

Family number, Wilson line phases and hidden supersymmetry

Yuhei GOTO,^{*} Yoshiharu KAWAMURA[†]
*Department of Physics, Shinshu University,
Matsumoto 390-8621, Japan*

and

Takashi MIURA[‡]
*Department of Physics, Kobe University,
Kobe 657-8501, Japan*

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Abstract

We study the relationship between the family number of chiral fermions and the Wilson line phases, based on the orbifold family unification. We find that flavor numbers are independent of the Wilson line phases relating extra-dimensional components of gauge bosons, as far as the standard model gauge symmetry is respected. This feature originates from a hidden quantum-mechanical supersymmetry.

1 Introduction

The origin of the chiral fermions and the family replication has been a big mystery. On a higher-dimensional space-time including an orbifold as an extra space, the family unification based on a large gauge group or the structure of extra dimensions can provide a possible solution [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12].

Four-dimensional (4D) chiral fermions come from a higher-dimensional fermion, after the elimination of mirror particles by orbifolding upon compactification. The symmetry breaking mechanism on the orbifold has been originally used in superstring theory [13, 14]. The family replication emerges from a few multiplets of a large gauge group including the family group as a subgroup. Hence, it is interesting to explore a nature of the family number based on the orbifold family unification, in the expectation that it offers a hint on the origin of three families in the standard model (SM).

^{*}E-mail: 14st302a@shinshu-u.ac.jp

[†]E-mail: haru@azusa.shinshu-u.ac.jp

[‡]E-mail: takashi.miura@people.kobe-u.ac.jp

In this paper, we study the relationship between the family number of chiral fermions and the Wilson line phases, based on the orbifold family unification. We find that flavor numbers are independent of the Wilson line phases relating extra-dimensional components of gauge bosons, as far as the SM gauge symmetry is respected. This feature originates from a hidden quantum-mechanical supersymmetry (SUSY).

The contents of this paper are as follows. In Sec. II, we review a feature of the family number in the orbifold family unification, and present a conjecture on flavor numbers. In Sec. III, we give an example to support the conjecture and show that it is understood from the viewpoint of the hidden SUSY. Section IV is devoted to conclusions. In Appendices A and B, we present several formulas concerning the combination ${}_n C_l$, derived from the feature that flavor numbers are independent of the Wilson line phases.

2 Family number in orbifold family unification

2.1 A feature

In our previous work [11], we have studied the orbifold family unification in $SU(N)$ gauge theory on 6D space-time, $M^4 \times T^2/Z_M$ ($M = 2, 3, 4, 6$). Here, M^4 is the 4D Minkowski space-time and T^2/Z_M is the 2D orbifold. We have derived enormous numbers of models with three families of $SU(5)$ multiplets and the SM multiplets from a pair of 6D Weyl fermions with different chiralities, using the orbifold breaking mechanism, after the breakdown of gauge symmetry such that $SU(N) \rightarrow SU(5) \times SU(p_2) \times \cdots \times SU(p_n) \times U(1)^{n-1-m}$ and $SU(N) \rightarrow SU(3) \times SU(2) \times SU(p_3) \times \cdots \times SU(p_n) \times U(1)^{n-1-m}$, respectively. Here and hereafter, m is the number of zero in $\{p_i\}$ and “ $SU(1)$ ” unconventionally stands for $U(1)$, $SU(0)$ means nothing.

Through the analysis, we have found the feature that *each flavor number obtained from a 6D Weyl fermion with $[N, k]$ is invariant under the change $\{p_i\}$ into $\{p'_i\}$ among the equivalent boundary conditions (BCs)*. Here, $[N, k]$ is the rank k totally antisymmetric tensor representation whose dimension is ${}_N C_k$.

Let us present several illustrations.

On T^2/Z_2 , the numbers of 4D left-handed Weyl fermions with the representations $\bar{\mathbf{5}}$ and $\mathbf{10}$ obtained from the breaking pattern $SU(N) \rightarrow SU(5) \times SU(p_2) \times \cdots \times SU(p_8) \times U(1)^{7-m}$ are same as those from $SU(N) \rightarrow SU(5) \times SU(p'_2) \times \cdots \times SU(p'_8) \times U(1)^{7-m}$, if the following relations are satisfied,

$$\begin{aligned} p'_2 - p_2 &= p'_7 - p_7 = p_3 - p'_3 = p_6 - p'_6, \\ p'_4 &= p_4, \quad p'_5 = p_5, \quad p'_8 = p_8, \end{aligned} \tag{1}$$

or

$$\begin{aligned} p'_2 - p_2 &= p'_7 - p_7 = p_4 - p'_4 = p_5 - p'_5, \\ p'_3 &= p_3, \quad p'_6 = p_6, \quad p'_8 = p_8, \end{aligned} \tag{2}$$

or

$$p'_3 - p_3 = p'_6 - p_6 = p_4 - p'_4 = p_5 - p'_5,$$

$$p'_2 = p_2, \quad p'_7 = p_7, \quad p'_8 = p_8. \quad (3)$$

In the same way, the flavor numbers of the SM fermions obtained from $SU(N) \rightarrow SU(3) \times SU(2) \times SU(p_3) \times \cdots \times SU(p_8) \times U(1)^{7-m}$ are same as those from $SU(N) \rightarrow SU(3) \times SU(2) \times SU(p'_3) \times \cdots \times SU(p'_8) \times U(1)^{7-m}$, if the following relations are satisfied,

$$p'_3 - p_3 = p'_6 - p_6 = p_4 - p'_4 = p_5 - p'_5, \quad p'_7 = p_7, \quad p'_8 = p_8. \quad (4)$$

On T^2/Z_3 , the numbers of 4D left-handed Weyl fermions with $\bar{\mathbf{5}}$ and $\mathbf{10}$ obtained from $SU(N) \rightarrow SU(5) \times SU(p_2) \times \cdots \times SU(p_9) \times U(1)^{8-m}$ are same as those from $SU(N) \rightarrow SU(5) \times SU(p'_2) \times \cdots \times SU(p'_9) \times U(1)^{8-m}$, if the following relations are satisfied,

$$p'_2 - p_2 = p'_6 - p_6 = p'_7 - p_7 = p_3 - p'_3 = p_4 - p'_4 = p_8 - p'_8, \quad p'_5 = p_5, \quad p'_9 = p_9. \quad (5)$$

2.2 A conjecture

In the above cases with the relations (1) – (5), the BCs relating $\{p_i\}$ are connected with those relating $\{p'_i\}$ by singular gauge transformations, and they are regarded as equivalent in the presence of the Wilson line phases relating extra-dimensional components of gauge bosons. This equivalence originates from the dynamical rearrangement in the Hosotani mechanism [15, 16, 17, 18].

For cases on T^2/Z_2 , the equivalence of BCs is shown by the following relations among the diagonal representatives for 2×2 submatrices of (P_0, P_1, P_2) [19],

$$(\tau_3, \tau_3, \tau_3) \sim (\tau_3, \tau_3, -\tau_3) \sim (\tau_3, -\tau_3, \tau_3) \sim (\tau_3, -\tau_3, -\tau_3), \quad (6)$$

where P_0, P_1 and P_2 are representation matrices for the Z_2 reflections, and τ_3 is the third component of the Pauli matrices. For case on T^2/Z_3 , it is shown by the following relations among the diagonal representatives for 3×3 submatrices of (Θ_0, Θ_1) [19],

$$(X, X) \sim (X, \bar{\omega}X) \sim (X, \omega X), \quad (7)$$

where Θ_0 and Θ_1 are representation matrices for the Z_3 rotations, and $X = \text{diag}(1, \omega, \bar{\omega})$ with $\omega = e^{2\pi i/3}$ and $\bar{\omega} = e^{4\pi i/3}$.

In [11], we assume that the BCs are chosen as physical ones, i.e., the system with the physical vacuum is realized with the vanishing Wilson line phases after a suitable gauge transformation is performed. Then, the feature is expressed by

$$N_{\mathbf{r}}|_{(\{p_i\}, a_k=0)} = N_{\mathbf{r}}|_{(\{p'_i\}, a_k=0)}, \quad (8)$$

where $N_{\mathbf{r}}$ is a net chiral fermion number (flavor number) for 4D fermions with the representation \mathbf{r} of the gauge group, unbroken even in the presence of the Wilson line phases ($2\pi a_k$), and it is defined by

$$N_{\mathbf{r}} \equiv n_{L\mathbf{r}}^0 - n_{R\mathbf{r}}^0 - n_{L\bar{\mathbf{r}}}^0 + n_{R\bar{\mathbf{r}}}^0. \quad (9)$$

Here, $n_{L\mathbf{r}}^0$, $n_{R\mathbf{r}}^0$, $n_{L\bar{\mathbf{r}}}^0$ and $n_{R\bar{\mathbf{r}}}^0$ are the numbers of 4D left-handed massless fermions with \mathbf{r} , 4D right-handed one with \mathbf{r} , 4D left-handed one with the complex conjugate representation $\bar{\mathbf{r}}$ and 4D right-handed one with $\bar{\mathbf{r}}$, respectively. Note that 4D right-handed

fermion with \bar{r} and 4D left-handed one with r are transformed into each other under the charge conjugation.

On the other hand, the equivalence due to the dynamical rearrangement is expressed by

$$N_{\mathbf{r}}|_{(\{p_i\}, a_k \neq 0)} = N_{\mathbf{r}}|_{(\{p'_i\}, a_k = 0)} . \quad (10)$$

From (8) and (10), we obtain the relation,

$$N_{\mathbf{r}}|_{(\{p_i\}, a_k = 0)} = N_{\mathbf{r}}|_{(\{p_i\}, a_k \neq 0)} , \quad (11)$$

and find that each flavor number obtained from $[N, k]$ does not change even though the vacuum changes different ones in the presence of the Wilson line phases.

In this way, we arrive at the conjecture that *each flavor number in the SM is independent of the Wilson line phases that respect the SM gauge group*. If there were a Wilson line phase with a non-vanishing SM gauge quantum number, (a part of) the SM gauge symmetry can be broken down. Hence, we assume that such a Wilson line phase is vanishing or switched off.

3 Fermion numbers and hidden supersymmetry

On a higher-dimensional space-time $M^4 \times K^{D-4}$, a massless fermion $\Psi = \Psi(x, y)$ satisfies the equation,

$$i\Gamma^M D_M \Psi = 0 , \quad (12)$$

where K^{D-4} is an $(D-4)$ -dimensional extra space, Γ^M ($M = 0, 1, 2, 3, 5, \dots, D$) are matrices that satisfy the Clifford algebra $\Gamma^M \Gamma^N + \Gamma^N \Gamma^M = 2\eta^{MN}$, $D_M \equiv \partial_M + ig A_M$ and Ψ is a fermion with $2^{\lfloor D/2 \rfloor}$ -components. Here, g is a gauge coupling constant, $A_M (= A_M^\alpha T^\alpha)$ are gauge bosons, and $[*]$ is the Gauss symbol. The coordinates x^μ ($\mu = 0, 1, 2, 3$) on M^4 and x^m ($m = 5, \dots, D$) on K^{D-4} are denoted by x and y , respectively.

After the breakdown of gauge symmetry, Ψ is decomposed as

$$\Psi(x, y) = \sum_{r_H} \sum_{\{n_i\}} \left[\psi_{Lr_H}^{\{n_i\}}(x) \phi_{Lr_H}^{\{n_i\}}(y) + \psi_{Rr_H}^{\{n_i\}}(x) \phi_{Rr_H}^{\{n_i\}}(y) \right] , \quad (13)$$

where $\psi_{Lr_H}^{\{n_i\}}(x)$ and $\psi_{Rr_H}^{\{n_i\}}(x)$ are 4D left-handed spinors and right-handed ones, respectively. The subscript r_H stands for some representation of the unbroken gauge group H , and the superscript $\{n_i\}$ represents a set of numbers relating massive modes and those concerning components of multiplet r_H . The functions $\phi_{Lr_H}^{\{n_i\}}(y)$ and $\phi_{Rr_H}^{\{n_i\}}(y)$ form complete sets on K^{D-4} .

We define the chiral fermion number relating r as

$$n_{\mathbf{r}} \equiv n_{Lr}^0 - n_{Rr}^0 , \quad (14)$$

where r is a representation of the subgroup unbroken in the presence of the Wilson line phases. The net chiral fermion number $N_{\mathbf{r}}$ is given by $N_{\mathbf{r}} = n_{\mathbf{r}} - n_{\bar{\mathbf{r}}}$.

In case that n_r is independent of the Wilson line phases $(2\pi a_k)$, n_{Lr}^0 and n_{Rr}^0 must be expressed as

$$n_{Lr}^0 = n_{Lr}^{\prime 0} + f_r(a_k) \quad \text{and} \quad n_{Rr}^0 = n_{Rr}^{\prime 0} + f_r(a_k), \quad (15)$$

respectively. Here, $n_{Lr}^{\prime 0}$ and $n_{Rr}^{\prime 0}$ are some constants irrelevant to a_k and $f_r(a_k)$ is a function of a_k .

3.1 An example

Let us calculate n_{Lr}^0 and n_{Rr}^0 , and verify the relations (15), using an $SU(3)$ gauge theory on $M^4 \times S^1/Z_2$.

On 5D space-time, Ψ is expressed as

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \quad (16)$$

where ψ_L and ψ_R are components containing 4D left-handed fermions and 4D right-handed ones, respectively.

The equation (12) is divided into two parts,

$$i\bar{\sigma}^\mu D_\mu \psi_L - D_y \psi_R = 0, \quad i\sigma^\mu D_\mu \psi_R + D_y \psi_L = 0, \quad (17)$$

where $D_y \equiv \partial_y + igA_y$. For ψ_L and ψ_R , the BCs are given by

$$\psi_L(x, -y) = \eta^0 P_0 \psi_L(x, y), \quad \psi_L(x, 2\pi R - y) = \eta^1 P_1 \psi_L(x, y), \quad (18)$$

$$\psi_R(x, -y) = -\eta^0 P_0 \psi_R(x, y), \quad \psi_R(x, 2\pi R - y) = -\eta^1 P_1 \psi_R(x, y), \quad (19)$$

where P_0 and P_1 are the representation matrices for the Z_2 transformation $y \rightarrow -y$ and the Z_2 transformation $y \rightarrow 2\pi R - y$, respectively. η^0 and η^1 are the intrinsic Z_2 parities for the left-handed component. Note that Z_2 parities for the right-handed one are opposite to those of the left-handed one. For the gauge bosons, the BCs are given by

$$A_\mu(x, -y) = P_0 A_\mu(x, y) P_0^\dagger, \quad A_\mu(x, 2\pi R - y) = P_1 A_\mu(x, y) P_1^\dagger, \quad (20)$$

$$A_y(x, -y) = -P_0 A_y(x, y) P_0^\dagger, \quad A_y(x, 2\pi R - y) = -P_1 A_y(x, y) P_1^\dagger. \quad (21)$$

We take the representation matrices,

$$P_0 = \text{diag}(1, 1, -1), \quad P_1 = \text{diag}(1, 1, -1). \quad (22)$$

Then $SU(3)$ is broken down to $SU(2) \times U(1)$. We consider the fermion with the representation $\mathbf{3}$ of $SU(3)$ and $(\eta^0, \eta^1) = (1, 1)$. Then, ψ_L and ψ_R are expanded as

$$\psi_L = \begin{pmatrix} \sum_{n=0}^{\infty} \psi_{Ln}^1(x) \cos \frac{n}{R} y \\ \sum_{n=0}^{\infty} \psi_{Ln}^2(x) \cos \frac{n}{R} y \\ \sum_{n=1}^{\infty} \psi_{Ln}^3(x) \sin \frac{n}{R} y \end{pmatrix}, \quad \psi_R = \begin{pmatrix} \sum_{n=1}^{\infty} \psi_{Rn}^1(x) \sin \frac{n}{R} y \\ \sum_{n=1}^{\infty} \psi_{Rn}^2(x) \sin \frac{n}{R} y \\ \sum_{n=0}^{\infty} \psi_{Rn}^3(x) \cos \frac{n}{R} y \end{pmatrix}. \quad (23)$$

After a suitable $SU(2)$ gauge transformation, the vacuum expectation value (VEV) of A_y is parameterized as

$$\langle A_y \rangle = \frac{-i}{gR} \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ -a & 0 & 0 \end{pmatrix}, \quad (24)$$

where $2\pi a$ is the Wilson line phase. From the periodicity, we limit the domain of definition for a as $0 \leq a < 1$. In case with $a \neq 0$, $SU(2)$ is broken down to $U(1)$, and then every 4D fermion becomes a singlet.

Inserting (23) and (24) into (17), we obtain a set of 4D equations,

$$i\bar{\sigma}^\mu D_\mu \psi_{L0}^1 - \frac{a}{R} \psi_{R0}^3 = 0, \quad i\sigma^\mu D_\mu \psi_{R0}^3 - \frac{a}{R} \psi_{L0}^1 = 0, \quad (25)$$

$$i\bar{\sigma}^\mu D_\mu \psi_{L0}^2 = 0, \quad (26)$$

$$i\bar{\sigma}^\mu D_\mu \psi_{Ln}^1 - \frac{n}{R} \psi_{Rn}^1 - \frac{a}{R} \psi_{Rn}^3 = 0 \quad (n = 1, 2, \dots), \quad (27)$$

$$i\bar{\sigma}^\mu D_\mu \psi_{Ln}^2 - \frac{n}{R} \psi_{Rn}^2 = 0 \quad (n = 1, 2, \dots), \quad (28)$$

$$i\bar{\sigma}^\mu D_\mu \psi_{Ln}^3 + \frac{n}{R} \psi_{Rn}^3 + \frac{a}{R} \psi_{Rn}^1 = 0 \quad (n = 1, 2, \dots), \quad (29)$$

$$i\sigma^\mu D_\mu \psi_{Rn}^1 - \frac{n}{R} \psi_{Ln}^1 + \frac{a}{R} \psi_{Ln}^3 = 0 \quad (n = 1, 2, \dots), \quad (30)$$

$$i\sigma^\mu D_\mu \psi_{Rn}^2 - \frac{n}{R} \psi_{Ln}^2 = 0 \quad (n = 1, 2, \dots), \quad (31)$$

$$i\sigma^\mu D_\mu \psi_{Rn}^3 + \frac{n}{R} \psi_{Ln}^3 - \frac{a}{R} \psi_{Ln}^1 = 0 \quad (n = 1, 2, \dots). \quad (32)$$

Using the equations (27), (29), (30) and (32), we derive a set of 4D equations,

$$i\bar{\sigma}^\mu D_\mu (\psi_{Ln}^1 + \psi_{Ln}^3) - \frac{n-a}{R} (\psi_{Rn}^1 - \psi_{Rn}^3) = 0 \quad (n = 1, 2, \dots), \quad (33)$$

$$i\bar{\sigma}^\mu D_\mu (\psi_{Ln}^1 - \psi_{Ln}^3) - \frac{n+a}{R} (\psi_{Rn}^1 + \psi_{Rn}^3) = 0 \quad (n = 1, 2, \dots), \quad (34)$$

$$i\sigma^\mu D_\mu (\psi_{Rn}^1 + \psi_{Rn}^3) - \frac{n+a}{R} (\psi_{Ln}^1 - \psi_{Ln}^3) = 0 \quad (n = 1, 2, \dots), \quad (35)$$

$$i\sigma^\mu D_\mu (\psi_{Rn}^1 - \psi_{Rn}^3) - \frac{n-a}{R} (\psi_{Ln}^1 + \psi_{Ln}^3) = 0 \quad (n = 1, 2, \dots). \quad (36)$$

From (25), ψ_{L0}^1 and ψ_{R0}^3 form a 4D Dirac fermion. In the same way, we find that $(\psi_{Ln}^2, \psi_{Rn}^2)$, $(\psi_{Ln}^1 + \psi_{Ln}^3, \psi_{Rn}^1 - \psi_{Rn}^3)$ and $(\psi_{Ln}^1 - \psi_{Ln}^3, \psi_{Rn}^1 + \psi_{Rn}^3)$ form 4D Dirac fermions for $n = 1, 2, \dots$ from (28) and (31), (33) and (36), and (34) and (35), respectively.

The numbers of 4D massless fermions are evaluated as

$$n_L^0 = 1 + \delta_{0a}, \quad n_R^0 = \delta_{0a}, \quad (37)$$

where δ_{0a} represents the Kronecker delta. From (37), we confirm that the fermion number n ($\equiv n_L^0 - n_R^0 = 1$) does not depend on the Wilson line phase. The mass spectrum for 4D fermions in this model is depicted as Figure 1. Strictly speaking, the figure describes the case with $0 < a < 1/2$.

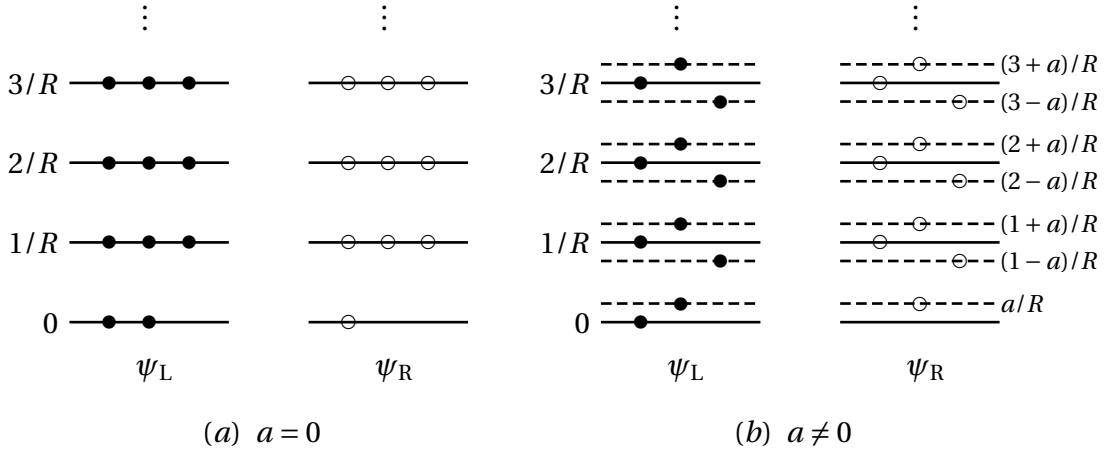


Figure 1: Mass spectrum of 4D fermions. The filled circles and the open ones represent left-handed fermions and right-handed ones, respectively.

3.2 Hidden quantum-mechanical supersymmetry

We explore a physics behind the feature that the fermion numbers are independent of the Wilson line phases.

From Figure 1, we anticipate that the feature originates from a hidden quantum-mechanical SUSY. Here, the quantum-mechanical SUSY means the symmetry generated by the supercharge Q that satisfies the algebraic relations [20, 21],

$$H = Q^2, \quad \{Q, (-1)^F\} = 0, \quad ((-1)^F)^2 = I, \quad (38)$$

where H , F and I are the Hamiltonian, the “fermion” number operator and the identity operator, respectively. The eigenvalue of $(-1)^F$ is given by $+1$ for “bosonic” states and -1 for “fermionic” states, and $\text{Tr}(-1)^F$ is a topological invariant, called the Witten index [22].

It is known that the system with 4D fermions has the hidden SUSY where the 4D Dirac operator plays the role of Q [23, 24]. The correspondences are given by

$$Q \leftrightarrow i\gamma^\mu D_\mu = \begin{pmatrix} 0 & i\sigma^\mu D_\mu \\ i\bar{\sigma}^\mu D_\mu & 0 \end{pmatrix}, \quad (-1)^F \leftrightarrow \gamma_5, \quad (39)$$

where γ_5 is the chirality operator defined by $\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$. The trace of γ_5 is the index of the 4D Dirac operator, and the following relations hold,

$$\begin{aligned} \text{Tr} \gamma_5|_r &= n_{Rr}^0[A_\mu] - n_{Lr}^0[A_\mu] = \dim \ker \sigma^\mu D_\mu|_r - \dim \ker \bar{\sigma}^\mu D_\mu|_r \\ &= \frac{1}{32\pi^2} \int \text{tr}_r \epsilon_{\mu\nu\alpha\beta} F^{\mu\nu} F^{\alpha\beta} d^4x, \end{aligned} \quad (40)$$

from the Atiyah-Singer index theorem. Here, $n_{Rr}^0[A_\mu]$ and $n_{Lr}^0[A_\mu]$ are the numbers of normalizable solutions (massless fermions) satisfying $i\sigma^\mu D_\mu \psi_{Rr} = 0$ and $i\bar{\sigma}^\mu D_\mu \psi_{Lr} = 0$, respectively. Note that massive fermions exist in pairs (ψ_{Rr} and ψ_{Lr}) and do not contribute to the index. The integral quantity in (40) is called the Pontryagin number, and it is deeply connected to the configuration of gauge bosons A_μ on 4D space-time.

It is pointed out that higher-dimensional theories with extra dimensions also possess the hidden SUSY [25, 26]. In the system with a 5D fermion, the Dirac operator relating the fifth-coordinate plays the role of Q and there are the correspondences,

$$Q \leftrightarrow \tilde{D}_y = \begin{pmatrix} 0 & D_y \\ -D_y & 0 \end{pmatrix}, \quad (-1)^F \leftrightarrow \tilde{\Gamma} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (41)$$

Note that $\tilde{\Gamma} = -\gamma_5$. The counterpart of the Witten index is given by

$$\text{Tr } \tilde{\Gamma} \Big|_r = \tilde{n}_{Rr}^0(a) - \tilde{n}_{Lr}^0(a), \quad (42)$$

where $\tilde{n}_{Rr}^0(a)$ and $\tilde{n}_{Lr}^0(a)$ are the numbers of eigenfunctions, that satisfy the equations,

$$\tilde{D}_y \begin{pmatrix} 0 \\ \psi_R \end{pmatrix} = \begin{pmatrix} D_y \psi_R \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (43)$$

and

$$\tilde{D}_y \begin{pmatrix} \psi_L \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -D_y \psi_L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (44)$$

respectively. Note that the eigenvalue equations are given by $D_y \psi_R = \lambda \psi_R$ and $D_y \psi_L = \lambda' \psi_L$, eigenfunctions with non-zero eigenvalues exist in pairs, which correspond to 4D massive fermions as seen from (17), and they do not contribute to the index. From the equations (17), there is a one-to-one correspondence such that

$$D_y \psi_R = 0 \leftrightarrow i\bar{\sigma}^\mu D_\mu \psi_L = 0, \quad D_y \psi_L = 0 \leftrightarrow i\sigma^\mu D_\mu \psi_R = 0. \quad (45)$$

Let us generalize to a system with a fermion on a higher-dimensional space-time. For the case that $D = 2n$ ($n = 3, 4, \dots$), the correspondences are given by

$$Q \leftrightarrow \tilde{D} \equiv \sum_{m=5}^D i\Gamma^m D_m, \quad (-1)^F \leftrightarrow \tilde{\Gamma} \equiv -\Gamma_{D+1}, \quad (46)$$

where Γ_{D+1} is the chirality operator defined by $\Gamma_{D+1} = (-i)^{n+1} \Gamma^0 \Gamma^1 \dots \Gamma^D$.

For the case that $D = 2n + 1$ ($n = 2, 3, \dots$), the correspondences are given by

$$Q \leftrightarrow \tilde{D} \equiv U^\dagger \sum_{m=5}^D i\Gamma^m D_m U, \quad (-1)^F \leftrightarrow \tilde{\Gamma} \equiv i\Gamma^D, \quad (47)$$

where U is the unitary matrix that satisfies the relation $i\Gamma^D = U^\dagger \Gamma^1 U$, and $i\Gamma^D$ is a diagonal matrix with the same form as the chirality operator on $D(=2n)$ -dimensions up to a sign factor.

The equation (12) is written by

$$i\Gamma^\mu D_\mu \Psi + \sum_{m=5}^D i\Gamma^m D_m \Psi = 0. \quad (48)$$

For the case that $D = 2n + 1$, after the unitary transformation $\Gamma'^M = U^\dagger \Gamma^M U$ and $\Psi' = U^\dagger \Psi$ is performed, Γ'^M and Ψ' are again denoted as Γ^M and Ψ in (48). The counterpart of the Witten index is given by

$$\text{Tr } \tilde{\Gamma}|_r = \tilde{n}_{Rr}^0(a_k) - \tilde{n}_{Lr}^0(a_k), \quad (49)$$

where $\tilde{n}_{Rr}^0(a_k)$ and $\tilde{n}_{Lr}^0(a_k)$ are the numbers of eigenfunctions, which satisfy $\tilde{D}\psi_R = 0$ and $\tilde{D}\psi_L = 0$, respectively. $\text{Tr } \tilde{\Gamma}|_r$ is a topological invariant. As seen from the Atiyah-Singer index theorem relating the Dirac operator for extra-dimensions, fermion numbers are deeply connected to the topological structure on K^{D-4} including the configurations of A_m on K^{D-4} . From this point of view, the family number has been studied in the Kaluza-Klein theory [27] and superstring theory [28].

From (48), there is a one-to-one correspondence such that

$$\tilde{D}\psi_R = 0 \leftrightarrow i\Gamma^\mu D_\mu \psi_L = 0, \quad \tilde{D}\psi_L = 0 \leftrightarrow i\Gamma^\mu D_\mu \psi_R = 0. \quad (50)$$

Here ψ_R and ψ_L are a 4D right-handed spinor component and a 4D left-handed one in Ψ , that are eigenspinors of the 4D chirality operator $\tilde{\Gamma}_5 \equiv i\Gamma^0\Gamma^1\Gamma^2\Gamma^3$ whose eigenvalues are 1 and -1 , respectively. Note that components with a different 4D chirality involve each other through the equation (48), because $\tilde{\Gamma}_5$ is anti-commutable to $i\Gamma^\mu D_\mu$ but it is commutable to \tilde{D} .

In our orbifold family unification models, the extra space is flat and the Wilson line phases determine the vacuum with $\langle F_{mn} \rangle = 0$ globally. Then, we derive the relation,

$$\text{Tr } \tilde{\Gamma}|_r = \tilde{n}_{Rr}^0(a_k) - \tilde{n}_{Lr}^0(a_k) = n_{Lr}^0 - n_{Rr}^0. \quad (51)$$

using the following relations coming from (50),

$$\tilde{n}_{Rr}^0(a_k) = n_{Lr}^0, \quad \tilde{n}_{Lr}^0(a_k) = n_{Rr}^0. \quad (52)$$

As a remnant of the topological invariance of $\text{Tr } \tilde{\Gamma}|_r$, we find that $n_r (= n_{Lr}^0 - n_{Rr}^0)$ is independent of the Wilson line phases. Hence, $N_r (= n_r - n_{\bar{r}})$ is also independent of the Wilson line phases.

4 Conclusions

We have studied the relationship between the family number of chiral fermions and the Wilson line phases, based on the orbifold family unification. We have found that flavor numbers are independent of the Wilson line phases relating extra-dimensional components of gauge bosons, as far as the SM gauge symmetry is respected. This feature originates from a hidden quantum-mechanical SUSY.

From our observation, the previous analyses [7, 9, 11], based on the assumption that the BCs are physical ones, are justified in the orbifold family unification. Concretely, even if the BCs are not physical, we can obtain the same result as that of the physical ones, because the family number is invariant under the change from the original BCs to the physical ones by singular gauge transformations.

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A Formulas based on equivalence relations

We present several formulas concerning the combination ${}_n C_l$, derived from the dynamical rearrangement and the feature that fermion numbers are independent of the Wilson line phases.

On S^1/Z_2 , we consider the representation matrices given by

$$P_0 = \text{diag}([+1]_{p_1}, [+1]_{p_2}, [-1]_{p_3}, [-1]_{p_4}), \quad (53)$$

$$P_1 = \text{diag}([+1]_{p_1}, [-1]_{p_2}, [+1]_{p_3}, [-1]_{p_4}), \quad (54)$$

where $[\pm 1]_{p_i}$ represents ± 1 for all p_i elements. Then, the following breakdown of $SU(N)$ gauge symmetry occurs:

$$SU(N) \rightarrow SU(p_1) \times SU(p_2) \times SU(p_3) \times SU(p_4) \times U(1)^{3-m}. \quad (55)$$

The Z_2 parities or BCs specified by integers $\{p_i\}$ are also denoted $[p_1; p_2, p_3; p_4]$.

After the breakdown of $SU(N)$, $[N, k]$ is decomposed as [7]

$$[N, k] = \sum_{l_1=0}^k \sum_{l_2=0}^{k-l_1} \sum_{l_3=0}^{k-l_1-l_2} ({}_{p_1} C_{l_1}, {}_{p_2} C_{l_2}, {}_{p_3} C_{l_3}, {}_{p_4} C_{l_4}), \quad (56)$$

where $p_4 = N - p_1 - p_2 - p_3$, $l_4 = k - l_1 - l_2 - l_3$, and we use ${}_p C_l$ instead of $[p, l]$. Our notation is that ${}_p C_l = 0$ for $l > p$ and $l < 0$.

The Z_2 parities of $({}_{p_1} C_{l_1}, {}_{p_2} C_{l_2}, {}_{p_3} C_{l_3}, {}_{p_4} C_{l_4})$ for 4D left-handed fermions are given by

$$\mathcal{P}_0 = (-1)^{l_3+l_4} \eta_k^0 = (-1)^{l_1+l_2} (-1)^k \eta_k^0 = (-1)^{l_1+l_2+\alpha}, \quad (57)$$

$$\mathcal{P}_1 = (-1)^{l_2+l_4} \eta_k^1 = (-1)^{l_1+l_3} (-1)^k \eta_k^1 = (-1)^{l_1+l_3+\beta}, \quad (58)$$

where the intrinsic Z_2 parities (η_k^0, η_k^1) take a value $+1$ or -1 by definition and are parameterized as $(-1)^k \eta_k^0 = (-1)^\alpha$ and $(-1)^k \eta_k^1 = (-1)^\beta$.

Zero modes for the left-handed fermions and the right-handed ones are picked out by operating the projection operators,

$$P^{(1,1)} = \frac{1 + \mathcal{P}_0}{2} \frac{1 + \mathcal{P}_1}{2} \quad \text{and} \quad P^{(-1,-1)} = \frac{1 - \mathcal{P}_0}{2} \frac{1 - \mathcal{P}_1}{2}, \quad (59)$$

respectively. Note that the intrinsic Z_2 parities for the right-handed fermions are opposite to those for the left-handed ones.

Then, the fermion number is given by

$$n = n_L^0 - n_R^0 = \sum_{l_1=0}^k \sum_{l_2=0}^{k-l_1} \sum_{l_3=0}^{k-l_1-l_2} (P^{(1,1)} - P^{(-1,-1)}) {}_{p_1} C_{l_1} {}_{p_2} C_{l_2} {}_{p_3} C_{l_3} {}_{p_4} C_{l_4}. \quad (60)$$

From the dynamical rearrangement, the following equivalence relations hold [18],

$$\begin{aligned} [p_1; p_2, p_3; p_4] &\sim [p_1 - 1; p_2 + 1, p_3 + 1; p_4 - 1] \quad (\text{for } p_1, p_4 \geq 1), \\ &\sim [p_1 + 1; p_2 - 1, p_3 - 1; p_4 + 1] \quad (\text{for } p_2, p_3 \geq 1). \end{aligned} \quad (61)$$

Using (61) and the feature that fermion numbers are independent of the Wilson line phases, the following formula is derived,

$$\begin{aligned} &\sum_{l_1=0}^k \sum_{l_2=0}^{k-l_1} \sum_{l_3=0}^{k-l_1-l_2} \left[(-1)^{l_1+l_2+\alpha} + (-1)^{l_1+l_3+\beta} \right] p_1 C_{l_1} p_2 C_{l_2} p_3 C_{l_3} p_4 C_{l_4} \\ &= \sum_{l_1=0}^k \sum_{l_2=0}^{k-l_1} \sum_{l_3=0}^{k-l_1-l_2} \left[(-1)^{l_1+l_2+\alpha} + (-1)^{l_1+l_3+\beta} \right] p_{1\mp 1} C_{l_1} p_{2\pm 1} C_{l_2} p_{3\pm 1} C_{l_3} p_{4\mp 1} C_{l_4}, \end{aligned} \quad (62)$$

where $p_4 = N - p_1 - p_2 - p_3$, $l_4 = k - l_1 - l_2 - l_3$, and we use the relation,

$$P^{(1,1)} - P^{(-1,-1)} = \frac{1}{2} (\mathcal{P}_0 + \mathcal{P}_1) = \frac{1}{2} \left[(-1)^{l_1+l_2+\alpha} + (-1)^{l_1+l_3+\beta} \right]. \quad (63)$$

Here and hereafter, we deal with the case that the inequality $p_i - 1 \geq 0$ is fulfilled in $p_{i-1} C_{l_i}$.

In the same way, the following formulas are derived from the feature of the fermion number on T^2/Z_2 ,

$$\begin{aligned} &\sum_{l_1=0}^k \sum_{l_2=0}^{k-l_1} \sum_{l_3=0}^{k-l_1-l_2} \sum_{l_4=0}^{k-l_1-l_2-l_3} \sum_{l_5=0}^{k-l_1-\dots-l_4} \sum_{l_6=0}^{k-l_1-\dots-l_5} \sum_{l_7=0}^{k-l_1-\dots-l_6} (P^{(1,1,1)} - P^{(-1,-1,-1)}) \\ &\quad \times p_1 C_{l_1} p_2 C_{l_2} p_3 C_{l_3} p_4 C_{l_4} p_5 C_{l_5} p_6 C_{l_6} p_7 C_{l_7} p_8 C_{l_8} \\ &= \sum_{l_1=0}^k \sum_{l_2=0}^{k-l_1} \sum_{l_3=0}^{k-l_1-l_2} \sum_{l_4=0}^{k-l_1-l_2-l_3} \sum_{l_5=0}^{k-l_1-\dots-l_4} \sum_{l_6=0}^{k-l_1-\dots-l_5} \sum_{l_7=0}^{k-l_1-\dots-l_6} (P^{(1,1,1)} - P^{(-1,-1,-1)}) \\ &\quad \times p_{1\mp 1} C_{l_1} p_{2\pm 1} C_{l_2} p_3 C_{l_3} p_4 C_{l_4} p_5 C_{l_5} p_6 C_{l_6} p_{7\pm 1} C_{l_7} p_{8\mp 1} C_{l_8} \\ &= \sum_{l_1=0}^k \sum_{l_2=0}^{k-l_1} \sum_{l_3=0}^{k-l_1-l_2} \sum_{l_4=0}^{k-l_1-l_2-l_3} \sum_{l_5=0}^{k-l_1-\dots-l_4} \sum_{l_6=0}^{k-l_1-\dots-l_5} \sum_{l_7=0}^{k-l_1-\dots-l_6} (P^{(1,1,1)} - P^{(-1,-1,-1)}) \\ &\quad \times p_1 C_{l_1} p_{2\mp 1} C_{l_2} p_{3\pm 1} C_{l_3} p_4 C_{l_4} p_5 C_{l_5} p_{6\pm 1} C_{l_6} p_{7\mp 1} C_{l_7} p_8 C_{l_8} \\ &= \sum_{l_1=0}^k \sum_{l_2=0}^{k-l_1} \sum_{l_3=0}^{k-l_1-l_2} \sum_{l_4=0}^{k-l_1-l_2-l_3} \sum_{l_5=0}^{k-l_1-\dots-l_4} \sum_{l_6=0}^{k-l_1-\dots-l_5} \sum_{l_7=0}^{k-l_1-\dots-l_6} (P^{(1,1,1)} - P^{(-1,-1,-1)}) \\ &\quad \times p_1 C_{l_1} p_{2\mp 1} C_{l_2} p_3 C_{l_3} p_{4\pm 1} C_{l_4} p_{5\pm 1} C_{l_5} p_6 C_{l_6} p_{7\mp 1} C_{l_7} p_8 C_{l_8} \\ &= \sum_{l_1=0}^k \sum_{l_2=0}^{k-l_1} \sum_{l_3=0}^{k-l_1-l_2} \sum_{l_4=0}^{k-l_1-l_2-l_3} \sum_{l_5=0}^{k-l_1-\dots-l_4} \sum_{l_6=0}^{k-l_1-\dots-l_5} \sum_{l_7=0}^{k-l_1-\dots-l_6} (P^{(1,1,1)} - P^{(-1,-1,-1)}) \\ &\quad \times p_1 C_{l_1} p_2 C_{l_2} p_{3\pm 1} C_{l_3} p_{4\mp 1} C_{l_4} p_{5\mp 1} C_{l_5} p_{6\pm 1} C_{l_6} p_7 C_{l_7} p_8 C_{l_8}, \end{aligned} \quad (64)$$

where $p_8 = N - p_1 - p_2 - \dots - p_7$ and $l_8 = k - l_1 - l_2 - \dots - l_7$. $P^{(a,b,c)}$ are the projection operators that pick out the Z_2 parities $(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2) = (a, b, c)$, defined by

$$P^{(a,b,c)} \equiv \frac{1 + a\mathcal{P}_0}{2} \frac{1 + b\mathcal{P}_1}{2} \frac{1 + c\mathcal{P}_2}{2}. \quad (65)$$

Here, a , b and c take 1 or -1 . \mathcal{P}_0 , \mathcal{P}_1 and \mathcal{P}_2 are given by

$$\mathcal{P}_0 = (-1)^{l_5+l_6+l_7+l_8} \eta_k^0 = (-1)^{l_1+l_2+l_3+l_4} (-1)^k \eta_k^0 = (-1)^{l_1+l_2+l_3+l_4+\alpha}, \quad (66)$$

$$\mathcal{P}_1 = (-1)^{l_3+l_4+l_7+l_8} \eta_k^1 = (-1)^{l_1+l_2+l_5+l_6} (-1)^k \eta_k^1 = (-1)^{l_1+l_2+l_5+l_6+\beta}, \quad (67)$$

$$\mathcal{P}_2 = (-1)^{l_2+l_4+l_6+l_8} \eta_k^2 = (-1)^{l_1+l_3+l_5+l_7} (-1)^k \eta_k^2 = (-1)^{l_1+l_3+l_5+l_7+\gamma}, \quad (68)$$

where α , β and γ take 0 or 1. Using (66), (67) and (68), $P^{(1,1,1)} - P^{(-1,-1,-1)}$ is calculated as

$$\begin{aligned} & P^{(1,1,1)} - P^{(-1,-1,-1)} \\ &= \frac{1}{4} \left[(-1)^{l_1+l_2+l_3+l_4+\alpha} + (-1)^{l_1+l_2+l_5+l_6+\beta} \right. \\ & \quad \left. + (-1)^{l_1+l_3+l_5+l_7+\gamma} + (-1)^{l_1+l_4+l_6+l_7+\alpha+\beta+\gamma} \right]. \end{aligned} \quad (69)$$

The following formulas are derived from the feature of the fermion numbers relating representations $p_1 C_{l_1}$ and $(p_1 C_{l_1}, p_2 C_{l_2})$,

$$\begin{aligned} & \sum_{l_2=0}^{k-l_1} \sum_{l_3=0}^{k-l_1-l_2} \sum_{l_4=0}^{k-l_1-l_2-l_3} \sum_{l_5=0}^{k-l_1-\dots-l_4} \sum_{l_6=0}^{k-l_1-\dots-l_5} \sum_{l_7=0}^{k-l_1-\dots-l_6} (P^{(1,1,1)} - P^{(-1,-1,-1)}) \\ & \quad \times p_2 C_{l_2} p_3 C_{l_3} p_4 C_{l_4} p_5 C_{l_5} p_6 C_{l_6} p_7 C_{l_7} p_8 C_{l_8} \\ &= \sum_{l_2=0}^{k-l_1} \sum_{l_3=0}^{k-l_1-l_2} \sum_{l_4=0}^{k-l_1-l_2-l_3} \sum_{l_5=0}^{k-l_1-\dots-l_4} \sum_{l_6=0}^{k-l_1-\dots-l_5} \sum_{l_7=0}^{k-l_1-\dots-l_6} (P^{(1,1,1)} - P^{(-1,-1,-1)}) \\ & \quad \times p_2 \mp 1 C_{l_2} p_3 \pm 1 C_{l_3} p_4 C_{l_4} p_5 C_{l_5} p_6 \pm 1 C_{l_6} p_7 \mp 1 C_{l_7} p_8 C_{l_8} \\ &= \sum_{l_2=0}^{k-l_1} \sum_{l_3=0}^{k-l_1-l_2} \sum_{l_4=0}^{k-l_1-l_2-l_3} \sum_{l_5=0}^{k-l_1-\dots-l_4} \sum_{l_6=0}^{k-l_1-\dots-l_5} \sum_{l_7=0}^{k-l_1-\dots-l_6} (P^{(1,1,1)} - P^{(-1,-1,-1)}) \\ & \quad \times p_2 \mp 1 C_{l_2} p_3 C_{l_3} p_4 \pm 1 C_{l_4} p_5 \pm 1 C_{l_5} p_6 C_{l_6} p_7 \mp 1 C_{l_7} p_8 C_{l_8} \\ &= \sum_{l_2=0}^{k-l_1} \sum_{l_3=0}^{k-l_1-l_2} \sum_{l_4=0}^{k-l_1-l_2-l_3} \sum_{l_5=0}^{k-l_1-\dots-l_4} \sum_{l_6=0}^{k-l_1-\dots-l_5} \sum_{l_7=0}^{k-l_1-\dots-l_6} (P^{(1,1,1)} - P^{(-1,-1,-1)}) \\ & \quad \times p_2 C_{l_2} p_3 \pm 1 C_{l_3} p_4 \mp 1 C_{l_4} p_5 \mp 1 C_{l_5} p_6 \pm 1 C_{l_6} p_7 C_{l_7} p_8 C_{l_8} \end{aligned} \quad (70)$$

and

$$\begin{aligned} & \sum_{l_3=0}^{k-l_1-l_2} \sum_{l_4=0}^{k-l_1-l_2-l_3} \sum_{l_5=0}^{k-l_1-\dots-l_4} \sum_{l_6=0}^{k-l_1-\dots-l_5} \sum_{l_7=0}^{k-l_1-\dots-l_6} (P^{(1,1,1)} - P^{(-1,-1,-1)}) \\ & \quad \times p_3 C_{l_3} p_4 C_{l_4} p_5 C_{l_5} p_6 C_{l_6} p_7 C_{l_7} p_8 C_{l_8} \\ &= \sum_{l_3=0}^{k-l_1-l_2} \sum_{l_4=0}^{k-l_1-l_2-l_3} \sum_{l_5=0}^{k-l_1-\dots-l_4} \sum_{l_6=0}^{k-l_1-\dots-l_5} \sum_{l_7=0}^{k-l_1-\dots-l_6} (P^{(1,1,1)} - P^{(-1,-1,-1)}) \\ & \quad \times p_3 \pm 1 C_{l_3} p_4 \mp 1 C_{l_4} p_5 \mp 1 C_{l_5} p_6 \pm 1 C_{l_6} p_7 C_{l_7} p_8 C_{l_8}. \end{aligned} \quad (71)$$

Furthermore, by changing $(p_3, p_4, p_5, p_6, p_7, p_8)$ into $(p_7, p_8, p_3, p_4, p_5, p_6)$ in the ordering of the summation and relabeling $(p_7, p_8, p_3, p_4, p_5, p_6)$ as $(p_3, p_4, p_5, p_6, p_7, p_8)$, the following formulas are derived from the feature of the fermion numbers relating representations $(p_1 C_{l_1}, p_2 C_{l_2}, p_3 C_{l_3})$ and $(p_1 C_{l_1}, p_2 C_{l_2}, p_3 C_{l_3}, p_4 C_{l_4})$,

$$\sum_{l_4=0}^{k-l_1-l_2-l_3} \sum_{l_5=0}^{k-l_1-\dots-l_4} \sum_{l_6=0}^{k-l_1-\dots-l_5} \sum_{l_7=0}^{k-l_1-\dots-l_6} (P^{(1,1,1)} - P^{(-1,-1,-1)})$$

$$\begin{aligned}
& \times p_4 C_{l_4} p_5 C_{l_5} p_6 C_{l_6} p_7 C_{l_7} p_8 C_{l_8} \\
= & \sum_{l_4=0}^{k-l_1-l_2-l_3} \sum_{l_5=0}^{k-l_1-\dots-l_4} \sum_{l_6=0}^{k-l_1-\dots-l_5} \sum_{l_7=0}^{k-l_1-\dots-l_6} (P^{(1,1,1)} - P^{(-1,-1,-1)}) \\
& \times p_4 C_{l_4} p_{5\mp 1} C_{l_5} p_{6\pm 1} C_{l_6} p_{7\pm 1} C_{l_7} p_{8\mp 1} C_{l_8} \quad (72)
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{l_5=0}^{k-l_1-\dots-l_4} \sum_{l_6=0}^{k-l_1-\dots-l_5} \sum_{l_7=0}^{k-l_1-\dots-l_6} (P^{(1,1,1)} - P^{(-1,-1,-1)}) p_5 C_{l_5} p_6 C_{l_6} p_7 C_{l_7} p_8 C_{l_8} \\
= & \sum_{l_5=0}^{k-l_1-\dots-l_4} \sum_{l_6=0}^{k-l_1-\dots-l_5} \sum_{l_7=0}^{k-l_1-\dots-l_6} (P^{(1,1,1)} - P^{(-1,-1,-1)}) p_{5\mp 1} C_{l_5} p_{6\pm 1} C_{l_6} p_{7\pm 1} C_{l_7} p_{8\mp 1} C_{l_8}, \quad (73)
\end{aligned}$$

where $P^{(1,1,1)} - P^{(-1,-1,-1)}$ is given by

$$\begin{aligned}
& P^{(1,1,1)} - P^{(-1,-1,-1)} \\
= & \frac{1}{4} \left[(-1)^{l_1+l_2+l_5+l_6+\alpha} + (-1)^{l_1+l_2+l_7+l_8+\beta} \right. \\
& \left. + (-1)^{l_1+l_3+l_5+l_7+\gamma} + (-1)^{l_1+l_3+l_6+l_8+\alpha+\beta+\gamma} \right]. \quad (74)
\end{aligned}$$

In the same way, the following formulas are derived from the feature of the fermion number on T^2/Z_3 ,

$$\begin{aligned}
& \sum_{l_1=0}^k \sum_{l_2=0}^{k-l_1} \sum_{l_3=0}^{k-l_1-l_2} \sum_{l_4=0}^{k-l_1-l_2-l_3} \sum_{l_5=0}^{k-l_1-\dots-l_4} \sum_{l_6=0}^{k-l_1-\dots-l_5} \sum_{l_7=0}^{k-l_1-\dots-l_6} \sum_{l_8=0}^{k-l_1-\dots-l_7} (P^{(1,1)} - P^{(\omega,\omega)}) \\
& \times p_1 C_{l_1} p_2 C_{l_2} p_3 C_{l_3} p_4 C_{l_4} p_5 C_{l_5} p_6 C_{l_6} p_7 C_{l_7} p_8 C_{l_8} p_9 C_{l_9} \\
= & \sum_{l_1=0}^k \sum_{l_2=0}^{k-l_1} \sum_{l_3=0}^{k-l_1-l_2} \sum_{l_4=0}^{k-l_1-l_2-l_3} \sum_{l_5=0}^{k-l_1-\dots-l_4} \sum_{l_6=0}^{k-l_1-\dots-l_5} \sum_{l_7=0}^{k-l_1-\dots-l_6} \sum_{l_8=0}^{k-l_1-\dots-l_7} (P^{(1,1)} - P^{(\omega,\omega)}) \\
& \times p_{1\pm 1} C_{l_1} p_2 C_{l_2} p_{3\mp 1} C_{l_3} p_{4\mp 1} C_{l_4} p_{5\pm 1} C_{l_5} p_6 C_{l_6} p_7 C_{l_7} p_{8\mp 1} C_{l_8} p_{9\pm 1} C_{l_9} \\
= & \sum_{l_1=0}^k \sum_{l_2=0}^{k-l_1} \sum_{l_3=0}^{k-l_1-l_2} \sum_{l_4=0}^{k-l_1-l_2-l_3} \sum_{l_5=0}^{k-l_1-\dots-l_4} \sum_{l_6=0}^{k-l_1-\dots-l_5} \sum_{l_7=0}^{k-l_1-\dots-l_6} \sum_{l_8=0}^{k-l_1-\dots-l_7} (P^{(1,1)} - P^{(\omega,\omega)}) \\
& \times p_{1\pm 1} C_{l_1} p_{2\mp 1} C_{l_2} p_3 C_{l_3} p_4 C_{l_4} p_{5\pm 1} C_{l_5} p_{6\mp 1} C_{l_6} p_{7\mp 1} C_{l_7} p_8 C_{l_8} p_{9\pm 1} C_{l_9} \\
= & \sum_{l_1=0}^k \sum_{l_2=0}^{k-l_1} \sum_{l_3=0}^{k-l_1-l_2} \sum_{l_4=0}^{k-l_1-l_2-l_3} \sum_{l_5=0}^{k-l_1-\dots-l_4} \sum_{l_6=0}^{k-l_1-\dots-l_5} \sum_{l_7=0}^{k-l_1-\dots-l_6} \sum_{l_8=0}^{k-l_1-\dots-l_7} (P^{(1,1)} - P^{(\omega,\omega)}) \\
& \times p_1 C_{l_1} p_{2\pm 1} C_{l_2} p_{3\mp 1} C_{l_3} p_{4\mp 1} C_{l_4} p_5 C_{l_5} p_{6\pm 1} C_{l_6} p_{7\pm 1} C_{l_7} p_{8\mp 1} C_{l_8} p_9 C_{l_9}, \quad (75)
\end{aligned}$$

where $p_9 = N - p_1 - p_2 - \dots - p_8$ and $l_9 = k - l_1 - l_2 - \dots - l_8$. $P^{(\xi,\eta)}$ are the projection operators that pick out the Z_3 elements $(\Theta_0, \Theta_1) = (\xi, \eta)$, defined by

$$P^{(\xi,\eta)} \equiv \frac{1 + \bar{\xi}\Theta_0 + \bar{\xi}^2\Theta_0^2}{3} \frac{1 + \bar{\eta}\Theta_1 + \bar{\eta}^2\Theta_1^2}{3}. \quad (76)$$

Here, ξ and η take 1, $\omega (= e^{2\pi i/3})$ or $\bar{\omega} (= e^{4\pi i/3})$, and $\bar{\xi}$ and $\bar{\eta}$ are the complex conjugates of ξ and η , respectively. Θ_0 and Θ_1 are given by

$$\Theta_0 = \omega^{l_4+l_5+l_6} \bar{\omega}^{l_7+l_8+l_9} \eta_k^0 = \omega^{l_1+l_2+l_3+2(l_4+l_5+l_6)} \bar{\omega}^k \eta_k^0 = \omega^{l_1+l_2+l_3+2(l_4+l_5+l_6)+\alpha}, \quad (77)$$

$$\Theta_1 = \omega^{l_2+l_5+l_8} \bar{\omega}^{l_3+l_6+l_9} \eta_k^1 = \omega^{l_1+l_4+l_7+2(l_2+l_5+l_8)} \bar{\omega}^k \eta_k^1 = \omega^{l_1+l_4+l_7+2(l_2+l_5+l_8)+\beta}, \quad (78)$$

where α and β take 0, 1 or 2.

In the same way, we can derive similar formulas from the feature of the fermion numbers relating representations ${}_{p_1}C_{l_1}$, $({}_{p_1}C_{l_1}, {}_{p_2}C_{l_2})$ and $({}_{p_1}C_{l_1}, {}_{p_2}C_{l_2}, {}_{p_3}C_{l_3})$ on T^2/Z_3 .

B Formulas based on independence from Wilson line phases

We derive other formulas concerning the combination ${}_n C_l$, counting the numbers of fermions irrelevant to the Wilson line phases and using the independence of fermion numbers from the Wilson line phases.

On S^1/Z_2 , we consider the representation matrices given by

$$P_0 = \text{diag}([+1]_p, [-1]_{N-p}), \quad P_1 = \text{diag}([+1]_p, [-1]_{N-p}). \quad (79)$$

Then, the following breakdown of $SU(N)$ gauge symmetry occurs:

$$SU(N) \rightarrow SU(p) \times SU(N-p) \times U(1)^{1-m}, \quad (80)$$

and $[N, k]$ is decomposed as

$$[N, k] = \sum_{l=0}^k ({}_p C_l, {}_{N-p} C_{k-l}). \quad (81)$$

The Z_2 parities of $({}_p C_l, {}_s C_{k-l})$ for 4D left-handed fermions are given and parameterized by

$$\mathcal{P}_0 = (-1)^{k-l} \eta_k^0 = (-1)^{l+\alpha}, \quad \mathcal{P}_1 = (-1)^{k-l} \eta_k^1 = (-1)^{l+\beta}, \quad (82)$$

where α and β take 0 or 1. Then, the fermion number is given by

$$n = n_L - n_R = \sum_{l=0}^k \frac{1}{2} \left[(-1)^{l+\alpha} + (-1)^{l+\beta} \right] {}_p C_l {}_{N-p} C_{k-l}. \quad (83)$$

The number of the Wilson line phases is $m \equiv \text{Min}(p, N-p)$ and, after a suitable $SU(p) \times SU(N-p)$ gauge transformation, $\langle A_y \rangle$ is parameterized as

$$\langle A_y \rangle = \frac{-i}{gR} \begin{pmatrix} \mathbf{0} & \Theta \\ -\Theta^T & \mathbf{0} \end{pmatrix}, \quad (84)$$

where Θ is the $p \times (N-p)$ matrix such that

$$\Theta = \begin{pmatrix} a_1 & & \mathbf{0} & \\ & a_2 & & \\ & & \ddots & \\ \mathbf{0} & & & a_m \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \quad (\text{for } p \geq N-p), \quad (85)$$

$$\Theta = \begin{pmatrix} a_1 & & \mathbf{0} & & 0 & \cdots & 0 \\ & a_2 & & & 0 & \cdots & 0 \\ & & \ddots & & \vdots & \ddots & \vdots \\ \mathbf{0} & & & a_m & 0 & \cdots & 0 \end{pmatrix} \quad (\text{for } p \leq N-p). \quad (86)$$

Here, $2\pi a_k$ ($k = 1, \dots, m; m \equiv \text{Min}(p, N-p)$) are the Wilson line phases.

For the fermion with $[N, 1]$, the number of components irrelevant to a_k is $p-m$ for $p \geq N-p$ and $N-p-m$ for $p \leq N-p$, and it is expressed as

$$\sum_{l'=0}^1 p-m C_{l'} N-p-m C_{1-l'} \Big|_{m=\text{Min}(p, N-p)}. \quad (87)$$

For the fermion with $[N, 2]$, the number of components irrelevant to a_k is $p-m C_2 + m$ for $p \geq N-p$ and $N-p-m C_2 + m$ for $p \leq N-p$, and it is expressed as

$$\sum_{l'=0}^2 p-m C_{l'} N-p-m C_{2-l'} + m C_1 \Big|_{m=\text{Min}(p, N-p)}, \quad (88)$$

where $m C_1$ comes from the components constructed from the tensor products between components in $[N, 1]$ with opposite values for the Wilson line phases, and the components corresponding $m C_1$ have odd Z_2 parities. In the iterative fashion, we find that the number of components irrelevant to a_k is given by

$$\sum_{n=0}^{[k/2]} \sum_{l'=0}^{k-2n} m C_n p-m C_{l'} N-p-m C_{k-2n-l'} \Big|_{m=\text{Min}(p, N-p)} \quad (89)$$

for the fermion with $[N, k]$.

Using the independence of fermion numbers from the Wilson line phases, the number of fermions is also calculated by counting the fermions irrelevant to a_k and the following formula is derived,

$$\sum_{l=0}^k (-1)^l {}_p C_l N-p C_{k-l} = \sum_{n=0}^{[k/2]} \sum_{l'=0}^{k-2n} (-1)^{n+l'} m C_n p-m C_{l'} N-p-m C_{k-2n-l'}, \quad (90)$$

where we use the assignment of Z_2 parities,

$$\mathcal{P}_0 = (-1)^{n+k-2n-l'} \eta_k^0 = (-1)^{n+l'+\alpha}, \quad \mathcal{P}_1 = (-1)^{n+k-2n-l'} \eta_k^1 = (-1)^{n+l'+\beta} \quad (91)$$

for the component corresponding $m C_n p-m C_{l'} N-p-m C_{k-2n-l'}$, and we take $\alpha = \beta$. The above formula (90) holds for the integer m satisfying $0 \leq m \leq \text{Min}(p, N-p)$, because the above argument is valid for m as the number of non-vanishing a_k even if some of a_k vanish.

Particularly, in case with $m = p$ and $m = N-p$, (90) reduces to

$$\sum_{l=0}^k (-1)^l {}_p C_l N-p C_{k-l} = \sum_{n=0}^{[k/2]} \sum_{l'=0}^{k-2n} (-1)^{n+l'} {}_p C_n N-2p C_{k-2n-l'}, \quad (92)$$

and

$$\sum_{l=0}^k (-1)^l {}_p C_l {}_{N-p} C_{k-l} = \sum_{n=0}^{\lfloor k/2 \rfloor} \sum_{l'=0}^{k-2n} (-1)^{n+l'} {}_{N-p} C_n {}_{2p-N} C_{k-2n-l'} , \quad (93)$$

respectively.

Based on the representation matrices (53) and (54), the following formula is derived,

$$\begin{aligned} & \sum_{l_1=0}^k \sum_{l_2=0}^{k-l_1} \sum_{l_3=0}^{k-l_1-l_2} \left[(-1)^{l_1+l_2+\alpha} + (-1)^{l_1+l_3+\beta} \right] {}_{p_1} C_{l_1} {}_{p_2} C_{l_2} {}_{p_3} C_{l_3} {}_{p_4} C_{l_4} \\ &= \sum_{n=0}^{\lfloor k/2 \rfloor} \sum_{n_1=0}^n \sum_{l'_1=0}^{k-2n} \sum_{l'_2=0}^{k-2n-l'_1} \sum_{l'_3=0}^{k-2n-l'_1-l'_2} \left[(-1)^{n+l'_1+l'_2+\alpha} + (-1)^{n+l'_1+l'_3+\beta} \right] \\ & \quad \times {}_{m_1} C_{n_1} {}_{m_2} C_{n-n_1} {}_{p_1-m_1} C_{l'_1} {}_{p_2-m_2} C_{l'_2} {}_{p_3-m_2} C_{l'_3} {}_{p_4-m_1} C_{l'_4} , \end{aligned} \quad (94)$$

where $p_4 = N - p_1 - p_2 - p_3$ and $l'_4 = k - 2n - l'_1 - l'_2 - l'_3$. The above formula (94) holds for the integers m_1 and m_2 satisfying $0 \leq m_1 \leq \text{Min}(p_1, p_4)$ and $0 \leq m_2 \leq \text{Min}(p_2, p_3)$.

In the same way, we can derive similar formulas using models on T^2/Z_M .

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