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# MULTIPLICITY OF THE LOWEST EIGENVALUE OF NON-COMMUTATIVE HARMONIC OSCILLATORS

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**Abstract.** The multiplicity of the lowest eigenvalue $E$ of the so-called non-commutative harmonic oscillator $Q(\alpha, \beta)$ is studied. It is shown that $E$ is simple for $\alpha$ and $\beta$ in some region.

## 1. Definition and main results

Recently special attention has been paid to studying the spectrum of self-adjoint operators with *non-commutative* coefficients. This is considered in not only mathematics but also physics experiments. A historically important model is the Dirac operator, and the Rabi model and the Jaynes–Cumming model are prevalent in cavity quantum electrodynamics (see [**HH12**] and references therein). The non-commutative harmonic oscillator is a quantum system defined by the Hamiltonian:

$$Q = Q(\alpha, \beta) = A \otimes \left(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2\right) + J \otimes \left(x\frac{d}{dx} + \frac{1}{2}\right), \tag{1.1}$$

where $A = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)$, $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and $\alpha, \beta > 0$ parameters with $\alpha\beta > 1$. Operator $Q$ defines a positive self-adjoint operator acting in the Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes L^2(\mathbb{R})$.

The non-commutative harmonic oscillator $Q$ has been introduced by Parmeggiani and Wakayama [**PW01, PW02a, PW02b, PW03**], and the spectral property of $Q$ is considered in [**Par04, Par06, Par08a**] from the pseudo-differential calculus point of view. It can be seen that $Q$ has purely discrete spectrum $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots \uparrow \infty$, where the eigenvalues are counted with multiplicity. One can define the so-called spectral zeta function associated with $Q$ as

$$\zeta_Q(s) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s}.$$

When $\alpha = \beta$, $Q$ is unitarily equivalent to the direct sum of harmonic oscillators, and $\lambda_{2m-1} = \lambda_{2m} = \sqrt{\alpha^2 - 1}(m + \frac{1}{2})$, and thus $\zeta_Q$ with $\alpha \ne \beta$ can be regarded as a $q$-deform of the Riemann zeta-function. Analytic properties of the spectral zeta-function are studied in [**IW05a, IW05b, IW07, KW06, KW07, KY09**]. Furthermore, it is also known that the set of *odd* eigenvectors of non-commutative harmonic oscillator is deeply related to the set of some

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solutions of the Heun differential equation [**IW05b, Och01, Och04**]:

$$\frac{\partial^2}{\partial w^2} f + \left( \frac{1-n}{w} + \frac{-n}{w-1} + \frac{n+3/2}{w-a} \right) \frac{\partial}{\partial w} f + \frac{-(3/2)nw - q}{w(w-1)(w-a)} f = 0,$$

where $n \in \mathbb{N} \cup \{0\}$, $a \in \mathbb{C}$ with $|a| < 1$ and $q \in \mathbb{C}$.

In this paper we concentrate on the study of the lowest eigenvalue $\lambda_1$ of $Q$. We set

$$E = \lambda_1 \tag{1.2}$$

and $p = -id/dx$. In particular, we are interested in determining the dimension of $\mathrm{Ker}(Q - E)$. The eigenvector associated with the lowest eigenvalue is called the ground state. In the case of $\alpha = \beta$, as is mentioned above, $Q$ can be diagonalized as $Q \cong \left(\begin{smallmatrix} h & 0 \\ 0 & h \end{smallmatrix}\right)$ with

$$h = \frac{1}{2} p^2 + \frac{\alpha^2 - 1}{2} x^2, \tag{1.3}$$

where $\cong$ denotes the unitary equivalence. Then all of the eigenvalues of $Q(\alpha, \alpha)$ are twofold degenerate. In particular, its lowest eigenvalue

$$E_0 = \tfrac{1}{2}\sqrt{\alpha^2 - 1} \tag{1.4}$$

is two-fold degenerate. In the general case, $\alpha \neq \beta$, the so-called Ichinose–Wakayama (IW) bound is established in [**IW07**] (see also [**Par10**]):

$$\left(j - \frac{1}{2}\right) \min\{\alpha, \beta\} \sqrt{\frac{\alpha\beta - 1}{\alpha\beta}} \leq \lambda_{2j-1} \leq \lambda_{2j} \leq \left(j - \frac{1}{2}\right) \max\{\alpha, \beta\} \sqrt{\frac{\alpha\beta - 1}{\alpha\beta}} \tag{1.5}$$

for $j \in \mathbb{N}$. By this inequality we see that the multiplicity of $E$ is at most two if $\beta < 3\alpha$ or $\alpha < 3\beta$.

Furthermore, beyond the above results one may expect that $E$ is simple for $\alpha \neq \beta$. In [**NNW02**] it is numerically shown that $E$ is simple for $\alpha \neq \beta$ and, in [**Par04**], the simplicity is proven but only for sufficiently large $\alpha\beta$. It is then mentioned in [**Par08b**, 8.3 Notes] that the determination of the multiplicity of the lowest eigenvalue should be explored.

In this paper we show that:

(a) $E$ is at most twofold degenerate for $(\alpha, \beta) \in (2, \infty) \times (2, \infty)$;

(b) $E$ is simple for some region of $\alpha$ and $\beta$.

In order to prove (a), we apply the method in [**Hir05**], where the so-called pull-through formula [**GJ68**] is a key ingredient. The second result (b) consists of two estimates. The first is for large $|\beta - \alpha|$ and the second for small $|\beta - \alpha|$ but $\alpha \neq \beta$. The first case is proven in a similar manner to (a) and the second by the regular perturbation theory of discrete spectrum.

Let $\mathscr{G} = \mathrm{Ker}(Q - E)$ be the set of ground states. Let $L_+ \subset L^2(\mathbb{R})$ (respectively $L_-$) be the set of even functions (respectively odd functions). We define $\mathcal{H}_\pm = \mathbb{C}^2 \otimes L_\pm$. Since $Q$ conserves the parity, $Q$ is reduced by $\mathcal{H}_\pm$. Set $Q\lceil_{\mathcal{H}_\pm} = Q_\pm$ and then $Q = Q_+ \oplus Q_-$.

THEOREM 1.1. *Suppose that* $(\alpha, \beta) \in (2, \infty) \times (2, \infty)$. *Then* $\dim \mathscr{G} \leq 2$ *and* $\mathscr{G} \subset \mathcal{H}_+$. *That is, the multiplicity of $E$ is at most two and ground states are even functions.*

We can furthermore show that $E$ is simple.

THEOREM 1.2. *Suppose that $\beta > \alpha > 2$ and*

$$\frac{1}{2} > \left(\frac{1}{2}\beta - E\right)^{-2} \frac{1}{\alpha^2 - 1} + \frac{1}{\alpha^2 - 1}. \tag{1.6}$$

*Then $E$ is simple.*

Condition (1.6) includes the implicit value $E$. Let

$$E_{\text{upper}} = \frac{\sqrt{\alpha\beta}\sqrt{\alpha\beta - 1}}{\alpha + \beta + |\alpha - \beta|((\alpha\beta - 1)^{1/4}/\sqrt{\alpha\beta}) \operatorname{Re} \rho},$$

where $\rho^2 = \sqrt{\alpha\beta - 1} - i$ with $\operatorname{Re} \rho > 0$, i.e. $\operatorname{Re} \rho = \sqrt{\sqrt{\alpha\beta}(\sqrt{\alpha\beta - 1} + 1)/2}$. The bound $E < E_{\text{upper}}$ holds. See [**Par08b**, Theorem 8.2.1] and [**Par10**]. This upper bound is better than the IW bound (1.5). Combining this with (1.6) we have the following corollary.

COROLLARY 1.3. *Suppose that $\beta > \alpha > 2$ and*

$$\frac{1}{2} > \left(\frac{1}{2}\beta - E_{\text{upper}}\right)^{-2} \frac{1}{\alpha^2 - 1} + \frac{1}{\alpha^2 - 1}. \tag{1.7}$$

*Then $E$ is simple.*

Theorem 1.2 does not hold for $(\alpha, \beta)$ in a neighborhood of the diagonal line on $\alpha - \beta$ plane. See Figure 1. However, we can show that $E$ is simple for $\alpha$ and $\beta$ in a suitable neighborhood of the diagonal line. We define $g_1, \dots, g_4$ by

$$g_1 = (\alpha - 1)^{-1}\left(3 + \frac{\sqrt{3}}{\sqrt{\alpha^2 - 1}}\right), \tag{1.8}$$

$$g_2 = \frac{\sqrt{\alpha^2 - 1}}{2\left|\sqrt{\alpha^2 - 1} - \lambda_2\right|} g_1^2, \tag{1.9}$$

$$g_3 = \frac{\alpha}{2\sqrt{\alpha^2 - 1}}, \tag{1.10}$$

$$g_4 = \frac{(\sqrt{\alpha^2 - 1})^{3/2}}{4\alpha^{3/2}}. \tag{1.11}$$

THEOREM 1.4. *Let $\varepsilon = \beta - \alpha$. Assume that $\beta > \alpha > 1$, $\sqrt{\beta^2 - 1} \leq 3\sqrt{\alpha^2 - 1}$ and $\varepsilon^2 g_2 < 1/2$. Let*

$$\kappa(\varepsilon) = E_0 g_1^2 + \varepsilon g_2(E_0 g_1 + g_3 + g_4) + \varepsilon^2 2E_0 g_1^2 g_2 + \varepsilon^3 2E_0 g_1 g_2^2, \tag{1.12}$$

$$\ell(\varepsilon) = (1 - \varepsilon^2 g_2)\sqrt{1 - 2\varepsilon^2 g_2^2}. \tag{1.13}$$

*Then*

$$|\lambda_1 - \lambda_2| \geq \frac{2\varepsilon}{\ell(\varepsilon)}(g_4 - \varepsilon\kappa(\varepsilon)). \tag{1.14}$$

*In particular when $\varepsilon\kappa(\varepsilon) < g_4$, $E$ is simple.*

Note that we know the bound $\lambda_2 \leq (\beta/2)(\sqrt{\alpha\beta - 1}/\sqrt{\alpha\beta})$ by the IW bound. Then, the region of $\alpha$, $\beta$ satisfying $\varepsilon\kappa(\varepsilon) < g_4$, includes a wedge-shaped region illustrated in Figure 2, where we also drew the case of $\alpha > \beta$.

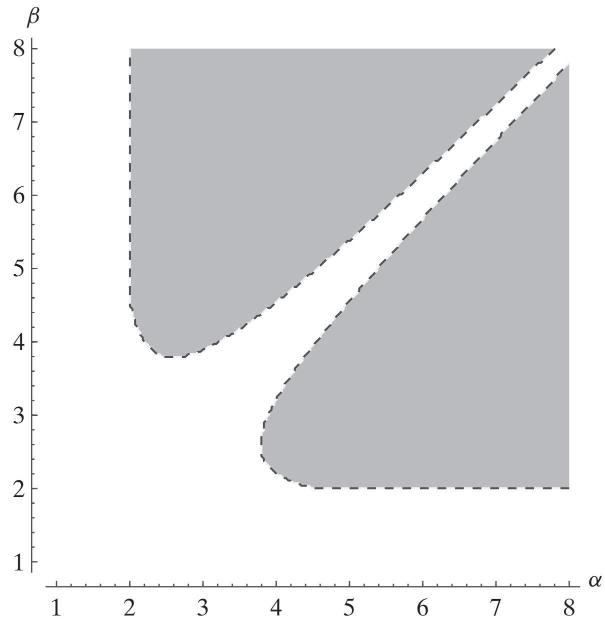


FIGURE 1. The region satisfying (1.7). (The case of $\alpha > \beta$ is also drawn.)

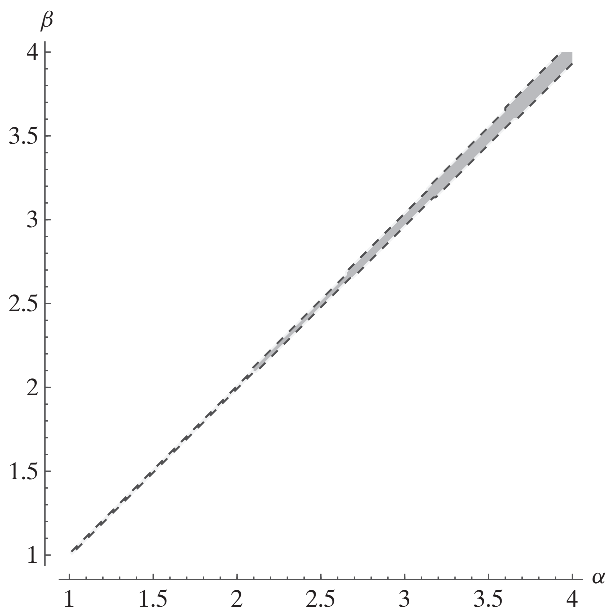


FIGURE 2. A wedge-shaped region included in the region satisfying $\varepsilon\kappa(\varepsilon) < g_4$.

## 2. Proofs of theorems

### *2.1. Proof of Theorem 1.1*

In the following we omit the symbol $\otimes$ for the notational simplicity, and we can suppose that $\alpha < \beta$ without loss of generality. Let

$$N = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}(p^2 + x^2 - 1).$$

The spectrum of $N$ is $\sigma(N) = \{0\} \cup \mathbb{N}$ and the multiplicity of each eigenvalue is two. Let $a = \frac{1}{\sqrt{2}}(x + ip)$ and $a^* = \frac{1}{\sqrt{2}}(x - ip)$. They satisfy canonical commutation relations $[a, a^*] = 1$ and $[a, a] = [a^*, a^*] = 0$, and we have $a^*a = N$.

*Proof of Theorem 1.1.* In terms of $a$ and $a^*$, the operator $Q$ can be realized as

$$Q = a^*Aa + \tfrac{1}{2}A + \tfrac{1}{2}(aJa - a^*Ja^*). \tag{2.1}$$

Let $\varphi_{\mathrm{g}} \in \mathscr{G}$. We have $(Q - E)a\varphi_{\mathrm{g}} = [Q, a]\varphi_{\mathrm{g}}$. Since $[Q, a] = -Aa + Ja^*$ by canonical commutation relation, we have $(Q - E)a\varphi_{\mathrm{g}} = (-Aa + Ja^*)\varphi_{\mathrm{g}}$ and then $(Q - E + A)a\varphi_{\mathrm{g}} = Ja^*\varphi_{\mathrm{g}}$. Note that $Q - E + A \geq \alpha > 0$. We have

$$a\varphi_{\mathrm{g}} = (Q - E + A)^{-1}Ja^*\varphi_{\mathrm{g}}. \tag{2.2}$$

Taking the norm on both sides above, we have

$$\|a\varphi_{\mathrm{g}}\|^2 \leq \frac{1}{\alpha^2}\|a^*\varphi_{\mathrm{g}}\|^2. \tag{2.3}$$

Since $\|a\varphi_{\mathrm{g}}\|^2 = \langle\varphi_{\mathrm{g}}, N\varphi_{\mathrm{g}}\rangle$ and $\|a^*\varphi_{\mathrm{g}}\|^2 = \langle\varphi_{\mathrm{g}}, N\varphi_{\mathrm{g}}\rangle + \|\varphi_{\mathrm{g}}\|^2$, we see that

$$\langle\varphi_{\mathrm{g}}, N\varphi_{\mathrm{g}}\rangle \leq \frac{1}{\alpha^2 - 1}\|\varphi_{\mathrm{g}}\|^2. \tag{2.4}$$

Let $\mathrm{P}_\Omega$ be the projection onto $\ker N = \ker a$. Note that $N + \mathrm{P}_\Omega \geq 1$. Let $\mathscr{G} = \mathscr{G}_+ \oplus \mathscr{G}_-$, where $\mathscr{G}_\pm = \mathscr{G} \cap \mathcal{H}_\pm$. Let $\mathrm{P}^\pm$ be the projection onto $\mathscr{G}_\pm$. Then, by (2.4), we have

$$\mathrm{P}^+\mathrm{P}_\Omega\mathrm{P}^+ \geq \mathrm{P}^+(1 - N)\mathrm{P}^+ \geq \left(1 - \frac{1}{\alpha^2 - 1}\right)\mathrm{P}^+. \tag{2.5}$$

Taking the trace of both sides, we have

$$2 \geq \mathrm{Tr}(\mathrm{P}^+\mathrm{P}_\Omega\mathrm{P}^+) \geq \frac{\alpha^2 - 2}{\alpha^2 - 1}\,\mathrm{Tr}\,\mathrm{P}^+. \tag{2.6}$$

Thus, we have the bound

$$\dim\ker\mathrm{P}^+ \leq 2\frac{\alpha^2 - 1}{\alpha^2 - 2}. \tag{2.7}$$

Then the right-hand side above is less than three for $\alpha > 2$. Then $\dim\mathscr{G}^+ \leq 2$. Similarly but replacing $\mathrm{P}^+$ with $\mathrm{P}^-$, we can also see that

$$\mathrm{P}^-\mathrm{P}_\Omega\mathrm{P}^- \geq \mathrm{P}^-(1 - N)\mathrm{P}^- \geq \left(1 - \frac{1}{\alpha^2 - 1}\right)\mathrm{P}^-. \tag{2.8}$$

Note that $\mathrm{P}^-\mathrm{P}_\Omega\mathrm{P}^- = 0$, since $\mathrm{P}_\Omega$ is the projection to the set of even functions. Then we have

$$0 = \mathrm{Tr}(\mathrm{P}^-\mathrm{P}_\Omega\mathrm{P}^-) \geq \frac{\alpha^2 - 2}{\alpha^2 - 1}\,\mathrm{Tr}\,\mathrm{P}^-. \tag{2.9}$$

In particular, for $\alpha > \sqrt{2}$, the dimension of $\mathscr{G}_-$ equals zero. Then the theorem follows. $\square$

*2.2. Proof of Theorem 1.2*

In this section we show that the lowest eigenvalue is simple. The strategy is parallel to that of the previous section but with $\mathrm{P}^+$ replaced by a projector R with dimension Ran $\mathrm{R} = 1$. Let $\sigma_1, \sigma_2, \sigma_3$ be the $2 \times 2$ Pauli matrices given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.10}$$

*Proof of Theorem 1.2.* The Hamiltonian $Q$ can be written in the form

$$Q = \tfrac{1}{2} A(p^2 + x^2) + \tfrac{1}{2} \sigma_2 (px + xp). \tag{2.11}$$

We set $M = \frac{1}{2}(1 + \sigma_3) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $M^\perp = \frac{1}{2}(1 - \sigma_3) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\mathrm{R} = M^\perp \mathrm{P}_\Omega$. Then we have

$$(Q - E)\mathrm{R}\varphi_{\mathrm{g}} = M^\perp [Q, \mathrm{P}_\Omega]\varphi_{\mathrm{g}} + [Q, M^\perp]\mathrm{P}_\Omega \varphi_{\mathrm{g}}. \tag{2.12}$$

The commutator $[Q, M^\perp]$ can be computed as

$$[Q, M^\perp] = -\tfrac{1}{2}(aa - a^* a^*)\sigma_1. \tag{2.13}$$

Thus, we have

$$[Q, \mathrm{R}] = M^\perp \frac{i}{2} \sigma_2 [aa - a^* a^*, \mathrm{P}_\Omega] + \frac{1}{2} a^* a^* \sigma_1 \mathrm{P}_\Omega, \tag{2.14}$$

where we used the fact that $a\mathrm{P}_\Omega = 0$. Hence, we have

$$\begin{aligned} \langle \mathrm{R}\varphi_{\mathrm{g}}, (Q - E)\mathrm{R}\varphi_{\mathrm{g}} \rangle &= \left\langle \varphi_{\mathrm{g}}, \frac{i}{2} \mathrm{R} M^\perp \sigma_2 \big[ aa - a^* a^*, \mathrm{P}_\Omega \big] \varphi_{\mathrm{g}} \right\rangle + \frac{1}{2} \langle \varphi_{\mathrm{g}}, \mathrm{R} a^* a^* \sigma_1 \mathrm{P}_\Omega \varphi_{\mathrm{g}} \rangle \\ &= \frac{i}{2} \langle \varphi_{\mathrm{g}}, \mathrm{R} \sigma_2 a^2 \varphi_{\mathrm{g}} \rangle. \end{aligned} \tag{2.15}$$

On the other hand, $\mathrm{R}(Q - E)\mathrm{R} = (\frac{1}{2}\beta - E)\mathrm{R}^2$. Then (2.4) and $\|a^* \mathrm{R}\varphi_{\mathrm{g}}\| = \|\mathrm{R}\varphi_{\mathrm{g}}\|$ yield that

$$\left( \frac{1}{2}\beta - E \right) \|\mathrm{R}\varphi_{\mathrm{g}}\|^2 \leq \frac{1}{2} \|a^* \mathrm{R}\varphi_{\mathrm{g}}\| \|\sigma_2 a \varphi_{\mathrm{g}}\| \leq \frac{1}{\sqrt{\alpha^2 - 1}} \|\mathrm{R}\varphi_{\mathrm{g}}\| \|\varphi_{\mathrm{g}}\|.$$

Therefore,

$$\|\mathrm{R}\varphi_{\mathrm{g}}\|^2 \leq \left( \frac{1}{2}\beta - E \right)^{-2} \frac{1}{\alpha^2 - 1} \|\varphi_{\mathrm{g}}\|^2.$$

Since $M + M^\perp = 1$, it holds that $\mathrm{P}_\Omega M + \mathrm{R} + N \geq 1$. Then, by using (2.4) we have

$$\mathrm{P}(\mathrm{P}_\Omega M)\mathrm{P} \geq \mathrm{P}(1 - \mathrm{P}_\Omega M^\perp - N)\mathrm{P} \geq \left( 1 - \left( \frac{1}{2}\beta - E \right)^{-2} \frac{1}{\alpha^2 - 1} - \frac{1}{\alpha^2 - 1} \right) \mathrm{P},$$

where $P = P^+ + P^-$ is the orthogonal projection onto $\mathscr{G}$. Taking the trace of both sides above, we have

$$1 \geq \left( 1 - \left( \frac{1}{2}\beta - E \right)^{-2} \frac{1}{\alpha^2 - 1} - \frac{1}{\alpha^2 - 1} \right) \operatorname{Tr} \mathrm{P},$$

and the theorem follows. □

### 2.3. *Proof of Theorem 1.4*

Recall that $\varepsilon = \beta - \alpha$. In this section, we fix an arbitrary $\alpha > 1$ and set

$$Q(\alpha, \beta) = Q = Q(\varepsilon) = Q_0 + \varepsilon V, \tag{2.16}$$

where $Q_0 = Q(\alpha, \alpha)$ and $V = \frac{1}{2}\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)(p^2 + x^2)$.

LEMMA 2.1. *For all $\Phi \in D(Q_0)$, it follows that*

$$\|V\Phi\| \le (\alpha - 1)^{-1}\|Q_0\Phi\| + \frac{\sqrt{3}}{2}(\alpha - 1)^{-1}\|\Phi\|. \tag{2.17}$$

*Proof.* One can show that $\|(px + xp)u\|^2 \le \|(p^2 + x^2)u\|^2 + 3\|u\|^2$. Since $\sigma_2^2 = 1$, we have

$$\begin{aligned}\|Q_0\Phi\| &\ge \alpha\|\tfrac{1}{2}(p^2 + x^2)\Phi\| - \|\tfrac{1}{2}(px + xp)\Phi\| \\ &\ge \alpha\|\tfrac{1}{2}(p^2 + x^2)\Phi\| - \tfrac{1}{2}(\|(p^2 + x^2)\Phi\|^2 + 3\|\Phi\|^2)^{1/2},\end{aligned}$$

and, hence,

$$\|Q_0\Phi\| \ge \frac{\alpha - 1}{2}\|(p^2 + x^2)\Phi\| - \frac{\sqrt{3}}{2}\|\Phi\|. \tag{2.18}$$

Noting that $\|V\Phi\| \le \frac{1}{2}\|(p^2 + x^2)\Phi\|$ we have the bound (2.17). □

By (1.5) or the sandwich estimate $Q(\alpha, \alpha) \le Q(\alpha, \beta) \le Q(\beta, \beta)$ we have the bounds

$$\frac{\sqrt{\alpha^2 - 1}}{2} \le \lambda_1 \le \lambda_2 \le \frac{\sqrt{\beta^2 - 1}}{2} \quad \text{and} \quad \frac{3}{2}\sqrt{\alpha^2 - 1} \le \lambda_3$$

(with repetition according to multiplicity). When $\sqrt{\beta^2 - 1} \le 3\sqrt{\alpha^2 - 1}$, $Q$ has exactly two eigenvalues in the interval $[\frac{1}{2}\sqrt{\alpha^2 - 1}, \frac{1}{2}\sqrt{\beta^2 - 1}]$. Let $C$ be the closed disk centered at $\sqrt{\alpha^2 - 1}/2$ with the radius $\sqrt{\alpha^2 - 1}/2$ in the complex plane:

$$C = \left\{ \frac{\sqrt{\alpha^2 - 1}}{2} + re^{i\theta} \in \mathbb{C} \,\middle|\, 0 \le r \le \frac{\sqrt{\alpha^2 - 1}}{2}, 0 \le \theta \le 2\pi \right\}. \tag{2.19}$$

Thus, $Q$ has exactly two eigenvalues, $\lambda_1$ and $\lambda_2$, inside of $C$.

LEMMA 2.2. *We have $\|V(Q_0 - z)^{-1}\| \le g_1$ for all $z \in \partial C$.*

*Proof.* By Lemma 2.1, we have

$$\|V(Q_0 - z)^{-1}\| \le (\alpha - 1)^{-1}\|Q_0(Q_0 - z)^{-1}\| + \frac{\sqrt{3}}{2}(\alpha - 1)^{-1}\|(Q_0 - z)^{-1}\|. \tag{2.20}$$

Since the eigenvalues of $Q_0$ are $\{(\frac{1}{2} + n)\sqrt{\alpha^2 - 1}\}_{n=0}^{\infty}$ we have $\sup_{z\in\partial C} \|Q_0(Q_0 - z)^{-1}\| = 3$ and $\sup_{z\in\partial C} \|(Q_0 - z)^{-1}\| = 2/\sqrt{\alpha^2 - 1}$. Then the lemma follows. □

The two-dimensional subspace spanned by the eigenvectors associated with the eigenvalues $\lambda_1$ and $\lambda_2$ is denoted by $\mathscr{F}$. The orthogonal projection onto $\mathscr{F}$ is then given by

$$P = P(\varepsilon) = -\frac{1}{2\pi i}\oint_{\partial C} (Q - z)^{-1}\, dz. \tag{2.21}$$

We expand $P(\varepsilon)$ with respect to $\varepsilon$ up to the second order:

$$P = P_0 + \varepsilon P_1 + \varepsilon^2 R, \tag{2.22}$$

where $P_0$ is the orthogonal projection onto the ground states of $Q_0$ and

$$P_1 = -\frac{1}{2\pi i}\oint_{\partial C}(Q_0 - z)^{-1}V(Q_0 - z)^{-1}\,dz, \tag{2.23}$$

$$R = R(\varepsilon) = -\frac{1}{2\pi i}\oint_{\partial C}(Q_0 - z)^{-1}V(Q_0 - z)^{-1}V(Q - z)^{-1}\,dz. \tag{2.24}$$

LEMMA 2.3. *We have $\|R\| \le g_2$ and $\|VP_1\| \le E_0 g_1^2$.*

*Proof.* By Lemma 2.2, we have

$$\|R\| \le \frac{|C|}{2\pi}\sup_{z\in\partial C}\|V(Q_0 - z)^{-1}\|^2\|(Q - z)^{-1}\| \le \frac{\sqrt{\alpha^2 - 1}}{2}g_1^2\|(Q - \sqrt{\alpha^2 - 1})^{-1}\|. \tag{2.25}$$

Since $\lambda_3 \ge \frac{3}{2}\sqrt{\alpha^2 - 1}$, we have $\|(Q - \sqrt{\alpha^2 - 1})^{-1}\| = |\lambda_2 - \sqrt{\alpha^2 - 1}|^{-1}$. Hence, $\|R\| \le g_2$ holds. Similarly one can prove the second bound. □

Let $v_0 \in L^2(\mathbb{R})$ be the normalized ground state of $h = \frac{1}{2}p^2 + ((\alpha^2 - 1)/2)x^2$. Namely

$$v_0(x) = \left(\frac{\sqrt{\alpha^2 - 1}}{\pi}\right)^{1/4} e^{-\sqrt{\alpha^2 - 1}x^2/2}. \tag{2.26}$$

Let $S_a$ be the dilation defined by $S_a f(x) = (1/\sqrt{a})f(ax)$ for $a > 0$. We define the unitary operator $U$ on $\mathcal{H}$ by

$$U = \frac{1}{\sqrt{2}}S_{\sqrt{\alpha}}\begin{pmatrix} e^{ix^2/(2\alpha)} & 0 \\ 0 & e^{-ix^2/(2\alpha)}\end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i\end{pmatrix}. \tag{2.27}$$

Then $UQ_0U^* = \left(\begin{smallmatrix} h & 0 \\ 0 & h\end{smallmatrix}\right)$ and the vectors $u_1 = U^*\left(\begin{smallmatrix} v_0 \\ 0\end{smallmatrix}\right)$ and $u_2 = U^*\left(\begin{smallmatrix} 0 \\ v_0\end{smallmatrix}\right)$ are twofold ground states of $Q(\alpha, \alpha)$. Since $P$ is a projection onto $\mathscr{F}$, the ground state and the first excited state can be expressed as a linear combination of $Pu_1$ and $Pu_2$ as long as both $Pu_1$ and $Pu_2$ are linearly independent, which is proven in the lemma below.

LEMMA 2.4. *Assume that $\varepsilon^2 g_2 < 1$. Then $Pu_1$ and $Pu_2$ are non-zero vectors. Moreover, if $\varepsilon^2 g_2 < 1/2$, then $Pu_1$ and $Pu_2$ are linearly independent.*

*Proof.* By (2.22) we have

$$\|Pu_1\|^2 = 1 + \varepsilon\langle u_1, P_1u_1\rangle + \varepsilon^2\langle u_1, Ru_1\rangle. \tag{2.28}$$

The second term on the right-hand side above is zero, since

$$\langle u_1, P_1u_1\rangle = \frac{-1}{2\pi i}\oint_{\partial C}\frac{1}{(E_0 - z)^2}\,dz\langle u_1, Vu_1\rangle = 0. \tag{2.29}$$

Hence, $\|Pu_1\|^2 \ge 1 - \varepsilon^2 g_2^2 > 0$ holds by Lemma 2.3 and the assumption. Since the same holds for $\|Pu_2\|$, thus $Pu_i\,(i = 1, 2)$ are non-zero vectors. Next we assume that $\varepsilon^2 g_2 < 1/2$.

Then we have

$$\begin{aligned}\|Pu_1\|^2\|Pu_2\|^2 - |\langle Pu_1, Pu_2\rangle|^2 &\geq (1-\varepsilon^2 g_2)(1-\varepsilon^2 g_2) - \varepsilon^4 g_2^2 \\ &= 1 - 2\varepsilon^2 g_2 > 0 \end{aligned} \tag{2.30}$$

which implies that $Pu_1$ and $Pu_2$ are linearly independent. □

LEMMA 2.5. *Let $g_3$ and $g_4$ be given in (1.10) and (1.11), respectively. Then it follows that where we used the bound*

$$\|Vu_i\| \leq g_1 E_0, \quad i = 1, 2, \tag{2.31}$$

$$|\langle u_1, Vu_1\rangle| = g_3, \tag{2.32}$$

$$|\langle u_1, Vu_2\rangle| \leq g_4. \tag{2.33}$$

*Proof.* Equation (2.31) follows from

$$\|Vu_i\| \leq \|V(Q_0 - \sqrt{\alpha^2-1})^{-1}\|\|(Q_0 - \sqrt{\alpha^2-1})u_i\| \leq g_1 E_0. \tag{2.34}$$

The proofs of (2.32) and (2.33) are given in the Appendix. □

*Proof of Theorem 1.4.* Suppose that $\varepsilon^2 g_2 < 1/2$. We define

$$\Phi_1 = \frac{Pu_1}{\|Pu_1\|}, \quad \Phi_2 = \frac{Pu_2 - \langle \Phi_1, u_2\rangle \Phi_1}{\|Pu_2 - \langle \Phi_1, u_2\rangle\Phi_1\|}. \tag{2.35}$$

Then, by Lemma 2.4, $\Phi_1$ and $\Phi_2$ are orthogonal vectors in $\mathscr{F}$. Let $\mathscr{V} = \mathrm{Span}\{\Phi_1, \Phi_2\}$ be the two-dimensional vector space and $Q : \mathscr{V} \to \mathscr{V}$ can be regarded as a linear operator and its matrix representation is given by

$$m = \begin{pmatrix} \langle \Phi_1, Q\Phi_1\rangle & \langle \Phi_1, Q\Phi_2\rangle \\ \langle \Phi_2, Q\Phi_1\rangle & \langle \Phi_2, Q\Phi_2\rangle \end{pmatrix}. \tag{2.36}$$

Thus, the eigenvalues $\lambda_1$ and $\lambda_2$ of $Q$ are also the eigenvalue of $m$. Therefore, the difference of $\lambda_1$ and $\lambda_2$ can be computed by

$$(\lambda_2 - \lambda_1)^2 = (\langle \Phi_1, Q\Phi_1\rangle - \langle \Phi_2, Q\Phi_2\rangle)^2 + 4|\langle \Phi_1, Q\Phi_2\rangle|^2, \tag{2.37}$$

which implies that

$$|\lambda_2 - \lambda_1| \geq 2|\langle \Phi_1, Q\Phi_2\rangle|. \tag{2.38}$$

We estimate $|\langle \Phi_1, Q\Phi_2\rangle|$ from below. Inserting the definition of $\Phi_j$ into $\langle \Phi_i, Q\Phi_j\rangle$ we have

$$\langle \Phi_1, Q\Phi_2\rangle = \frac{\langle Pu_1, Q(\|Pu_1\|^2 Pu_2 - (Pu_1, Pu_2)Pu_1)\rangle}{\|Pu_1\|^3\|Pu_2 - (Pu_1, Pu_2)Pu_1/\|Pu_1\|^2\|}. \tag{2.39}$$

Note that

$$\langle Pu_i, QPu_j\rangle = (Pu_i, Qu_j) = E_0\delta_{ij} + \varepsilon\langle V\rangle_{ij} + \varepsilon^2(E_0\langle R\rangle_{ij} + \langle P_1 V\rangle_{ij}) + \varepsilon^3\langle RV\rangle_{ij},$$

where $\langle K\rangle_{ij} = (u_i, Ku_j)$. We see that the denominator of (2.39) is expanded as

$$\begin{aligned}&\|Pu_1\|\|\|Pu_1\|^2 Pu_2 - (Pu_1, Pu_2)Pu_1\| \\ &\quad = (1+\varepsilon^2\langle R\rangle_{11})\sqrt{(1+\varepsilon^2\langle R\rangle_{11})(1+\varepsilon^2\langle R\rangle_{22}) - \varepsilon^4|\langle R\rangle_{12}|^2}.\end{aligned}$$

By the bound $\|R\| \leq g_2$ we have the lower bound

$$\|Pu_1\| \| \|Pu_1\|^2 Pu_2 - (Pu_1, Pu_2) Pu_1 \| \geq (1 - \varepsilon^2 g_2)\sqrt{1 - 2\varepsilon^2 g_2^2}. \tag{2.40}$$

The numerator of (2.39) can be also expanded as

$$\begin{aligned}
&(Pu_1, Q(\|Pu_1\|^2 Pu_2 - (Pu_1, Pu_2)Pu_1)) \\
&\quad = \varepsilon \langle V \rangle_{12} + \varepsilon^2 \langle P_1 V \rangle_{12} + \varepsilon^3 (\langle RV \rangle_{12} + \langle V \rangle_{12} \langle R \rangle_{11} - \langle R \rangle_{12} \langle V \rangle_{11}) \\
&\qquad + \varepsilon^4 (\langle P_1 V \rangle_{12} \langle R \rangle_{11} - \langle R \rangle_{12} \langle P_1 V \rangle_{11}) + \varepsilon^5 (\langle R \rangle_{11} \langle RV \rangle_{12} - \langle R \rangle_{12} \langle RV \rangle_{12}).
\end{aligned}$$

By using Lemmas 2.3 and 2.5, each term can be evaluated as

$$\begin{aligned}
&\varepsilon^2 |\langle P_1 V \rangle_{12}| \leq \varepsilon^2 E_0 g_1^2 \\
&\varepsilon^3 |\langle RV \rangle_{12} + \langle V \rangle_{12} \langle R \rangle_{11} - \langle R \rangle_{12} \langle V \rangle_{11}| \leq \varepsilon^3 (E_0 g_1 g_2 + g_2 g_4 + g_2 g_3) \\
&\varepsilon^4 |\langle P_1 V \rangle_{12} \langle R \rangle_{11} - \langle R \rangle_{12} \langle P_1 V \rangle_{11}| \leq \varepsilon^4 2 E_0 g_1^2 g_2 \\
&\varepsilon^5 |\langle R \rangle_{11} \langle RV \rangle_{12} - \langle R \rangle_{12} \langle RV \rangle_{12}| \leq \varepsilon^5 2 E_0 g_1 g_2^2.
\end{aligned}$$

By combining all of the estimates stated above, we have

$$|\lambda_1 - \lambda_2| \geq \frac{2\varepsilon}{\ell(\varepsilon)} (g_4 - \varepsilon \kappa(\varepsilon)), \tag{2.41}$$

where $\ell(\varepsilon) = (1 - \varepsilon^2 g_2)\sqrt{1 - 2\varepsilon^2 g_2^2}$. Hence, the theorem follows. □

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## A. Appendix: computation of $\langle V \rangle_{ij}$

We recall that $v_0 = (\omega/\pi)^{1/4} e^{-\omega x^2/2}$ with $\omega = \sqrt{\alpha^2 - 1}$, $V = \frac{1}{2}\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)(p^2 + x^2)$ and $U$ is given by (2.27). We directly see that

$$\begin{aligned}
UVU^{-1} &= \frac{1}{4} S_{\sqrt{\alpha}} \begin{pmatrix} e^{ix^2/2\alpha} & 0 \\ 0 & e^{-ix^2/2\alpha} \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \\
&\qquad \times (p^2 + x^2) \begin{pmatrix} e^{-ix^2/2\alpha} & 0 \\ 0 & e^{ix^2/2\alpha} \end{pmatrix} S_{1/\sqrt{\alpha}} \\
&= \frac{1}{4} \begin{pmatrix} e^{ix^2/2\alpha} (\frac{p^2}{\alpha} + \alpha x^2) e^{-ix^2/2\alpha} & e^{-ix^2/2\alpha} (\frac{p^2}{\alpha} + \alpha x^2) e^{ix^2/2\alpha} \\ -e^{-ix^2/2\alpha} (\frac{p^2}{\alpha} + \alpha x^2) e^{-ix^2/2\alpha} & e^{ix^2/2\alpha} (\frac{p^2}{\alpha} + \alpha x^2) e^{ix^2/2\alpha} \end{pmatrix}.
\end{aligned}$$

Then we have

$$\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, UVU^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle_{\mathbb{C}^2} = \frac{1}{4} e^{ix^2/2\alpha} \left( \frac{p^2}{\alpha} + \alpha x^2 \right) e^{-ix^2/2\alpha} \tag{A.1}$$

$$\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, UVU^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle_{\mathbb{C}^2} = -\frac{1}{4} e^{ix^2/2\alpha} \left( \frac{p^2}{\alpha} + \alpha x^2 \right) e^{ix^2/2\alpha}. \tag{A.2}$$

LEMMA A.1. *We have that $\langle u_1, Vu_1 \rangle = \alpha/4\omega$, and then $|\langle u_1, Vu_1 \rangle| = g_3$.*

*Proof.* By (A.1) we have

$$\langle u_1, Vu_1 \rangle = \frac{1}{4} \left\langle v_0, e^{ix^2/2} \left( \frac{p^2}{\alpha} + \alpha x^2 \right) e^{-ix^2/2} v_0 \right\rangle = \frac{1}{4} \left\langle v_0, \left( \frac{(p-x)^2}{\alpha} + \alpha x^2 \right) v_0 \right\rangle.$$

Since $\langle v_0, (px + xp) v_0 \rangle = 0$, we obtain that

$$\langle u_1, Vu_1 \rangle = \frac{1}{4} \left\langle v_0, \left( \frac{p^2 + x^2}{\alpha} + \alpha x^2 \right) v_0 \right\rangle.$$

By $\langle v_0, x^2 v_0 \rangle = 1/(2\omega)$ and $\langle v_0, p^2 v_0 \rangle = \omega/2$ we have $\langle u_1, Vu_1 \rangle = \alpha/(4\omega)$. □

LEMMA A.2. *We have that $\langle u_1, Vu_2 \rangle = -(\omega^{3/2}/4\alpha)(\omega - i)^{-1/2}$. In particular $|\langle u_1, Vu_2 \rangle| \leq \omega^{3/2}/4\alpha^{3/2}$.*

*Proof.* By (A.2) we have

$$\begin{aligned} \langle u_1, Vu_2 \rangle &= -\frac{1}{4} \left\langle v_0, e^{ix^2/2} \left( \frac{p^2}{\alpha} + \alpha x^2 \right) e^{ix^2/2} v_0 \right\rangle \\ &= -\frac{1}{4\alpha} \langle (-i - \omega) x e^{-ix^2/2} v_0, (i - \omega) x e^{ix^2/2} v_0 \rangle - \frac{\alpha}{4} \langle v_0, x^2 e^{ix^2} v_0 \rangle \\ &= -\frac{1}{4\alpha} ((\omega - i)^2 + \alpha^2) \langle v_0, x^2 e^{ix^2} v_0 \rangle. \end{aligned}$$

Since $\langle v_0, x^2 e^{ix^2} v_0 \rangle = (\omega/\pi)^{1/2} \int_{\mathbb{R}} x^2 e^{-(\omega - i)x^2} \, dx = \frac{1}{2} \omega^{1/2} (\omega - i)^{-3/2}$, we have the lemma. □

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