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Ulam type stability problems for alternative homomorphisms

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Abstract

We introduce an alternative homomorphism with respect to binary operations and investigate the Ulam type stability problem for such a mapping. The obtained results apply to Ulam type stability problems for several important functional equations.

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1 Introduction

In 1940, SM Ulam proposed the following stability problem: Given an approximately additive mapping, can one find the strictly additive mapping near it? A year later, DH Hyers gave an affirmative answer to this problem for additive mappings between Banach spaces. Subsequently many mathematicians came to deal with this problem (cf. [1–5]).

We introduce an alternative homomorphism from a set X with two binary operations \circ and $*$ to another set E with two binary operations \diamond and \star defined by

$$f(x \circ y) \star f(x * y) = f(x) \diamond f(y) \quad (\forall x, y \in X),$$

and we investigate the Ulam type stability problem for such a mapping when E is a complete metric space. In particular, if $s \star t = s$ for all $s, t \in E$, then our results imply the stability results obtained in [6]. Also the method used in the paper have already applied for some other equations (cf. [7–15]).

One consequence of Banach's fixed point theorem

A fixed point theorem has played an important role in the stability problem (cf. [16]). The authors used an easy consequence of Banach's fixed point theorem in [6]. It will serve again in this paper. Here we review it.

Let X be a set and (E, d) a complete metric space. Fix two mappings $f : X \rightarrow E$ and $\varphi : X \rightarrow \mathbb{R}^+$, where \mathbb{R}^+ denotes the set of all nonnegative real numbers. Denote by $\Delta_{f, \varphi}$ the set of all mappings $u : X \rightarrow E$ such that there exists a finite constant K_u satisfying

$$d(u(x), f(x)) \leq K_u \varphi(x) \quad (\forall x \in X).$$

For any $u, v \in \Delta_{f,\varphi}$, we define

$$\rho_{f,\varphi}(u, v) = \inf\{K \geq 0 : d(u(x), v(x)) \leq K\varphi(x) \ (\forall x \in X)\}.$$

Then $(\Delta_{f,\varphi}, \rho_{f,\varphi})$ is a complete metric space which contains f .

Now, fix three mappings $\sigma : X \rightarrow X$, $\tau : E \rightarrow E$ and $\varepsilon : X \times X \rightarrow \mathbb{R}^+$. For any mapping $u : X \rightarrow E$, we define the mapping $T_{\sigma,\tau}u : X \rightarrow E$ by

$$(T_{\sigma,\tau}u)(x) = \tau(u(\sigma x)) \quad (x \in X).$$

Also, we consider three quantities:

$$\alpha_{\sigma,\varepsilon} = \inf\{K \geq 0 : \varepsilon(\sigma x, \sigma y) \leq K\varepsilon(x, y) \ (x, y \in X)\},$$

$$\beta_{\sigma,\varphi} = \inf\{K \geq 0 : \varphi(\sigma x) \leq K\varphi(x) \ (x \in X)\},$$

$$\gamma_\tau = \inf\{K \geq 0 : d(\tau s, \tau t) \leq Kd(s, t) \ (s, t \in E)\}.$$

If $\alpha_{\sigma,\varepsilon} < \infty$, $\beta_{\sigma,\varphi} < \infty$ and $\gamma_\tau < \infty$, then we have

$$\varepsilon(\sigma x, \sigma y) \leq \alpha_{\sigma,\varepsilon}\varepsilon(x, y) \quad (\forall x, y \in X),$$

$$\varphi(\sigma x) \leq \beta_{\sigma,\varphi}\varphi(x) \quad (\forall x \in X),$$

$$d(\tau s, \tau t) \leq \gamma_\tau d(s, t) \quad (\forall s, t \in E),$$

respectively. We will use these inequalities throughout this paper.

We now state our fixed point theorem.

Lemma A ([6, Proposition 2.1]) *Let X be a set and (E, d) a complete metric space. Suppose that four mappings $f : X \rightarrow E$, $\varphi : X \rightarrow \mathbb{R}^+$, $\sigma : X \rightarrow X$ and $\tau : E \rightarrow E$ satisfy*

$$T_{\sigma,\tau}f \in \Delta_{f,\varphi}, \quad \beta_{\sigma,\varphi} < \infty, \quad \gamma_\tau < \infty \quad \text{and} \quad \beta_{\sigma,\varphi}\gamma_\tau < 1.$$

Then $T_{\sigma,\tau}(\Delta_{f,\varphi}) \subseteq \Delta_{f,\varphi}$ and $T_{\sigma,\tau}$ has a unique fixed point f_∞ in $\Delta_{f,\varphi}$. Moreover,

$$\lim_{n \rightarrow \infty} d((T_{\sigma,\tau}^n f)(x), f_\infty(x)) = 0 \quad \text{and} \quad d(f(x), f_\infty(x)) \leq \frac{\rho_{f,\varphi}(T_{\sigma,\tau}f, f)}{1 - \beta_{\sigma,\varphi}\gamma_\tau} \varphi(x)$$

for all $x \in X$.

2 A stability of alternative homomorphisms

Let $(X, \circ, *)$ be a set X with two binary operations \circ and $*$. Let (E, d, \diamond, \star) be a complete metric space (E, d) with two binary operations \diamond and \star . Given $f : X \rightarrow E$, we consider the following commutative diagram:

$$\begin{array}{ccc} X \times X & \xrightarrow{(f \circ) \times (f *)} & E \times E \\ f \times f \downarrow & & \downarrow \star \\ E \times E & \xrightarrow{\diamond} & E. \end{array} \tag{1}$$

This means that

$$f(x \circ y) \star f(x \ast y) = f(x) \diamond f(y) \quad (\forall x, y \in X). \tag{2}$$

In particular, if $s \star t = s$ for all $s, t \in E$, then (1) and (2) become

$$\begin{array}{ccc} X \times X & \xrightarrow{\circ} & X \\ f \times f \downarrow & & \downarrow f \\ E \times E & \xrightarrow{\diamond} & E \end{array}$$

and

$$f(x \circ y) = f(x) \diamond f(y) \quad (\forall x, y \in X).$$

In other words, f is a homomorphism from X to E . Thus, if a mapping $f : X \rightarrow E$ satisfies (2), then we say that f is an *alternative homomorphism*.

In this section, we establish two general settings, on which we can give an affirmative answer to the Ulam type stability problem for the commutative diagram (1). These settings have a property such as duality, that is, each of them works as a complement of the other.

Let us describe the first setting. For $\varepsilon : X \times X \rightarrow \mathbb{R}^+$ and $\delta : X \rightarrow \mathbb{R}^+$, we consider the following three conditions:

- (i) The square operator $x \mapsto x \circ x$ is an automorphism of X with respect to \circ and \ast . We denote by σ the inverse mapping of this automorphism.
- (ii) The binary operations \diamond and \star on E are continuous. The square operator $\tau : s \mapsto s \diamond s$ is an endomorphism of E with respect to \diamond and \star .
- (iii) $\alpha \equiv \alpha_{\sigma, \varepsilon} < \infty$, $\beta \equiv \beta_{\sigma, \delta} < \infty$, $\gamma \equiv \gamma_{\tau} < \infty$ and $\gamma \max\{\alpha, \beta\} < 1$.

Under the above conditions, we show the Ulam type stability for the commutative diagram (1), as follows.

Theorem 1 *Let (X, \circ, \ast) and (E, d, \diamond, \star) be as above. Suppose that four mappings $\sigma : X \rightarrow X$, $\tau : E \rightarrow E$, $\varepsilon : X \times X \rightarrow \mathbb{R}^+$ and $\delta : X \rightarrow \mathbb{R}^+$ satisfy (i), (ii), and (iii). If a mapping $f : X \rightarrow E$ satisfies*

$$d(f(x \circ y) \star f(x \ast y), f(x) \diamond f(y)) \leq \varepsilon(x, y) \quad (\forall x, y \in X), \tag{3}$$

$$d(f(x) \star f(\sigma x \ast \sigma x), f(x)) \leq \delta(x) \quad (\forall x \in X), \tag{4}$$

then there exists a mapping $f_{\infty} : X \rightarrow E$ such that

$$f_{\infty}(x \circ y) \star f_{\infty}(x \ast y) = f_{\infty}(x) \diamond f_{\infty}(y) \quad (\forall x, y \in X), \tag{5}$$

$$f_{\infty}(x) \star f_{\infty}(\sigma x \ast \sigma x) = f_{\infty}(x) \quad (\forall x \in X), \tag{6}$$

$$d(f(x), f_{\infty}(x)) \leq \frac{\alpha \varepsilon(x, x) + \delta(x)}{1 - \gamma \max\{\alpha, \beta\}} \quad (\forall x \in X). \tag{7}$$

Moreover, if a mapping $g : X \rightarrow E$ satisfies (5), (6), and

$$\exists K_g \geq 0 : d(f(x), g(x)) \leq K_g \{\alpha \varepsilon(x, x) + \delta(x)\} \quad (\forall x \in X), \tag{8}$$

then $g = f_{\infty}$.

Proof For simplicity, we write $T = T_{\sigma, \tau}$. We note that α, β , and γ are finite by (iii). Suppose that $f : X \rightarrow E$ satisfies (3) and (4). Put $\varphi(x) = \alpha\varepsilon(x, x) + \delta(x)$ for all $x \in X$. To apply Lemma A to f and φ , we first observe that $Tf \in \Delta_{f, \varphi}$. Fix $x \in X$. Replacing x and y in (3) by σx , we get

$$d(f(\sigma x \circ \sigma x) \star f(\sigma x * \sigma x), f(\sigma x) \diamond f(\sigma x)) \leq \varepsilon(\sigma x, \sigma x).$$

Since

$$\begin{aligned} \sigma x \circ \sigma x &= \sigma^{-1}(\sigma x) = x, \\ f(\sigma x) \diamond f(\sigma x) &= \tau(f(\sigma x)) = (Tf)(x), \end{aligned}$$

and

$$\varepsilon(\sigma x, \sigma x) \leq \alpha\varepsilon(x, x),$$

it follows that

$$d(f(x) \star f(\sigma x * \sigma x), (Tf)(x)) \leq \alpha\varepsilon(x, x).$$

Using this and (4), we have

$$\begin{aligned} d((Tf)(x), f(x)) &\leq d((Tf)(x), f(x) \star f(\sigma x * \sigma x)) + d(f(x) \star f(\sigma x * \sigma x), f(x)) \\ &\leq \alpha\varepsilon(x, x) + \delta(x) \\ &= \varphi(x). \end{aligned}$$

Hence $Tf \in \Delta_{f, \varphi}$ and $\rho_{f, \varphi}(Tf, f) \leq 1$.

We next estimate the quantity $\beta_{\sigma, \varphi}$. For $x \in X$, we have

$$\begin{aligned} \varphi(\sigma x) &= \alpha\varepsilon(\sigma x, \sigma x) + \delta(\sigma x) \\ &\leq \alpha^2\varepsilon(x, x) + \beta\delta(x) \\ &\leq \max\{\alpha, \beta\}(\alpha\varepsilon(x, x) + \delta(x)) \\ &= \max\{\alpha, \beta\}\varphi(x). \end{aligned}$$

Hence $\beta_{\sigma, \varphi} \leq \max\{\alpha, \beta\}$ and $\beta_{\sigma, \varphi}\gamma_\tau \leq \gamma \max\{\alpha, \beta\} < 1$ by (iii).

Thus we can apply Lemma A. As a consequence, T has a unique fixed point $f_\infty \in \Delta_{f, \varphi}$. Moreover,

$$\lim_{n \rightarrow \infty} d((T^n f)(x), f_\infty(x)) = 0 \tag{9}$$

and

$$d(f(x), f_\infty(x)) \leq \frac{\rho_{f, \varphi}(Tf, f)}{1 - \beta_{\sigma, \varphi}\gamma_\tau} \varphi(x) \tag{10}$$

for all $x \in X$. Since $\rho_{f, \varphi}(Tf, f) \leq 1$ and $\beta_{\sigma, \varphi}\gamma_\tau \leq \gamma \max\{\alpha, \beta\} < 1$, (10) implies (7).

Here we show (5). If $x, y \in X$ and $n \in \mathbb{N}$, then we have

$$\begin{aligned}
 & d(f_\infty(x \circ y) \star f_\infty(x * y), f_\infty(x) \diamond f_\infty(y)) \\
 & \leq d(f_\infty(x \circ y) \star f_\infty(x * y), (T^n f)(x \circ y) \star (T^n f)(x * y)) \\
 & \quad + d((T^n f)(x \circ y) \star (T^n f)(x * y), (T^n f)(x) \diamond (T^n f)(y)) \\
 & \quad + d((T^n f)(x) \diamond (T^n f)(y), f_\infty(x) \diamond f_\infty(y)). \tag{11}
 \end{aligned}$$

We will see that the right hand side of (11) tends to 0 as $n \rightarrow \infty$. The first and third terms on the right hand side tend to 0 as $n \rightarrow \infty$, because of (9) and the continuity of \star and \diamond in (ii). Moreover, the second term, say $A_n(x, y)$, is estimated as follows: By (i), (ii), and (3), we have

$$\begin{aligned}
 A_n(x, y) &= d(\tau^n(f(\sigma^n(x \circ y))) \star \tau^n(f(\sigma^n(x * y))), \tau^n(f(\sigma^n x)) \diamond \tau^n(f(\sigma^n y))) \\
 &= d(\tau^n(f(\sigma^n x \circ \sigma^n y)) \star \tau^n(f(\sigma^n x * \sigma^n y)), \tau^n(f(\sigma^n x) \diamond f(\sigma^n y))) \\
 &= d(\tau^n(f(\sigma^n x \circ \sigma^n y) \star f(\sigma^n x * \sigma^n y)), \tau^n(f(\sigma^n x) \diamond f(\sigma^n y))) \\
 &\leq \gamma^n d(f(\sigma^n x \circ \sigma^n y) \star f(\sigma^n x * \sigma^n y), f(\sigma^n x) \diamond f(\sigma^n y)) \\
 &\leq \gamma^n \varepsilon(\sigma^n x, \sigma^n y) \\
 &\leq \gamma^n \alpha^n \varepsilon(x, y),
 \end{aligned}$$

where τ^n and σ^n denote the n -fold compositions of endomorphisms τ and σ , respectively. Since $\gamma \alpha < 1$ by (iii), it follows that $A_n(x, y) \rightarrow 0$ as $n \rightarrow \infty$. Thus the right hand side of (11) tends to 0, and we obtain (5).

Next, we show (6). For $x \in X$, we replace x and y in (5) by σx to get

$$f_\infty(\sigma x \circ \sigma x) \star f_\infty(\sigma x * \sigma x) = f_\infty(\sigma x) \diamond f_\infty(\sigma x).$$

Since $\sigma x \circ \sigma x = x$ and

$$f_\infty(\sigma x) \diamond f_\infty(\sigma x) = \tau(f_\infty(\sigma x)) = (Tf_\infty)(x) = f_\infty(x),$$

we obtain (6).

Finally, we show the last statement. Since g satisfies (5) and (6), we have

$$\begin{aligned}
 (Tg)(x) &= \tau(g(\sigma x)) = g(\sigma x) \diamond g(\sigma x) \\
 &= g(\sigma x \circ \sigma x) \star g(\sigma x * \sigma x) \\
 &= g(x) \star g(\sigma x * \sigma x) \\
 &= g(x)
 \end{aligned}$$

for all $x \in X$. This says that g is a fixed point of T . Also, by (8), we have $g \in \Delta_{f, \varphi}$. Thus the uniqueness of a fixed point of T in $\Delta_{f, \varphi}$ implies that $g = f_\infty$. \square

The next corollary is obtained in [6].

Corollary 1 ([6, Corollary 3.2]) *Let X be a set with a binary operation \circ such that the square operation $x \mapsto x \circ x$ is an automorphism of X with respect to \circ and E a complete metric space with a continuous binary operation \diamond such that the square operation $\tau : s \mapsto s \diamond s$ is an endomorphism of E with respect to \diamond . Let $\varepsilon : X \times X \rightarrow \mathbb{R}^+$ and suppose that $\alpha \equiv \alpha_{\sigma, \varepsilon} < \infty$, $\gamma \equiv \gamma_{\tau} < \infty$ and $\gamma\alpha < 1$, where σ denotes the inverse mapping of the square operation $x \mapsto x \circ x$. If a mapping $f : X \rightarrow E$ satisfies*

$$d(f(x \circ y), f(x) \diamond f(y)) \leq \varepsilon(x, y) \quad (\forall x, y \in X),$$

then there exists a unique mapping $f_{\infty} : X \rightarrow E$ such that

$$f_{\infty}(x \circ y) = f_{\infty}(x) \diamond f_{\infty}(y) \quad \text{and} \quad d(f(x), f_{\infty}(x)) \leq \frac{\alpha}{1 - \alpha\gamma} \varepsilon(x, x)$$

for all $x, y \in X$.

Proof Consider the case that $\circ = \diamond$ and $s \star t = s$ for $s, t \in E$, in Theorem 1. In this case, τ is clearly an endomorphism of E with respect to \star . Therefore the corollary follows immediately from Theorem 1 with $\delta = 0$. \square

Now we turn to another setting. Let (X, \circ, \circ) and (E, d, \diamond, \star) be as in the first part of this section. For $\varepsilon : X \times X \rightarrow \mathbb{R}^+$ and $\delta : X \rightarrow \mathbb{R}^+$, we consider the following three conditions:

- (iv) The square operator $\tilde{\sigma} : x \mapsto x \circ x$ is an endomorphism of X with respect to \circ and \circ .
- (v) The binary operations \diamond and \star on E are continuous. The square operator $s \mapsto s \diamond s$ is an automorphism of E with respect to \diamond and \star . We denote by $\tilde{\tau}$ the inverse mapping of this automorphism.
- (vi) $\tilde{\alpha} \equiv \alpha_{\tilde{\sigma}, \varepsilon} < \infty$, $\tilde{\beta} \equiv \beta_{\tilde{\sigma}, \delta} < \infty$, $\tilde{\gamma} \equiv \gamma_{\tilde{\tau}} < \infty$, and $\tilde{\gamma} \max\{\tilde{\alpha}, \tilde{\beta}\} < 1$.

Under the above conditions, we show the Ulam type stability for the commutative diagram (1), as follows.

Theorem 2 *Let (X, \circ, \circ) and (E, d, \diamond, \star) be as above. Suppose that four mappings $\tilde{\sigma} : X \rightarrow X$, $\tilde{\tau} : E \rightarrow E$, $\varepsilon : X \times X \rightarrow \mathbb{R}^+$ and $\delta : X \rightarrow \mathbb{R}^+$ satisfy (iv), (v), and (vi). If a mapping $f : X \rightarrow E$ satisfies (3) and*

$$d(f(x \circ x) \star f(x \circ x), f(x \circ x)) \leq \delta(x) \quad (\forall x \in X), \tag{12}$$

then there exists a mapping $f_{\infty} : X \rightarrow E$ satisfying (5)

$$f_{\infty}(x \circ x) \star f_{\infty}(x \circ x) = f_{\infty}(x \circ x) \quad (\forall x \in X), \tag{13}$$

$$d(f(x), f_{\infty}(x)) \leq \frac{\tilde{\gamma} \{\varepsilon(x, x) + \delta(x)\}}{1 - \tilde{\gamma} \max\{\tilde{\alpha}, \tilde{\beta}\}} \quad (\forall x \in X). \tag{14}$$

Moreover, if a mapping $g : X \rightarrow E$ satisfies (13), (14), and

$$\exists K_g \geq 0 : d(f(x), g(x)) \leq K_g \tilde{\gamma} \{\varepsilon(x, x) + \delta(x)\} \quad (\forall x \in X), \tag{15}$$

then $g = f_{\infty}$.

Proof For simplicity, we write $\tilde{T} = T_{\tilde{\sigma}, \tilde{\tau}}$, that is, $(\tilde{T}f)(x) = \tilde{\tau}(f(\tilde{\sigma}x))$ for $x \in X$. We note that $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ are finite by (vi). Suppose that $f : X \rightarrow E$ satisfies (3) and (12). Put $\tilde{\varphi}(x) = \tilde{\gamma}\{\varepsilon(x, x) + \delta(x)\}$ for all $x \in X$. To apply Lemma A to f and $\tilde{\varphi}$, we first observe that $\tilde{T}f \in \Delta_{f, \tilde{\varphi}}$. Fix $x \in X$. Since $\tilde{\tau}(f(x) \diamond f(x)) = f(x)$, it follows from (3) and (12) that

$$\begin{aligned} & d((\tilde{T}f)(x), f(x)) \\ &= d(\tilde{\tau}(f(\tilde{\sigma}x)), f(x)) \\ &= d(\tilde{\tau}(f(x \circ x)), \tilde{\tau}(f(x) \diamond f(x))) \\ &\leq \tilde{\gamma}d(f(x \circ x), f(x) \diamond f(x)) \\ &\leq \tilde{\gamma}\{d(f(x \circ x), f(x \circ x) \star f(x \ast x)) + d(f(x \circ x) \star f(x \ast x), f(x) \diamond f(x))\} \\ &\leq \tilde{\gamma}\{\delta(x) + \varepsilon(x, x)\} \\ &= \tilde{\varphi}(x). \end{aligned}$$

Hence $\tilde{T}f \in \Delta_{f, \tilde{\varphi}}$ and $\rho_{f, \tilde{\varphi}}(\tilde{T}f, f) \leq 1$.

We next estimate the quantity $\beta_{\tilde{\sigma}, \tilde{\varphi}}$. For $x \in X$, we have

$$\begin{aligned} \tilde{\varphi}(\tilde{\sigma}x) &= \tilde{\gamma}\{\varepsilon(\tilde{\sigma}x, \tilde{\sigma}x) + \delta(\tilde{\sigma}x)\} \\ &\leq \tilde{\gamma}\{\tilde{\alpha}\varepsilon(x, x) + \tilde{\beta}\delta(x)\} \\ &\leq \tilde{\gamma}\max\{\tilde{\alpha}, \tilde{\beta}\}\{\varepsilon(x, x) + \delta(x)\} \\ &= \max\{\tilde{\alpha}, \tilde{\beta}\}\tilde{\varphi}(x). \end{aligned}$$

Hence $\beta_{\tilde{\sigma}, \tilde{\varphi}} \leq \max\{\tilde{\alpha}, \tilde{\beta}\}$ and $\beta_{\tilde{\sigma}, \tilde{\varphi}}\tilde{\gamma}\tilde{\tau} \leq \tilde{\gamma}\max\{\tilde{\alpha}, \tilde{\beta}\} < 1$ by (vi).

Thus we can apply Lemma A. As a consequence, \tilde{T} has a unique fixed point $f_\infty \in \Delta_{f, \tilde{\varphi}}$. Moreover,

$$\lim_{n \rightarrow \infty} d((\tilde{T}^n f)(x), f_\infty(x)) = 0 \tag{16}$$

and

$$d(f(x), f_\infty(x)) \leq \frac{\rho_{f, \tilde{\varphi}}(\tilde{T}f, f)}{1 - \beta_{\tilde{\sigma}, \tilde{\varphi}}\tilde{\gamma}\tilde{\tau}}\tilde{\varphi}(x) \tag{17}$$

for all $x \in X$. Since $\rho_{f, \tilde{\varphi}}(\tilde{T}f, f) \leq 1$ and $\beta_{\tilde{\sigma}, \tilde{\varphi}}\tilde{\gamma}\tilde{\tau} \leq \tilde{\gamma}\max\{\tilde{\alpha}, \tilde{\beta}\} < 1$, (17) implies (14).

Here we show (5). If $x, y \in X$ and $n \in \mathbb{N}$, then we have

$$\begin{aligned} & d(f_\infty(x \circ y) \star f_\infty(x \ast y), f_\infty(x) \diamond f_\infty(y)) \\ &\leq d(f_\infty(x \circ y) \star f_\infty(x \ast y), (\tilde{T}^n f)(x \circ y) \star (\tilde{T}^n f)(x \ast y)) \\ &\quad + d((\tilde{T}^n f)(x \circ y) \star (\tilde{T}^n f)(x \ast y), (\tilde{T}^n f)(x) \diamond (\tilde{T}^n f)(y)) \\ &\quad + d((\tilde{T}^n f)(x) \diamond (\tilde{T}^n f)(y), f_\infty(x) \diamond f_\infty(y)). \end{aligned}$$

Letting $n \rightarrow \infty$, the first and third terms on the right hand side tend to 0, because of (16) and the continuity of \star and \diamond in (v). Moreover, the second term, say $\tilde{A}_n(x, y)$, is estimated

as follows: By (iv), (v), and (3),

$$\begin{aligned} \tilde{A}_n(x, y) &= d(\tilde{\tau}^n(f(\tilde{\sigma}^n(x \circ y))) \star \tilde{\tau}^n(f(\tilde{\sigma}^n(x \ast y))), \tilde{\tau}^n(f(\tilde{\sigma}^n x)) \diamond \tilde{\tau}^n(f(\tilde{\sigma}^n y))) \\ &= d(\tilde{\tau}^n(f(\tilde{\sigma}^n x \circ \tilde{\sigma}^n y)) \star \tilde{\tau}^n(f(\tilde{\sigma}^n x \ast \tilde{\sigma}^n y)), \tilde{\tau}^n(f(\tilde{\sigma}^n x)) \diamond f(\tilde{\sigma}^n y)) \\ &= d(\tilde{\tau}^n(f(\tilde{\sigma}^n x \circ \tilde{\sigma}^n y)) \star f(\tilde{\sigma}^n x \ast \tilde{\sigma}^n y), \tilde{\tau}^n(f(\tilde{\sigma}^n x)) \diamond f(\tilde{\sigma}^n y)) \\ &\leq \tilde{\gamma}^n d(f(\tilde{\sigma}^n x \circ \tilde{\sigma}^n y) \star f(\tilde{\sigma}^n x \ast \tilde{\sigma}^n y), f(\tilde{\sigma}^n x) \diamond f(\tilde{\sigma}^n y)) \\ &\leq \tilde{\gamma}^n \varepsilon(\tilde{\sigma}^n x, \tilde{\sigma}^n y) \\ &\leq \tilde{\gamma}^n \tilde{\alpha}^n \varepsilon(x, y), \end{aligned}$$

where $\tilde{\tau}^n$ and $\tilde{\sigma}^n$ denote the n -fold compositions of endomorphisms $\tilde{\tau}$ and $\tilde{\sigma}$, respectively. Since $\tilde{\gamma} \tilde{\alpha} < 1$ by (vi), it follows that $\tilde{A}_n(x, y) \rightarrow 0$ as $n \rightarrow \infty$. Thus we obtain (5).

Next, we show (13). Replacing y in (5) by x , we have

$$f_\infty(x \circ x) \star f_\infty(x \ast x) = f_\infty(x) \diamond f_\infty(x). \tag{18}$$

Also since

$$\tilde{\tau}(f_\infty(x \circ x)) = \tilde{\tau}(f_\infty(\tilde{\sigma} x)) = (\tilde{T}f_\infty)(x) = f_\infty(x) = \tilde{\tau}(f_\infty(x) \diamond f_\infty(x)),$$

it follows that

$$f_\infty(x \circ x) = f_\infty(x) \diamond f_\infty(x).$$

Combining with (18), we obtain (13).

Finally, we show the last statement. Since g satisfies (14) and (13), we have

$$g(\tilde{\sigma} x) = g(x \circ x) = g(x \circ x) \star g(x \ast x) = g(x) \diamond g(x) = \tilde{\tau}^{-1}(g(x)),$$

that is, $(\tilde{T}g)(x) = g(x)$ for all $x \in X$. This says that g is a fixed point of \tilde{T} . Also, by (15), we have $g \in \Delta_{f, \tilde{\varphi}}$. Hence the uniqueness of a fixed point of \tilde{T} in $\Delta_{f, \tilde{\varphi}}$ implies that $g = f_\infty$. \square

The next corollary is obtained in [6].

Corollary 2 ([6, Corollary 3.5]) *Let X be a set with a binary operation \circ such that the square operation $\tilde{\sigma} : x \mapsto x \circ x$ is an endomorphism of X with respect to \circ and E a complete metric space with a continuous binary operation \diamond such that the square operation $s \mapsto s \diamond s$ is an automorphism of E with respect to \diamond . Let $\varepsilon : X \times X \rightarrow \mathbb{R}^+$ and suppose that $\tilde{\alpha} \equiv \alpha_{\tilde{\sigma}, \varepsilon} < \infty$, $\tilde{\gamma} \equiv \gamma_{\tilde{\tau}} < \infty$ and $\tilde{\gamma} \tilde{\alpha} < 1$, where $\tilde{\tau}$ denotes the inverse mapping of the square operation $s \mapsto s \diamond s$. If a mapping $f : X \rightarrow E$ satisfies*

$$d(f(x \circ y), f(x) \diamond f(y)) \leq \varepsilon(x, y) \quad (\forall x, y \in X),$$

then there exists a unique mapping $f_\infty : X \rightarrow E$ such that

$$f_\infty(x \circ y) = f_\infty(x) \diamond f_\infty(y) \quad \text{and} \quad d(f(x), f_\infty(x)) \leq \frac{\tilde{\gamma}}{1 - \tilde{\alpha} \tilde{\gamma}} \varepsilon(x, x)$$

for all $x, y \in X$.

Proof Consider the case that $\circ = \circ$ and $s \star t = s$ for $s, t \in E$, in Theorem 2. Then $\tilde{\tau}$ is clearly an endomorphism of E with respect to \star . Therefore the corollary follows immediately from Theorem 2 with $\delta = 0$. \square

3 Application I

The Ulam type stability problem for Euler-Lagrange type additive mappings has been investigated in [17]. Here we take up the following Euler-Lagrange type mapping $f : X \rightarrow E$ satisfying

$$f(ax + by) + f(bx + ay) + (a + b)(f(-x) + f(-y)) = 0 \quad (\forall x, y \in X), \quad (19)$$

where X is a complex normed space, E a complex Banach space and $a, b \in \mathbb{C}$ with $a + b \neq 0$. The following is an Ulam type stability result for this mapping.

Corollary 3 (cf. [17, Theorem 2.1]) *Let $\varepsilon : X \times X \rightarrow \mathbb{R}^+$ and suppose that*

$$(vii) \exists K \geq 0 : |a + b|K < 1 \text{ and } \varepsilon(x, y) \leq K\varepsilon(-(a + b)x, -(a + b)y) \quad (\forall x, y \in X).$$

If a mapping $f : X \rightarrow E$ satisfies

$$\|f(ax + by) + f(bx + ay) + (a + b)(f(-x) + f(-y))\| \leq \varepsilon(x, y) \quad (\forall x, y \in X), \quad (20)$$

then there exists a unique mapping $f_\infty : X \rightarrow E$ satisfying (19) and

$$\|f(x) - f_\infty(x)\| \leq \frac{K}{2(1 - |a + b|K)} \varepsilon(-x, -x) \quad (\forall x \in X). \quad (21)$$

Proof Put $u = -x, v = -y$ for each $x, y \in X$. Under these transformations, (20) changes into the following estimate:

$$\left\| \frac{1}{2} \{f(-au - bv) + f(-bu - av)\} + \frac{a + b}{2} \{f(u) + f(v)\} \right\| \leq \varepsilon_1(u, v) \quad (\forall u, v \in X), \quad (22)$$

where $\varepsilon_1(u, v) = \frac{1}{2}\varepsilon(-u, -v)$ ($\forall u, v \in X$).

Now we define $u \circ v = -au - bv, u \star v = -bu - av$ for each $u, v \in X$. In this case, we can easily see that the square operator $u \mapsto u \circ u$ is an endomorphism of X with respect to \circ and \star . Also since $a + b \neq 0$, this endomorphism is bijective and so automorphic. We denote by σ the inverse mapping of this automorphism. Moreover, we define $s \diamond t = -\frac{1}{2}(a + b)(s + t), s \star t = \frac{1}{2}(s + t)$ for each $s, t \in E$. Then we can also see that the binary operations \diamond and \star on E are continuous and the square operator $\tau : s \mapsto s \diamond s$ is an automorphism of E with respect to \diamond and \star . Note that (22) changes into the following:

$$\|f(u \circ v) \star f(u \star v) - f(u) \diamond f(v)\| \leq \varepsilon_1(u, v) \quad (\forall u, v \in X). \quad (23)$$

Since $x \circ x = x \star x$ for all $x \in X$, it follows that $\sigma x \star \sigma x = \sigma x \circ \sigma x = \sigma^{-1} \sigma x = x$ for all $x \in X$. Also, since $s \star s = s$ for all $s \in E$, it follows that $f(x) \star f(\sigma x \star \sigma x) = f(x) \star f(x) = f(x)$ for all $x \in X$ and then (4) holds with $\delta = 0$. Moreover, $\beta_{\sigma, \delta} = 0$ must hold with $\delta = 0$. It is also obvious that $\gamma_\tau = |a + b|$ from the definition of τ . We also note that $\alpha_{\sigma, \varepsilon_1} \leq K$ from the

second condition of (vii) and hence $\gamma_\tau \alpha_{\sigma, \varepsilon_1} \leq |a + b|K < 1$ from the first condition of (vii). Therefore, by Theorem 1, there exists a unique mapping $f_\infty : X \rightarrow E$ such that

$$f_\infty(u \circ v) \star f_\infty(u * v) = f_\infty(u) \diamond f_\infty(v) \quad (\forall u, v \in X),$$

namely, (19) holds and

$$\|f(u) - f_\infty(u)\| \leq \frac{\alpha_{\sigma, \varepsilon_1} \varepsilon_1(u, u)}{1 - \gamma_\tau \max\{\alpha_{\sigma, \varepsilon_1}, \beta_{\sigma, \delta}\}} \leq \frac{K}{2(1 - |a + b|K)} \varepsilon(-u, -u) \quad (\forall u \in X),$$

and so (21) holds. □

The following is also an Ulam type stability result for the mapping satisfying (19).

Corollary 4 (cf. [17, Theorem 2.2]) *Let $\varepsilon : X \times X \rightarrow \mathbb{R}^+$ and suppose that*

(viii) $\exists K \geq 0 : K < |a + b|$ and $\varepsilon(-(a + b)x, -(a + b)y) \leq K\varepsilon(x, y) \quad (\forall x, y \in X)$.

If a mapping $f : X \rightarrow E$ satisfies (20), then there exists a unique mapping $f_\infty : X \rightarrow E$ satisfying (19) and

$$\|f(x) - f_\infty(x)\| \leq \frac{1}{2(|a + b| - K)} \varepsilon(-x, -x) \quad (\forall x \in X). \tag{24}$$

Proof As observed in the proof of Corollary 3, (20) changes into (22). Now we define $u \circ v = -au - bv$, $u * v = -bu - av$ for each $u, v \in X$. In this case, we can easily see that the square operator $\tilde{\sigma} : u \mapsto u \circ u$ is an endomorphism of X with respect to \circ and $*$. Moreover, we define $s \diamond t = -\frac{1}{2}(a + b)(s + t)$, $s \star t = \frac{1}{2}(s + t)$ for each $s, t \in E$. Then we can also see that the binary operations \diamond and \star on E are continuous and the square operator $s \mapsto s \diamond s$ is an endomorphism of E with respect to \diamond and \star . Also since $a + b \neq 0$, this endomorphism is bijective and so automorphic. We denote by $\tilde{\tau}$ the inverse mapping of this automorphism. Note that (22) changes into (23). Since $x \circ x = x * x \quad (\forall x \in X)$ and $s \star s = s \quad (\forall s \in E)$, it follows that $f(x \circ x) \star f(x * x) = f(x \circ x)$ for all $x \in X$ and then (12) holds with $\delta = 0$.

Moreover, $\beta_{\tilde{\sigma}, \delta} = 0$ must hold with $\delta = 0$. It is also obvious that $\gamma_{\tilde{\tau}} = |a + b|^{-1}$ from the definition of $\tilde{\tau}$. We also note that $\alpha_{\tilde{\sigma}, \varepsilon_1} \leq K$ from the second condition of (viii) and hence $\gamma_{\tilde{\tau}} \alpha_{\tilde{\sigma}, \varepsilon_1} \leq |a + b|^{-1}K < 1$ from the first condition of (viii).

Therefore, by Theorem 2, there exists a unique mapping $f_\infty : X \rightarrow E$ such that

$$f_\infty(u \circ v) \star f_\infty(u * v) = f_\infty(u) \diamond f_\infty(v) \quad (\forall u, v \in X),$$

namely, (19) holds and

$$\begin{aligned} \|f(u) - f_\infty(u)\| &\leq \frac{\gamma_{\tilde{\tau}} \varepsilon_1(u, u)}{1 - \gamma_{\tilde{\tau}} \max\{\alpha_{\tilde{\sigma}, \varepsilon_1}, \beta_{\tilde{\sigma}, \delta}\}} \\ &\leq \frac{|a + b|^{-1}}{2(1 - |a + b|^{-1}K)} \varepsilon(-u, -u) \\ &= \frac{1}{2(|a + b| - K)} \varepsilon(-u, -u) \quad (\forall u \in X), \end{aligned}$$

and so (24) holds. □

Corollary 5 (cf. [17, Corollary 2.3]) *Suppose that $|a + b| \neq 1$, $\delta, p, q \geq 0$ and $p + q \neq 1$. If a mapping $f : X \rightarrow E$ satisfies*

$$\|f(ax + by) + f(bx + ay) + (a + b)\{f(-x) + f(-y)\}\| \leq \delta \|x\|^p \|y\|^q$$

for all $x, y \in X$, then there exists a unique mapping $f_\infty : X \rightarrow E$ satisfying (19) and

$$\|f(x) - f_\infty(x)\| \leq \frac{\delta}{2(|a + b|^{p+q} - |a + b|)} \|x\|^{p+q} \quad (\forall x \in X).$$

Proof Put $\varepsilon(x, y) = \delta \|x\|^p \|y\|^q$ for each $x, y \in X$.

(a) The case where either

$$\begin{cases} |a + b| > 1, \\ p + q > 1, \end{cases}$$

or

$$\begin{cases} |a + b| < 1, \\ p + q < 1. \end{cases}$$

Put $K = |a + b|^{-(p+q)}$. Then K satisfies (vii). Note also that

$$\frac{K}{2(1 - |a + b|K)} \varepsilon(-x, -x) = \frac{\delta}{2(|a + b|^{p+q} - |a + b|)} \|x\|^{p+q}$$

for all $x \in X$. Then the desired result follows from Corollary 3.

(b) The case where either

$$\begin{cases} |a + b| > 1, \\ p + q < 1, \end{cases}$$

or

$$\begin{cases} |a + b| < 1, \\ p + q > 1. \end{cases}$$

Put $K = |a + b|^{p+q}$. Then K satisfies (viii). Note also that

$$\frac{1}{2(|a + b| - K)} \varepsilon(-x, -x) = \frac{\delta}{2(|a + b| - |a + b|^{p+q})} \|x\|^{p+q}$$

for all $x \in X$. Then the desired result follows from Corollary 4. □

4 Application II

Let $(X, +)$ be an Abelian group. In [18], the following result has been shown by A. Simon and P. Volkman.

Lemma B ([18, Théorème 1]) *A mapping $f : X \rightarrow \mathbb{R}$ satisfies*

$$\max\{f(x+y), f(x-y)\} = f(x) + f(y) \quad (\forall x, y \in X), \tag{25}$$

if and only if $f(x) = |\pi(x)|$ ($\forall x \in X$) for some additive function $\pi : X \rightarrow \mathbb{R}$.

In this section, we deal with the Ulam type stability problem for Equation (25). Put $x \circ y = x + y$ and $x * y = x - y$ for each $x, y \in X$. Moreover, put $s \diamond t = s + t$ and $s \star t = \max\{s, t\}$ for each $s, t \in \mathbb{R}$. Then (25) changes into (2). Also we can easily see that the square operation $\tilde{\sigma} : x \mapsto x \circ x$ is endomorphic with respect to \circ and $*$ and that the square operator $s \mapsto s \diamond s$ is automorphic with respect to \diamond and \star . Denote by $\tilde{\tau}$ the inverse mapping of this automorphism. In this case, it is obvious that $\tilde{\tau}(s) = \frac{1}{2}s$ for each $s \in \mathbb{R}$ and hence $\gamma_{\tilde{\tau}} = 1/2$.

Now let ε be a nonnegative constant and suppose that $f : X \rightarrow \mathbb{R}$ satisfies

$$\left| \max\{f(x+y), f(x-y)\} - \{f(x) + f(y)\} \right| \leq \varepsilon \quad (\forall x, y \in X). \tag{26}$$

Putting $x = y = 0$ in (26), we obtain

$$|f(0)| \leq \varepsilon. \tag{27}$$

Also, putting $x = y$ in (26), we obtain

$$-\varepsilon + f(0) \leq -\varepsilon + \max\{f(x+x), f(0)\} \leq 2f(x) \quad (\forall x \in X). \tag{28}$$

Combining (27) and (28), we obtain

$$-\varepsilon \leq f(x) \quad (\forall x \in X). \tag{29}$$

Put $\delta = 2\varepsilon$. By (27) and (28), we obtain

$$0 \leq \max\{f(x+x), f(0)\} - f(x+x) \leq \varepsilon + \varepsilon = \delta \quad (\forall x \in X),$$

and hence (12) holds. Moreover, note that $\alpha_{\tilde{\sigma}, \varepsilon} = \beta_{\tilde{\sigma}, \delta} = 1$ since ε and δ are constant. Then Lemma B and Theorem 2 easily imply the following.

Corollary 6 *Let X be an Abelian group and ε a nonnegative constant. If $f : X \rightarrow \mathbb{R}$ satisfies (26), then there exists an additive mapping $\pi : X \rightarrow \mathbb{R}$ such that*

$$|f(x) - |\pi(x)|| \leq 3\varepsilon \quad (\forall x \in X).$$

For the related results, see [19, 20].

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to this paper. They read and approved the final manuscript.

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