdots\cdots+\mathbf{x}^{2n}),$$

then two symbols $\mathbf{x}^{13\cdots\cdots(2n)}$ and $\mathbf{x}^{24\cdots\cdots(2n)}$ are the position vectors from origin O to the centroids of the two sets consisting of n points $P_1, P_3, \cdots\cdots$, P_{2n-1} and $P_2, P_4, \cdots\cdots\cdots, P_{2n}$, respectively.

Put $$\phi(s_1)=\frac{1}{n}(\mathbf{x}^1+\mathbf{x}^3+\cdots\cdots+\mathbf{x}^{2n-1})-\frac{1}{n}(\mathbf{x}^2+\mathbf{x}^4+\cdots\cdots+\mathbf{x}^{2n})$$

and $\phi(s_1)$ is a continuous function of s_1.

If we replace s_1 by s_2, we find easily that $\phi(s_1)=-\phi(s_2)$.

Based on the same theory we have the following.

Theorem 4. *Among the system $\mathfrak{S}=\{P_i, i=1, 2\cdots\cdots\cdots, 2n\}$ ordered along the curve C, dividing a rectifiable closed curve C into 2n parts of equal length, there exists at least one system that the centroid of n points $P_1, P_3\cdots\cdots\cdots, P_{2n-1}$ is coincident with the centroid of n points $P_2, P_4, \cdots\cdots\cdots, P_{2n}$.*

Let $\mathfrak{S}=\{P_i(s_i), i=1, 2, \cdots\cdots, 6\}$ be a system of points ordered along the curve C, dividing it into six parts of equal length, and $\mathbf{x}^i(s_i)$ be the position vectors to the points P_i, $i=1, 2, 3, 4, 5, 6$, respectively.

By similar considerations, let us deal with the two triple scalar products: $\mathbf{x}^1(s_1)\cdot(\mathbf{x}^3(s_3)\times\mathbf{x}^5(s_5))$ and $\mathbf{x}^2(s_2)\cdot(\mathbf{x}^4(s_4)\times\mathbf{x}^6(s_6))$.

These scalars represent the two volumes of the parallelepipeds formed by the coterminous sides $\mathbf{x}^1(s_1)$, $\mathbf{x}^3(s_3)$, $\mathbf{x}^5(s_5)$ and $\mathbf{x}^2(s_2)$, $\mathbf{x}^4(s_4)$, $\mathbf{x}^6(s_6)$, respectively. Since $\mathbf{x}^1(s_1)\cdot(\mathbf{x}^3(s_3)\times\mathbf{x}^5(s_5))$ and $\mathbf{x}^2(s_2)\cdot(\mathbf{x}^4(s_4)\times\mathbf{x}^6(s_6))$ are continuous functions of s_1, put

$\Psi(s_1)=\mathbf{x}^1(s_1)\cdot(\mathbf{x}^3(s_3)\times\mathbf{x}^5(s_5))-\mathbf{x}^2(s_2)\cdot(\mathbf{x}^4(s_4)\times\mathbf{x}^6(s_6))$, and $\Psi(s_1)$ is written as follows:

$$\Psi(s_1)=\begin{vmatrix} x_1^1(s_1) & x_2^1(s_1) & x_3^1(s_1) \\ x_1^3(s_3) & x_2^3(s_3) & x_3^3(s_3) \\ x_1^5(s_5) & x_3^5(s_5) & x_5^3(s_5) \end{vmatrix} - \begin{vmatrix} x_1^2(s_2) & x_2^2(s_2) & x_3^2(s_2) \\ x_1^4(s_4) & x_2^4(s_4) & x_3^4(s_4) \\ x_1^6(s_6) & x_2^6(s_6) & x_3^6(s_6) \end{vmatrix}.$$

here, replacing s_1 by S_2 we have easily $\Psi(s_1) = -\Psi(s_2)$.

Therefore, we have at least one such ζ as satisfies both $\Psi(\zeta) = o$ and $s_1 < \zeta < s_2$. For this ζ, the volumes of the two parallelepipeds are equal. We have the following.

Theorem 5. *Among the system $\mathfrak{S} = \{P_i,\ i=1, 2, 3, \cdots\cdots, 6\}$, ordered along the curve C, dividing a rectifiable closed curve C into six parts of equal length, there exists at least one system that the two parallelepipeds $\mathbf{x}^1(s_1)\cdot(\mathbf{x}^3(s_3)\times \mathbf{x}^5(s_5))$ and $\mathbf{x}^2(s_2)\cdot(\mathbf{x}^4(s_4)\times\mathbf{x}^6(s_6))$ have equal volume.*

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