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Clifford Modules, Finite-Dimensional Approximation and Twisted K-Theory

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Abstract. A twisted version of Furuta's generalized vector bundle provides a finite-dimensional model of twisted K-theory. We generalize this fact involving actions of Clifford algebras. As an application, we show that an analogy of the Atiyah-Singer map for the generalized vector bundles is bijective. Furthermore, a finite-dimensional model of twisted K-theory with coefficients $\mathbb{Z}/p$ is given.

1. Introduction

Furuta's generalized vector bundle [9], which we call a *vectorial bundle* in this paper, arises naturally as a geometric object approximating a family of Fredholm operators. This means that there is a natural homomorphism of groups

$$\alpha: \ [X, \mathcal{F}(\mathcal{H})] \longrightarrow KF(X),$$

where $[X, \mathcal{F}(\mathcal{H})]$ is the group of homotopy classes of continuous maps from a topological space X to the space $\mathcal{F}(\mathcal{H})$ of Fredholm operators on a separable Hilbert space $\mathcal{H}$, and $KF(X)$ is the group of homotopy classes of ($\mathbb{Z}/2$-graded) vectorial bundles on X. Usual vector bundles are examples of vectorial bundles, so that there exists a natural homomorphism from the K-group $K(X)$ to $KF(X)$. It is shown [9] that this homomorphism $K(X) \to KF(X)$ is an isomorphism on a compact Hausdorff space X. In this case, the K-group of X is also realized as $[X, \mathcal{F}(\mathcal{H})]$, as is well-known [1]. Hence the homomorphism α, coming from a "finite-dimensional approximation", turns out to be bijective.

In [10], the construction above is generalized to

$$\alpha: \ K^{\tau}(X) \longrightarrow KF^{\tau}(X),$$

where $K^{\tau}(X)$ stands for the *twisted K-group* [5, 7] twisted by a principal bundle τ over X whose structure group is the projective unitary group of $\mathcal{H}$,

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and $KF^\tau(X)$ consists of homotopy classes of *τ-twisted vectorial bundles* on X. The homomorphism α again comes from an idea of finite-dimensional approximation of a family of Fredholm operators, and turns out to be bijective for any CW complex X. It should be noticed that a general description of a class in $K^\tau(X)$ usually involves some infinite-dimensional objects. The isomorphism above provides a way to describe $K^\tau(X)$ in terms of finite-dimensional objects.

The aim of this paper is to generalize the isomorphisms α involving actions of Clifford algebras: let $Cl(n) = Cl(\mathbb{R}^n)$ be the Clifford algebra associated to $\mathbb{R}^n$ equipped with the standard metric, $\mathcal{H}_n$ a separable infinite-dimensional $\mathbb{Z}/2$-graded Hilbert space which contains each irreducible $\mathbb{Z}/2$-graded module of $Cl(n)$ infinitely many, and $\mathcal{F}_n$ the non-contractible connected component of the space of self-adjoint Fredholm operators on $\mathcal{H}_n$ which are degree 1 (i.e. switching the gradings) and anti-commute with the actions of generators of $Cl(n)$. As is known [6], $\mathcal{F}_n$ classifies the K-cohomology K^{-n}, so that $[X, \mathcal{F}_n] \cong K^{-n}(X)$. On the other hand, vectorial bundles with $Cl(n)$-actions are also introduced in [9]. Their homotopy classes constitute a group $KF_{Cl(n)}(X)$, providing a model of the K-cohomology $K^{-n}(X)$. As before, we can construct a natural homomorphism

$$\alpha: \ [X, \mathcal{F}_n] \longrightarrow KF_{Cl(n)}(X).$$

Taking a "twist" into account, we also have a natural homomorphism

$$K^{\tau-n}(X) \longrightarrow KF^\tau_{Cl(n)}(X).$$

Then we will prove:

THEOREM 1. *For any twist τ on a CW complex X, the homomorphism $K^{\tau-n}(X) \to KF^\tau_{Cl(n)}(X)$ is bijective.*

The idea of the proof of Theorem 1 is parallel to that in [10]: we lift $K^{\tau-n}(X)$ and $KF^\tau_{Cl(n)}(X)$ to certain generalized cohomology theories, and compare these theories by using a natural transformation induced from α. Then the problems reduce to the case of a single point: The key fact that the natural transformation is bijective in this case again relies on a result of Furuta [9].

The main result in [10] allows us to describe classes in $K^{\tau-n}(X)$ by using ordinary twisted vectorial bundles on $X \times [0,1]^n$, whereas Theorem

1 provides a different way to describe classes in $K^{\tau-n}(X)$. The equivalence of these two options is useful in studying $K^{\tau-n}(X)$, and will be applied to a construction of *twisted differential K-cohomology* in a forthcoming paper.

A more simple application of Theorem 1 is the bijectivity of a homomorphism

$$\mathrm{AS}: \ KF^{\tau}_{Cl(n)}(X) \longrightarrow KF^{\tau-1}_{Cl(n-1)}(X),$$

whose construction is similar to that of the homotopy equivalence $\mathcal{F}_n \to \Omega\mathcal{F}_{n-1}$ of Atiyah-Singer [6]. Another application of Theorem 1 is an introduction of a finite-dimensional model of twisted mod p K-theory, or twisted K-theory with coefficients in $\mathbb{Z}/p$, based on twisted vectorial bundles with Clifford action.

The organization of this paper is as follows: In Section 2, we recall Clifford modules [2, 8], and the classifying space $\mathcal{F}_n$ of the K-cohomology constructed out of the space of Fredholm operators [6]. In Section 3, we briefly review a definition of twisted K-theory, and summarize axioms of the induced cohomology theory. In Section 4, we introduce twisted vectorial bundles with Clifford action, generalizing an idea in [9]. The definition is quite parallel to that of twisted vectorial bundles without Clifford action [10]. In this section, we also summarize axioms of certain cohomology theory induced from $KF^{\tau}_{Cl(n)}(X)$: its proof is skipped, because the argument in [10] is straightly generalized to the present case. Then, in Section 5, we introduce the homomorphisms α and prove our main theorem (Theorem 5.2), from which Theorem 1 is derived as a corollary. In the proof of the main theorem, we refrain from reproducing the same argument as that in [10], and only details a proof of a key proposition. Finally, in Section 6, we introduce the counterpart of the Atiyah-Singer map to twisted vectorial bundles with Clifford action, and prove its bijectivity. Our finite-dimensional model of twisted mod p K-theory is also provided in this section.

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2. Review of Clifford Modules and Fredholm Operators

2.1. Clifford modules

For $n > 0$, we let $Cl(n) = Cl(\mathbb{R}^n)$ be the Clifford algebra associated to the standard $\mathbb{R}^n$, that is, the algebra over $\mathbb{R}$ generated by the generators e_i,

$(i = 1, \dots, n)$ subject to the relation $e_i e_j + e_j e_i = -2\delta_{i,j}$.

By a (unitary) module of $Cl(n)$, we mean a $\mathbb{Z}/2$-graded Hermitian vector space $V = V^0 \hat{\oplus} V^1$ over $\mathbb{C}$ equipped with an algebra homomorphism $\rho : Cl(n) \to \mathrm{End}_{\mathbb{C}}(V)$ such that $\rho(e_i) : V \to V$, $(i = 1, \dots, n)$ are skew-Hermitian maps of degree 1. (As a convention of this paper, we put a hat on the symbol of the direct sum to distinguish the grading of a $\mathbb{Z}/2$-graded vector space V: the even part appears on the left of $\hat{\oplus}$ and the odd part on the right.)

Finite-rank irreducible modules of $Cl(n)$ are classified as follows: if n is odd, then $Cl(n)$ has essentially a unique irreducible module Δ_n; if n is even, then $Cl(n)$ has essentially two distinct irreducible modules $\Delta_n^{\pm}$. One irreducible module is obtained by switching the grading of the other. These irreducible modules are distinguished by the action of the volume element, that is,

$$\rho_{\Delta_n^{\pm}}(e_1 \cdots e_n) = \pm(\sqrt{-1})^{n/2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with respect to the decomposition $\Delta_n^{\pm} = (\Delta_n^{\pm})^0 \hat{\oplus} (\Delta_n^{\pm})^1$. For convenience, we put $\Delta_n = \Delta_n^+ \oplus \Delta_n^-$.

Under the natural isomorphism $Cl(n) \otimes Cl(n') \cong Cl(n+n')$, a $Cl(n)$-module V and a $Cl(n')$-module V' give a $Cl(n+n')$-module $V \otimes V'$, where the tensor product is taken in the $\mathbb{Z}/2$-graded sense. If n or n' is even, and both V and V' are irreducible, then $V \otimes V'$ is also irreducible. In particular, $\Delta_{2m}^+ \otimes \Delta_{2m'}^+ \cong \Delta_{2(m+m')}^+$.

The above behaviour of irreducible modules under tensor products implies:

LEMMA 2.1 ([8, 9]). *Let n and m be positive integers.*

(1) *The category of $Cl(n)$-modules and that of $Cl(n+2m)$-modules are equivalent under the functor assigning $V \otimes \Delta_{2m}^+$ to a $Cl(n)$-module V and $f \otimes \mathrm{id}$ to a homomorphism f of $Cl(n)$-modules.*

(2) *The functor induces an isomorphism $H_{\mathbb{Z}/2}(V) \cong H_{\mathbb{Z}/2}(V \otimes \Delta_{2m}^+)$, where $H_{\mathbb{Z}/2}(V)$ is the following vector space introduced to any $Cl(n)$-module V:*

$$H_{\mathbb{Z}/2}(V) = \left\{ \gamma : V \to V \,\middle|\, \begin{array}{l} \text{degree 1, Hermitian,} \\ \rho_V(e_i)\gamma + \gamma\rho_V(e_i) = 0 \text{ for } i = 1, \dots, n \end{array} \right\},$$

and $H_{\mathbb{Z}/2}(V \otimes \Delta_{2m}^+)$ is defined similarly.

Notice that this lemma also makes sense in the case of $n = 0$. (In this case, we forget Clifford actions, and regard a $Cl(0)$-module V as just a $\mathbb{Z}/2$-graded Hermitian vector space, and $H_{\mathbb{Z}/2}(V)$ as the space of degree 1 Hermitian maps on V.)

For $n = 1, 2$, we describe the irreducible $Cl(n)$-modules explicitly. In the case of $n = 1$, the irreducible module is $\Delta_1 = \mathbb{C}\hat{\oplus}\mathbb{C}$ and $\rho_{\Delta_1}(e_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. In the case of $n = 2$, the irreducible $Cl(2)$-module Δ_2^+ is $\Delta_2^+ = \mathbb{C}\hat{\oplus}\mathbb{C}$ and

$$\rho_{\Delta_2^+}(e_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \rho_{\Delta_2^+}(e_2) = \begin{pmatrix} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}.$$

We easily see $H_{\mathbb{Z}/2}(\Delta_1) = \mathbb{C}$, with its basis $\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $H_{\mathbb{Z}/2}(\Delta_2^+) = 0$.

2.2. Fredholm operators

For $n > 0$, let $\mathcal{H}_n$ be a separable infinite-dimensional $\mathbb{Z}/2$-graded Hilbert space which contains each irreducible $Cl(n)$-modules infinitely many. A particular construction of $\mathcal{H}_n$ is $\mathcal{H}_n = H \otimes \Delta_n$, where H is an ungraded separable Hilbert space of infinite-dimension. We also let $\tilde{\mathcal{F}}_n$ be the space of degree 1 self-adjoint Fredholm operators on $\mathcal{H}_n$ anti-commuting with the actions of $e_i \in Cl(n)$, $(i = 1, \ldots, n)$:

$$\tilde{\mathcal{F}}_n = \left\{ A : \mathcal{H}_n \to \mathcal{H}_n \,\middle|\, \begin{array}{l} \text{degree 1, Fredholm, } A^* = A \\ Ae_i + e_iA = 0 \text{ for } i = 1, \ldots, n \end{array} \right\}.$$

We topologize this space by the operator norm. In the case that n is odd, $\tilde{\mathcal{F}}_n$ has three connected components [6]. Two of them are contractible, and we will denote the remaining non-trivial component by $\mathcal{F}_n(\mathcal{H}_n) = \mathcal{F}_n$. In the case that n is even, we put $\mathcal{F}_n(\mathcal{H}_n) = \mathcal{F}_n = \tilde{\mathcal{F}}_n$. In the case of $n = 0$, we also define $\mathcal{F}_0 = \tilde{\mathcal{F}}_0$ to be the space of degree 1 self-adjoint Fredholm operators on a separable infinite-dimensional $\mathbb{Z}/2$-graded Hilbert space.

Notice that there exists a homotopy equivalence [6]:

$$\mathrm{AS}: \ \mathcal{F}_n(\mathcal{H}_n) \longrightarrow \Omega\mathcal{F}_{n-1}(\mathcal{H}_n),$$

where $\Omega\mathcal{F}_{n-1}(\mathcal{H}_n)$ stands for the space of maps $\tilde{A} : [-1, 1] \to \mathcal{F}_{n-1}(\mathcal{H}_n)$ such that $\tilde{A}(\pm 1)$ are invertible. For $A \in \mathcal{F}_n(\mathcal{H}_n)$, an explicit description of the map $\mathrm{AS}(A) : [-1, 1] \to \mathcal{F}_{n-1}(\mathcal{H}_n)$ is

$$\mathrm{AS}(A)(t) = A + \sqrt{-1} t e_n.$$

Notice also that there is a homeomorphism $\mathcal{F}_n \cong \mathcal{F}_{n+2m}$, ([6]). This homeomprhism $\mathcal{F}_n(\mathcal{H}_n) \to \mathcal{F}_{n+2m}(\mathcal{H}_{n+2m})$ is given by $A \mapsto A \otimes \mathrm{id}$ under the identification $\mathcal{H}_{n+2m} \cong \mathcal{H}_n \otimes \Delta^+_{2m}$.

Because of the homotopy equivalence $\mathcal{F}_n \to \Omega\mathcal{F}_{n-1}$, the space $\mathcal{F}_n$ provides a model of the classifying space of the K-theory of degree $-n$. Put differently, we may define the K-group $K^{-n}(X)$ of a CW complex X to be the homotopy classes of continuous maps from X to $\mathcal{F}_n$. Under this realization of K^{-n}, the homeomorphism $\mathcal{F}_n \cong \mathcal{F}_{n+2n}$ induces the Bott periodicity.

Remark 1. As a model of the classifying space of K^{-n}, the space of Fredholm operators $\mathcal{F}_n$ is chosen in this paper. We can also choose the model provided in [5]. With this choice, the subsequent argument is still valid.

3. Twisted K-Theory

3.1. Twisted K-theory

To twist usual topological K-theory, we will use a principal bundle whose structure group is a projective unitary group: For a separable infinite-dimensional Hilbert space H, the projective unitary group $PU(H)$ is defined by the quotient $PU(H) = U(H)/U(1)$. We topologize $PU(H)$ by using the the operator norm topology on $U(H)$. Then, for $n \geq 0$, the group $PU(H)$ acts on $\mathcal{F}_n(\mathcal{H}_n) = \mathcal{F}_n(H \otimes \Delta_n)$ by conjugation, and we can associate a fiber bundle $\mathcal{F}_n(\tau) = \tau \times_{PU(H)} \mathcal{F}_n$ to a given principal $PU(H)$-bundle τ over a space X. (In the case that we employ the model of the classifying space of K-theory in [5], we give $PU(H)$ a compact open topology.)

Let $\Gamma(X, \mathcal{F}_n(\tau))$ be the space of sections of this fiber bundle $\mathcal{F}_n(\tau) \to X$. For a section $\mathbb{A} \in \Gamma(X, \mathcal{F}_n(\tau))$, we define the support of $\mathbb{A}$ to be the closure of the set of points $x \in X$ at which $\mathbb{A}_x$ in not invertible:

$$\mathrm{Supp}(\mathbb{A}) = \overline{\{x \in X | \ \mathbb{A}_x \text{ is not invertible}\}}.$$

For a closed subspace $Y \subset X$, we denote by $\Gamma(X, Y, \mathcal{F}_n(\tau))$ the set of sections $\mathbb{A} \in \Gamma(X, \mathcal{F}_n(\tau))$ such that $\mathrm{Supp}(\mathbb{A}) \cap Y = \emptyset$.

Now, we define $K^{\tau}_{Cl(n)}(X, Y)$ to be the homotopy classes of $\mathbb{A} \in \Gamma(X, Y, \mathcal{F}_n(\tau))$. Two sections $\mathbb{A}_0, \mathbb{A}_1 \in \Gamma(X, Y, \mathcal{F}_n(\tau))$ are said to be *homotopic* if there exists a section $\tilde{\mathbb{A}} \in \Gamma(X \times I, Y \times I, \mathcal{F}_n(\tau) \times I)$ such that $\tilde{\mathbb{A}}|_{X \times \{i\}} = \mathbb{A}_i$, $(i = 0, 1)$. (We denote by $I = [0, 1]$ the unit interval.) A choice of an identification $\mathcal{H}_n \oplus \mathcal{H}_n \cong \mathcal{H}_n$ makes $K^{\tau}_{Cl(n)}(X, Y)$ into an abelian group. In view of the homotopy equivalence $\mathcal{F}_n \to \Omega\mathcal{F}_{n-1}$, the group $K^{\tau}_{Cl(n)}(X, Y)$ is isomorphic to $K^{\tau - n}(X, Y) = K^{\tau}(X \times I^n, Y \times I^n \cup X \times \partial I^n)$, the τ-twisted K-group [5, 7] of degree $-n$.

3.2. Axioms of twisted K-theory

To lift the group $K^{\tau}_{Cl(n)}(X, Y)$ into a generalized cohomology theory, we introduce a category $\hat{\mathcal{C}}$ as follows: an object in $\hat{\mathcal{C}}$ is a triple (X, Y, τ) consisting of a CW pair (X, Y) and a principal $PU(H)$-bundle $\tau \to X$. A morphism $(f, F) : (X', Y', \tau') \to (X, Y, \tau)$ in $\hat{\mathcal{C}}$ consists of a continuous map $f : X' \to X$ such that $f(Y') \subset Y$ and a bundle isomorphism $F : \tau' \to f^*\tau$ covering the identity of X'.

For $(X, Y, \tau) \in \hat{\mathcal{C}}$, we define the group $K^{\tau - j}_{Cl(n)}(X, Y)$ by

$$K^{\tau - j}_{Cl(n)}(X, Y) = \begin{cases} K^{\tau \times I^j}_{Cl(n)}(X \times I^j, Y \times I^j \cup X \times \partial I^j), & (j \geq 0) \\ K^{\tau + j}_{Cl(n)}(X, Y). & (j < 0) \end{cases}$$

A morphism $(f, F) : (X', Y', \tau') \to (X, Y, \tau)$ clearly induces a homomorphism $(f, F)^* : K^{\tau - j}_{Cl(n)}(X, Y) \to K^{\tau' - j}_{Cl(n)}(X', Y')$. Thus, the assignment $(X, Y, \tau) \to K^{\tau - j - n}(X, Y)$ gives rise to a functor from $\hat{\mathcal{C}}$ to the category of abelian groups. Since $K^{\tau - j}_{Cl(n)}(X, Y) \cong K^{\tau - j - n}(X, Y)$, we see the following properties from [7]:

Proposition 3.1. *The functors assigning $K^{\tau + j}_{Cl(n)}(X, Y)$ to $(X, Y, \tau) \in \hat{\mathcal{C}}$, $(j \in \mathbb{Z})$ have the following properties:*

(1) *(Homotopy axiom) If $(f_i, F_i) : (X', Y', \tau') \to (X, Y, \tau)$, $(i = 0, 1)$ are homotopic, then the induced homomorphisms coincide: $(f_0, F_0)^* = (f_1, F_1)^*$.*

(2) *(Excision axiom) For subcomplexs $A, B \subset X$, the inclusion map induces the isomorphism:*

$$K^{\tau+j}_{Cl(n)}(A \cup B, B) \cong K^{\tau+j}_{Cl(n)}(A, A \cap B).$$

(3) *(Exactness axiom) There is the natural long exact sequence:*

$$\cdots \to K^{\tau+j-1}_{Cl(n)}(Y) \overset{\delta_{j-1}}{\to} K^{\tau+j}_{Cl(n)}(X, Y) \to K^{\tau+j}_{Cl(n)}(X) \to K^{\tau+j}_{Cl(n)}(Y) \overset{\delta_j}{\to} \cdots.$$

(4) *(Additivity axiom) For any family $\{(X_\lambda, Y_\lambda, \tau_\lambda)\}_{\lambda \in \Lambda}$ in $\hat{\mathcal{C}}$, the inclusion maps $X_\lambda \to \coprod_\lambda X_\lambda$ induce the natural isomorphism:*

$$K^{\coprod_\lambda \tau_\lambda + j}_{Cl(n)}(\textstyle\coprod_\lambda X_\lambda, \coprod_\lambda Y_\lambda) \cong \displaystyle\prod_\lambda K^{\tau_\lambda + j}_{Cl(n)}(X_\lambda, Y_\lambda).$$

We notice that the proof of the exactness axiom uses the Bott periodicity

$$K^{\tau-j}_{Cl(n)}(X, Y) \cong K^{\tau-j-2}_{Cl(n)}(X, Y).$$

This isomorphism is given by multiplying a generator of $K^{-2}(\mathrm{pt}) = K^0(D^2, S^1) \cong \mathbb{Z}$. (For $k > 0$, we denote by D^k the unit disk in $\mathbb{R}^k$, and by $S^{k-1} = \partial D^k$ the unit sphere.): In general, there exists a multiplication

$$K^{\tau-j}_{Cl(n)}(X, Y) \times K^{-k}_{Cl(m)}(X, Y') \longrightarrow K^{\tau-j-k}_{Cl(n+m)}(X, Y \cup Y').$$

This is induced from the map $\mathcal{F}_n(\mathcal{H}_n) \times \mathcal{F}_m(\mathcal{H}_m) \to \mathcal{F}_{n+m}(\mathcal{H}_n \otimes \mathcal{H}_m)$ given by $(A, A') \mapsto A \otimes 1 + 1 \otimes A'$, where the tensor products are taken in the graded sense.

4. Vectorial Bundles with Clifford Actions

4.1. Definitions

DEFINITION 4.1. Let n be a positive integer and X a topological space. For a subset $U \subset X$, we define the category $\mathcal{HF}_{Cl(n)}(U)$ as follows. An object in $\mathcal{HF}_{Cl(n)}(U)$ is a pair (E, h) consisting of a finite-rank $\mathbb{Z}/2$-graded Hermitian vector bundle $E \to U$ equipped with bundle maps $e_i : E \to E$, $(i = 1, \ldots, n)$ of degree 1 satisfying $e_i e_j + e_j e_i = -2\delta_{i,j}$ and of a Hermitian map $h : E \to E$ of degree 1 satisfying $he_i + e_i h = 0$, $(i = 1, \dots, n)$. The homomorphisms in $\mathcal{HF}_{Cl(n)}(U)$ are

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{HF}_{Cl(n)}(U)}((E,h),(E',h')) \\ &\qquad = \left\{ \phi : E \to E' \;\middle|\; \begin{array}{l} \text{degree } 0,\ \phi h = h'\phi, \\ e_i\phi = \phi e_i \text{ for } i = 1, \dots, n \end{array} \right\} / \doteqdot,\end{aligned}$$

where the meaning of the equivalence relation $\phi \doteqdot \phi'$ is as follows:

> For each point $x \in U$, there are a positive number $\mu > 0$ and an open subset $V \subset U$ containing x such that: for all $y \in V$ and $\xi \in (E,h)_{y,<\mu}$, we have $\phi(\xi) = \phi'(\xi)$.

In the above, we put

$$(E,h)_{y,<\mu} = \bigoplus_{\lambda<\mu} \mathrm{Ker}(h_y^2 - \lambda) = \bigoplus_{\lambda<\mu} \{\xi \in E_y |\ h_y^2\xi = \lambda\xi\}.$$

We will just write ϕ to mean the homomorphism in the category $\mathcal{HF}_{Cl(n)}(U)$ represented by $\phi : (E,h) \to (E',h')$.

Definition 4.2. Let X be a topological space, $\tau \to X$ a principal $PU(H)$-bundle, and $U \subset X$ a subset.

(a) We define the category $\mathcal{P}^\tau(U)$ as follows. The objects in $\mathcal{P}^\tau(U)$ consist of sections $s : U \to \tau|_U$. The morphisms in $\mathcal{P}^\tau(U)$ are defined by

$$\mathrm{Hom}_{\mathcal{P}^\tau(U)}(s,s') = \{g : U \to U(\mathcal{H})|\ s'\pi(g) = s\},$$

where $\pi : PU(H) \to U(H)$ is the projection. The composition of morphisms is defined by the pointwise multiplication.

(b) We define the category $\mathcal{HF}^\tau_{Cl(n)}(U)$ as follows. The objects in $\mathcal{HF}^\tau_{Cl(n)}(U)$ are the same as those in $\mathcal{P}^\tau(U) \times \mathcal{HF}_{Cl(n)}(U)$:

$$\mathrm{Obj}(\mathcal{HF}^\tau_{Cl(n)}(U)) = \mathrm{Obj}(\mathcal{P}^\tau(U)) \times \mathrm{Obj}(\mathcal{HF}_{Cl(n)}(U)).$$

The homomorphisms in $\mathcal{HF}^\tau_{Cl(n)}(U)$ are defined by:

$$\begin{aligned}&Hom_{\mathcal{HF}^\tau_{Cl(n)}(U)}((s,(E,h)),(s',(E',h'))) \\ &\qquad = \mathrm{Hom}_{\mathcal{P}^\tau(U)}(s,s') \times \mathrm{Hom}_{\mathcal{HF}_{Cl(n)}(U)}((E,h),(E',h'))/\sim,\end{aligned}$$

where the equivalence relation $\sim$ identifies (g, ϕ) with $(g\zeta, \phi\zeta)$ for any $U(1)$-valued map $\zeta : U \to U(1)$.

DEFINITION 4.3. For a positive integer n and a principal $PU(H)$-bundle τ over a topological space X, we define the category $\mathcal{KF}^{\tau}_{Cl(n)}(X)$ as follows.

(1) An object $(\mathcal{U}, \mathcal{E}_\alpha, \Phi_{\alpha\beta})$ in $\mathcal{KF}^{\tau}_{Cl(n)}(X)$ consists of an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in\mathfrak{A}}$ of X, objects $\mathcal{E}_\alpha$ in $\mathcal{HF}^{\tau}_{Cl(n)}(U_\alpha)$, and homomorphisms $\Phi_{\alpha\beta} : \mathcal{E}_\beta \to \mathcal{E}_\alpha$ in $\mathcal{HF}^{\tau}_{Cl(n)}(U_{\alpha\beta})$ such that:

$$\begin{array}{rcll} \Phi_{\alpha\beta}\Phi_{\beta\alpha} & = & 1 & \text{in } \mathcal{HF}^{\tau}_{Cl(n)}(U_{\alpha\beta}); \\ \Phi_{\alpha\beta}\Phi_{\beta\gamma} & = & \Phi_{\alpha\gamma} & \text{in } \mathcal{HF}^{\tau}_{Cl(n)}(U_{\alpha\beta\gamma}), \end{array}$$

where $U_{\alpha\beta} = U_\alpha \cap U_\beta$ and $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$ as usual. We call an object in the category $\mathcal{KF}^{\tau}_{Cl(n)}(X)$ a *τ-twisted $Cl(n)$-vectorial bundle* over X.

(2) A homomorphism $(\{U'_{\alpha'}\}, \mathcal{E}'_{\alpha'}, \Phi'_{\alpha'\beta'}) \to (\{U_\alpha\}, \mathcal{E}_\alpha, \Phi_{\alpha\beta})$ consists of homomorphisms $\Psi_{\alpha\alpha'} : \mathcal{E}'_{\alpha'} \to \mathcal{E}_\alpha$ in $\mathcal{HF}^{\tau}_{Cl(n)}(U_\alpha \cap U'_{\alpha'})$ such that the following diagrams commute in $\mathcal{HF}^{\tau}_{Cl(n)}(U_\alpha \cap U'_{\alpha'} \cap U'_{\beta'})$ and $\mathcal{HF}^{\tau}_{Cl(n)}(U_\alpha \cap U_\beta \cap U'_{\beta'})$, respectively.

$$\begin{array}{ccc} \mathcal{E}'_{\alpha'} & \xrightarrow{\Psi_{\alpha\alpha'}} & \mathcal{E}_\alpha \\ {\scriptstyle \Phi'_{\alpha'\beta'}}\uparrow & \nearrow{\scriptstyle \Psi_{\alpha\beta'}} & \\ \mathcal{E}'_{\beta'} & & \end{array} \qquad \begin{array}{ccc} & & \mathcal{E}_\alpha \\ & {\scriptstyle \Psi_{\alpha\beta'}}\nearrow & \uparrow{\scriptstyle \Phi_{\alpha\beta}} \\ \mathcal{E}'_{\beta'} & \xrightarrow[\Psi_{\beta\beta'}]{} & \mathcal{E}_\beta \end{array}$$

In the case of $n = 0$, we can identify $\mathcal{KF}^{\tau}_{Cl(0)}(X) = \mathcal{KF}^{\tau}(X)$ with the category of τ-twisted vectorial bundles ([10]) on X. Also, in the case that τ is the trivial $PU(H)$-bundle $\tau = X \times PU(H)$, we can identify $\mathcal{KF}^{\tau}_{Cl(n)}(X) = \mathcal{KF}_{Cl(n)}(X)$ with the category of ($\mathbb{Z}/2$-graded) $Cl(n)$-vectorial bundles ([9]) on X.

By definition, we can specify an object $\mathbb{E} \in \mathcal{KF}^{\tau}_{Cl(n)}(X)$ by the data

$$(\mathcal{U}, s_\alpha, g_{\alpha\beta}, (E_\alpha, h_\alpha), \phi_{\alpha\beta})$$

consisting of:

- an open cover $\mathcal{U} = \{U_\alpha\}$ of X;
- local sections $s_\alpha : U_\alpha \to \tau|_{U_\alpha}$, which define the transition functions $\bar{g}_{\alpha\beta} : U_{\alpha\beta} \to PU(H)$ by $s_\alpha \bar{g}_{\alpha\beta} = s_\beta$;
- functions $g_{\alpha\beta} : U_{\alpha\beta} \to U(H)$ such that $\pi \circ g_{\alpha\beta} = \bar{g}_{\alpha\beta}$, which define $z_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \to U(1)$ by $g_{\alpha\beta} g_{\beta\gamma} = z_{\alpha\beta\gamma} g_{\alpha\gamma}$;
- $\mathbb{Z}/2$-graded Hermitian vector bundles $E_\alpha \to U_\alpha$ of finite rank whose fibers are $Cl(n)$-modules by means of bundle maps $e_i : E_\alpha \to E_\alpha$, $(i = 1, \ldots, n)$ of degree 1 satisfying $e_i e_j + e_j e_i = -2\delta_{i,j}$.
- Hermitian maps $h_\alpha : E_\alpha \to E_\alpha$ of degree 1 such that $h_\alpha e_i + e_i h_\alpha = 0$ for all $i = 1, \ldots, n$;
- maps $\phi_{\alpha\beta} : E_\beta|_{U_{\alpha\beta}} \to E_\alpha|_{U_{\alpha\beta}}$ of degree 0 such that $h_\alpha \phi_{\alpha\beta} = \phi_{\alpha\beta} h_\beta$, $e_i \phi_{\alpha\beta} = \phi_{\alpha\beta} e_i$ for $i = 1, \ldots, n$ and:

$$\begin{array}{rcll} \phi_{\alpha\beta}\phi_{\beta\alpha} & \risingdotseq & 1 & \text{on } U_{\alpha\beta}; \\ \phi_{\alpha\beta}\phi_{\beta\gamma} & \risingdotseq & z_{\alpha\beta\gamma}\phi_{\alpha\gamma} & \text{on } U_{\alpha\beta\gamma}. \end{array}$$

The support of $\mathbb{E}$ is defined by

$$\mathrm{Supp}(\mathbb{E}) = \overline{\{x \in X |\ (h_\alpha)_x \text{ is not invertible for some } \alpha\}}.$$

For a subspace $Y \subset X$, we define $\mathcal{KF}^\tau_{Cl(n)}(X, Y)$ to be the full subcategory consisting of the objects $\mathbb{E} \in \mathcal{KF}^\tau_{Cl(n)}(X)$ such that $\mathrm{Supp}(\mathbb{E}) \cap Y = \emptyset$.

Now, for $(X, Y, \tau) \in \hat{\mathcal{C}}$, we define $KF^\tau_{Cl(n)}(X, Y)$ to be the homotopy classes of τ-twisted $Cl(n)$-vectorial bundles $\mathbb{E} \in \mathcal{KF}^\tau_{Cl(n)}(X, Y)$: we say $\mathbb{E}_0$ and $\mathbb{E}_1$ are *homotopic* if there exists $\tilde{\mathbb{E}} \in \mathcal{KF}^{\tau \times I}_{Cl(n)}(X \times I, Y \times I)$ such that $\mathbb{E}|_{X \times \{i\}}$ is isomorphic to $\mathbb{E}_i$ in $\mathcal{KF}^\tau_{Cl(n)}(X, Y)$ for each $i = 0, 1$. In the same way as in the case without $Cl(n)$-actions [9, 10], $KF^\tau_{Cl(n)}(X, Y)$ gives rise to an abelian group.

4.2. Axioms

For $(X, Y, \tau) \in \hat{\mathcal{C}}$ and $j \geq 0$, we put:

$$KF^{\tau - j}_{Cl(n)}(X, Y) = KF^{\tau \times I^j}_{Cl(n)}(X \times I^j, Y \times I^j \cup X \times \partial I^j).$$

We also put $KF^{\tau+1}_{Cl(n)}(X,Y) = KF^{\tau-1}_{Cl(n)}(X,Y)$. Then, the argument in [10] applies to $Cl(n)$-vectorial bundles, and we have a "cohomology theory":

PROPOSITION 4.4. *The functors assigning* $KF^{\tau+j}_{Cl(n)}(X,Y)$ *to* $(X,Y,\tau) \in \hat{\mathcal{C}}$, *(*$j \leq 1$*) have the following properties:*

(1) *(Homotopy axiom) If* $(f_i, F_i) : (X', Y', \tau') \to (X, Y, \tau)$, *(*$i = 0, 1$*) are homotopic, then the induced homomorphisms coincide:* $(f_0, F_0)^* = (f_1, F_1)^*$.

(2) *(Excision axiom) For subcomplexs* $A, B \subset X$, *the inclusion map induces the isomorphism:*

$$KF^{\tau+j}_{Cl(n)}(A \cup B, B) \cong KF^{\tau+j}_{Cl(n)}(A, A \cap B).$$

(3) *("Exactness" axiom) There is the natural complex of groups:*

$$\cdots \stackrel{\delta_{-1}}{\to} KF^{\tau+0}_{Cl(n)}(X,Y) \to KF^{\tau+0}_{Cl(n)}(X) \to KF^{\tau+0}_{Cl(n)}(Y) \stackrel{\delta_0}{\to} KF^{\tau+1}_{Cl(n)}(X,Y).$$

This complex is exact except at the term $KF^{\tau+0}_{Cl(n)}(Y)$.

(4) *(Additivity axiom) For a family* $\{(X_\lambda, Y_\lambda, \tau_\lambda)\}_{\lambda\in\Lambda}$ *in* $\hat{\mathcal{C}}$, *the inclusion maps* $X_\lambda \to \coprod_\lambda X_\lambda$ *induce the natural isomorphism:*

$$KF^{\coprod_\lambda \tau_\lambda + j}_{Cl(n)}(\textstyle\coprod_\lambda X_\lambda, \coprod_\lambda Y_\lambda) \cong \displaystyle\prod_\lambda KF^{\tau_\lambda + j}_{Cl(n)}(X_\lambda, Y_\lambda).$$

Notice that, in constructing δ_0 above, we use the multiplication

$$KF^{\tau}_{Cl(n)}(X,Y) \times KF(D^2, S^1) \to KF^{\tau}_{Cl(n)}(X \times D^2, Y \times D^2 \cup X \times S^1).$$

In general, we can define a multiplication

$$\otimes : \ KF^{\tau}_{Cl(n)}(X,Y) \times KF_{Cl(m)}(X,Y') \longrightarrow KF^{\tau}_{Cl(n+m)}(X, Y \cup Y').$$

This is induced from the functor $\otimes : \mathcal{HF}_{Cl(n)}(U) \times \mathcal{HF}_{Cl(m)}(U) \to \mathcal{HF}_{Cl(n+m)}(U)$ given by $(E,h) \otimes (E', h') = (E \otimes E', h \otimes 1 + 1 \otimes h')$, where the tensor products are taken in the $\mathbb{Z}/2$-graded sense.

5. Main Theorem

5.1. Finite-dimensional approximation

To begin with, we construct the following homomorphism via a "finite-dimensional approximation":

$$\alpha : \ K^{\tau}_{Cl(n)}(X) \longrightarrow KF^{\tau}_{Cl(n)}(X).$$

The construction is exactly the same as that performed in [10]: let $\mathbb{A} \in \Gamma(X, \mathcal{F}_n(\tau))$ be a section given. We then make the following choice:

- an open cover $\{U_\alpha\}$ of X;
- local sections $s_\alpha : U_\alpha \to \tau|_{U_\alpha}$ of τ, which define the transition functions $\bar{g}_{\alpha\beta} : U_{\alpha\beta} \to PU(H)$ by $s_\alpha \bar{g}_{\alpha\beta} = s_\beta$;
- lifts $g_{\alpha\beta} : U_{\alpha\beta} \to U(H)$ of the transition functions $\bar{g}_{\alpha\beta} : U_{\alpha\beta} \to PU(H)$;
- positive numbers μ_α such that the family of vector spaces

  $$E_\alpha = \bigcup_{x \in U_\alpha} (\mathcal{H}_n, (A_\alpha)_x)_{<\mu_\alpha} = \bigcup_{x \in U_\alpha} \bigoplus_{\lambda < \mu_\alpha} \{\xi \in \mathcal{H}_n |\ (A_\alpha)^2_x \xi = \lambda \xi\}$$

  becomes a vector bundle of finite rank.

By means of the trivializations s_α, the section $\mathbb{A}$ induces maps $A_\alpha : U_\alpha \to \mathcal{F}_n(\mathcal{H}_n)$ such that $g_{\alpha\beta} A_\beta g_{\alpha\beta}^{-1} = A_\alpha$. Now, for $i = 1, \dots, n$, the action of $e_i \in Cl(n)$ on $\mathcal{H}_n$ induces a vector bundle map $e_i : E_\alpha \to E_\alpha$ of degree 1 satisfying $e_i e_j + e_j e_i = -2\delta_{i,j}$. The restriction of A_α defines a Hermitian map $h_\alpha : E_\alpha \to E_\alpha$ of degree 1 anti-commuting with e_i. Finally, we define a map $\phi_{\alpha\beta} : E_\beta|_{U_{\alpha\beta}} \to E_\alpha|_{U_{\alpha\beta}}$ by the composition of the following maps:

$$E_\beta|_{U_{\alpha\beta}} \overset{\text{inclusion}}{\longrightarrow} U_{\alpha\beta} \times \mathcal{H}_n \overset{\mathrm{id}\times g_{\alpha\beta}}{\longrightarrow} U_{\alpha\beta} \times \mathcal{H}_n \overset{\text{projection}}{\longrightarrow} E_\alpha|_{U_{\alpha\beta}}.$$

Then $\mathbb{E} = (\{U_\alpha\}, s_\alpha, g_{\alpha\beta}, (E_\alpha, h_\alpha), \phi_{\alpha\beta})$ is a τ-twisted $Cl(n)$-vectorial bundle on X, and a well-defined homomorphism $\alpha : K^{\tau}_{Cl(n)}(X) \to KF^{\tau}_{Cl(n)}(X)$ is induced from the assignment $\mathbb{A} \mapsto \mathbb{E}$

The construction above also induces $\alpha : K^{\tau}_{Cl(n)}(X, Y) \to KF^{\tau}_{Cl(n)}(X, Y)$ as well as $\alpha_j : K^{\tau+j}_{Cl(n)}(X, Y) \to KF^{\tau+j}_{Cl(n)}(X, Y)$ for any $(X, Y, \tau) \in \hat{\mathcal{C}}$ and $j \leq 1$. Then, in the same way as in [10], we get:

Proposition 5.1. *The homomorphisms* $\alpha_j : K^{\tau+j}_{Cl(n)}(X, Y) \to KF^{\tau+j}_{Cl(n)}(X, Y)$, $(j \le 1)$ *give rise to natural transformations from the functors in Proposition 3.1 to those in Proposition 4.4.*

5.2. Main theorem and its corollary

Theorem 1 in Section 1 is a corollary (Corollary 5.4) to:

Theorem 5.2. *Let* τ *be any principal* $PU(H)$*-bundle over a CW complex* X*. For any* $n, j \ge 0$*, the homomorphism* $\alpha_{-j} : K^{\tau-j}_{Cl(n)}(X) \to KF^{\tau-j}_{Cl(n)}(X)$ *is bijective.*

The key to this theorem is the following proposition, which we will prove in the next subsection:

Proposition 5.3. *For any* $k, j \ge 0$*, the following homomorphism is bijective:*

$$\alpha_{-j} : K^{-j}_{Cl(n)}(D^k, S^{k-1}) \longrightarrow KF^{-j}_{Cl(n)}(D^k, S^{k-1}),$$

where (D^k, S^{k-1}) *means* $(\mathrm{pt}, \emptyset)$ *when* $k = 0$*.*

Proof of Theorem 5.2. In view of Proposition 3.1, 4.4, 5.1 and 5.3, the proof is exactly the same as that of the main result of [10]: First, in the case that X is a finite CW complex, we prove the bijectivity of α_{-j} by an induction on the number of cells in X. Then, the bijectivity of α_{-j} in the general case follows from that in the finite case through an argument by using the telescope of X. □

Corollary 5.4. *Suppose* $(X, Y, \tau) \in \hat{\mathcal{C}}$ *and* $j \ge 0$ *are given.*

(a) *The finite-dimensional approximation induces the bijection:*

$$\alpha_{-j} : K^{\tau-j}_{Cl(n)}(X, Y) \longrightarrow KF^{\tau-j}_{Cl(n)}(X, Y).$$

(b) *The multiplication of a generator of* $K(D^2, S^1) \cong KF(D^2, S^1) \cong \mathbb{Z}$ *induces the bijection:*

$$KF^{\tau-j}_{Cl(n)}(X, Y) \longrightarrow KF^{\tau-j-2}_{Cl(n)}(X, Y).$$

(c) *There exists a natural isomorphism*

$$K^{\tau-j-n}(X, Y) \cong KF^{\tau-j}_{Cl(n)}(X, Y).$$

5.3. Key proposition

This subsection is devoted to the proof of Proposition 5.3, which is clearly equivalent to:

Proposition 5.5. *For any $n, k \geq 0$, the following homomorphism is bijective:*

$$\alpha : \ K_{Cl(n)}(D^k, S^{k-1}) \longrightarrow KF_{Cl(n)}(D^k, S^{k-1}).$$

Notice that the principal $PU(H)$-bundle τ is absent (or trivial) in the present case. Therefore $K_{Cl(n)}(D^k, S^{k-1})$ is identified with the homotopy classes of maps from the k-dimensional disk D^k to $\mathcal{F}_n$ which carry all points in the sphere $S^{k-1} = \partial D^k$ into the subspace $\mathcal{F}_n^* \subset \mathcal{F}_n$ consisting of invertible operators:

$$K_{Cl(n)}(D^k, S^{k-1}) = [(D^k, S^{k-1}), (\mathcal{F}_n, \mathcal{F}_n^*)].$$

To prove Proposition 5.5, recall the homeomprhism $\mathcal{F}_n(\mathcal{H}_n) \to \mathcal{F}_{n+2m}(\mathcal{H}_{n+2m})$ given by $A \mapsto A \otimes \mathrm{id}$ under the identification $\mathcal{H}_n \otimes \Delta_{2m}^+ \cong \mathcal{H}_{n+2m}$. Consequently, for any CW pair (X, Y), we have a natural isomorphism

$$K_{Cl(n)}(X, Y) \longrightarrow K_{Cl(n+2m)}(X, Y).$$

There is a similar "periodicity" for vectorial bundles:

Lemma 5.6. *Let n be a non-negative integer. For any CW pair (X, Y) and $m > 0$, the tensor product of the irreducible $Cl(2m)$-module Δ_{2m}^+ induces a natural isomorphism $KF_{Cl(n)}(X, Y) \to KF_{Cl(n+2m)}(X, Y)$ fitting in the commutative diagram:*

$$\begin{array}{ccc} K_{Cl(n)}(X, Y) & \longrightarrow & K_{Cl(n+2m)}(X, Y) \\ \alpha \downarrow & & \downarrow \alpha \\ KF_{Cl(n)}(X, Y) & \longrightarrow & KF_{Cl(n+2m)}(X, Y). \end{array}$$

Proof. The first part of this lemma, which is shown in [9], follows from Lemma 2.1. The second part is clear by construction. □

As a consequence of this lemma, it suffices to consider the case of $n = 0$ and $n = 1$ only in Proposition 5.5. In the case of $n = 0$, the proposition is established in [10]. Hence we are left with the case of $n = 1$. To deal with this case, we use the following fact (Remark 10.29 (2), [9]):

PROPOSITION 5.7 ([9]). *For $n, k > 0$, there is a natural isomorphism*

$$KF_{Cl(n)}(\mathrm{pt}) \longrightarrow KF_{Cl(k+n)}(D^k, S^{k-1})$$

given by the multiplication of the "symbol of the k-dimensional supersymmetric harmonic oscillator".

The symbol of the 1-*dimensional supersymmetric harmonic oscillator* ([8]) is the $Cl(1)$-vectorial bundle $(F, h) \in \mathcal{KF}_{Cl(1)}(I, \partial I)$ defined by:

$$F = I \times \Delta_1 = I \times (\mathbb{C} \hat{\oplus} \mathbb{C}), \qquad h = \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}, \quad (t \in I = [-1, 1]).$$

The symbol of the k-dimensional supersymmetric harmonic oscillator is the $Cl(k)$-vectorial bundle $\otimes_{i=1}^k \pi_i^*(F, h) \in \mathcal{KF}_{Cl(k)}(I^k, \partial I^k)$, where $\pi_i : I^k \to I$ is the projection onto the ith factor.

Proposition 5.7 leads to the following computational result:

COROLLARY 5.8. *For $k \geq 0$, we have:*

$$KF_{Cl(1)}(D^k, S^{k-1}) \cong \begin{cases} \mathbb{Z}, & (k : \mathit{odd}) \\ 0. & (k : \mathit{even}) \end{cases}$$

PROOF. First, we consider the case that k is an odd integer $k = 2m+1$. By means of Lemma 5.6 and Proposition 5.7, we have

$$KF_{Cl(1)}(D^{2m+1}, S^{2m}) \cong KF_{Cl(2m+1)}(D^{2m+1}, S^{2m}) \cong KF(\mathrm{pt}) \cong \mathbb{Z}.$$

In the even case $k = 2m$, we use Lemma 5.6 and Proposition 5.7 again to have

$$KF_{Cl(1)}(D^{2m}, S^{2m-1}) \cong KF_{Cl(2m+1)}(D^{2m}, S^{2m-1}) \cong KF_{Cl(1)}(\mathrm{pt}).$$

That $KF_{Cl(1)}(\mathrm{pt}) = 0$ is shown as follows: any element in $KF_{Cl(1)}(\mathrm{pt})$ can be represented by a pair (E, h) of a $Cl(1)$-module E and a Hermitian map $h : E \to E$ of degree 1 anti-commuting with the action of $e_1 \in Cl(1)$.

Since the irreducible $Cl(1)$-module is unique up to an equivalence, we can express E as $E = V \otimes \Delta_1$, where V is a vector space of finite rank. Now we define $(\tilde{E}, \tilde{h}) \in \mathcal{KF}_{Cl(1)}([0,1])$ by setting $\tilde{E} = I \times E$ and $\tilde{h}_t = (\cos \frac{\pi t}{2})h + \sqrt{-1}(\sin \frac{\pi t}{2})1 \otimes \gamma$, where γ is a basis of $H_{\mathbb{Z}/2}(\Delta_1) = \mathbb{C}$ such that $\gamma^2 = 1$. Then $(\tilde{E}, \tilde{h})$ is a homotopy between (E, h) and a $Cl(1)$-vectorial bundle representing $0 \in KF_{Cl(1)}(\mathrm{pt})$. □

As is well-known, we have

$$K_{Cl(1)}(D^k, S^{k-1}) = [(D^k, S^{k-1}), (\mathcal{F}_1, \mathcal{F}_1^*)] = \pi_k(\mathcal{F}_1) = \begin{cases} \mathbb{Z}, & (k : \text{odd}) \\ 0. & (k : \text{even}) \end{cases}$$

Therefore $\alpha : K_{Cl(1)}(D^k, S^{k-1}) \to KF_{Cl(1)}(D^k, S^{k-1})$ is apparently bijective in the case of k even.

Now, it remains the case of k odd. Since we have $K_{Cl(1)}(D^k, S^{k-1}) \cong \mathbb{Z}$ and $KF_{Cl(1)}(D^k, S^{k-1}) \cong \mathbb{Z}$ in this case, it suffices to see the correspondence of generators through α. As is well-known [4], a self-adjoint Fredholm operator whose *spectral flow* is 1 generates $[(I, \partial I), (\mathcal{F}_1, \mathcal{F}_1^*)] \cong \pi_1(\mathcal{F}_1) \cong \mathbb{Z}$. Hence the bijectivity of α in the case of $k = 1$ (and $n = 1$) follows from:

Lemma 5.9. *There is a continuous map $A : (I, \partial I) \to (\mathcal{F}_1, \mathcal{F}_1^*)$ such that:*

(1) *its spectral flow is* 1*;*

(2) $\alpha([A]) = [(F, h)]$ *in* $KF_{Cl(1)}(I, \partial I)$.

Proof. Let H be the Hilbert space with its complete orthonormal basis $\{e_\ell\}_{\ell \in \mathbb{Z}}$. For $t \in \mathbb{R}$, we define a bounded self-adjoint operator $a_t : H \to H$ by $a_t e_\ell = (t+\ell)/\sqrt{(t+\ell)^2+1}$. A computation shows

$$\begin{aligned} \|(a_t - a_{t'})e_\ell\| &\leq \left| \frac{t+\ell}{\sqrt{(t+\ell)^2+1}} - \frac{t+\ell}{\sqrt{(t'+\ell)^2+1}} \right| \\ &\qquad + \left| \frac{t+\ell}{\sqrt{(t'+\ell)^2+1}} - \frac{t'+\ell}{\sqrt{(t'+\ell)^2+1}} \right| \\ &\leq \frac{|t-t'|}{\sqrt{(t'+\ell)^2+1}} \frac{|t+\ell|}{\sqrt{(t+\ell)^2+1}} + \frac{|t-t'|}{\sqrt{(t'+\ell)^2+1}} \leq 2|t-t'|. \end{aligned}$$

Thus, we get $\|(a_t - a_{t'})u\| \le 2|t-t'|\|u\|$ for $u \in H$, so that $\|a_t - a_{t'}\| \le 2|t-t'|$. This means the map $a : \mathbb{R} \to B(H)$ is continuous. (Here $B(H)$ is topologized by the operator norm. In the case where the topology of $B(H)$ is the compact-open topology in the sense of [5], the map $a : \mathbb{R} \to B(H)$ is still continuous, since $\mathbb{R} \times H \to H$, $((t, u) \mapsto a_t u)$ is.) Now, we choose $\epsilon > 0$ sufficiently small. Then, setting $\mathcal{H}_1 = H \otimes \Delta_1 = H \hat{\oplus} H$, $A_t = \begin{pmatrix} 0 & a_t \\ a_t & 0 \end{pmatrix}$ and $I = [-\epsilon, \epsilon]$, we get $A : (I, \partial I) \to (\mathcal{F}_1, \mathcal{F}_1^*)$ such that its spectral flow is 1 and its finite-dimensional approximation is a $Cl(1)$-vectorial bundle on $(I, \partial I)$ homotopic to (F, h). □

To establish the bijectivity of α in the case of general odd number $k = 2m+1$, we recall the map $\mathcal{F}_p(\mathcal{H}_p) \times \mathcal{F}_q(\mathcal{H}_q) \to \mathcal{F}_{p+q}(\mathcal{H}_p \otimes \mathcal{H}_q)$ inducing the ring structure on the K-cohomology theory: the explicit description of the map is $(A, B) \mapsto A \otimes 1 + 1 \otimes B$. From this description and that of vectorial bundles, we see the commutative diagram

$$\begin{array}{ccc} K_{Cl(p)}(X, Y) \times K_{Cl(q)}(X, Y') & \longrightarrow & K_{Cl(p+q)}(X, Y \cup Y') \\ {\scriptstyle \alpha \times \alpha} \big\downarrow & & \big\downarrow {\scriptstyle \alpha} \\ KF_{Cl(p)}(X, Y) \times KF_{Cl(q)}(X, Y') & \xrightarrow{\otimes} & KF_{Cl(p+q)}(X, Y \cup Y'). \end{array}$$

This induces the following commutative diagram:

$$\begin{array}{ccc} \prod_{i=1}^{2m+1} K_{Cl(1)}(I, \partial I) & \longrightarrow & K_{Cl(2m+1)}(I^{2m+1}, \partial I^{2m+1}) \\ {\scriptstyle \Pi\ \alpha} \big\downarrow & & \big\downarrow {\scriptstyle \alpha} \\ \prod_{i=1}^{2m+1} KF_{Cl(1)}(I, \partial I) & \longrightarrow & KF_{Cl(2m+1)}(I^{2m+1}, \partial I^{2m+1}). \end{array}$$

By Proposition 5.7, $KF_{Cl(2m+1)}(I^{2m+1}, \partial I^{2m+1}) \cong \mathbb{Z}$ is generated by the symbol of the $(2m+1)$-dimensional supersymmetric harmonic oscillator, which is the product of $2m+1$ copies of $(F, h) \in \mathcal{KF}_{Cl(1)}(I, \partial I)$. Thus, by Lemma 5.9 and the commutative diagram above, the homomorphism

$$\alpha : \ K_{Cl(2m+1)}(I^{2m+1}, \partial I^{2m+1}) \longrightarrow KF_{Cl(2m+1)}(I^{2m+1}, \partial I^{2m+1})$$

is surjective. Since any surjective homomorphism $\mathbb{Z} \to \mathbb{Z}$ is bijective, we conclude that α above is bijective. Therefore the following homomorphism is also bijective by Lemma 5.6:

$$\alpha : \ K_{Cl(1)}(I^{2m+1}, \partial I^{2m+1}) \longrightarrow KF_{Cl(1)}(I^{2m+1}, \partial I^{2m+1}),$$

which completes the proof of Proposition 5.5.

6. Applications

6.1. The Atiyah-Singer map

As is mentioned, the map of Atiyah-Singer [6]

$$\mathrm{AS}: \ \mathcal{F}_n(\mathcal{H}_n) \longrightarrow \Omega\mathcal{F}_{n-1}(\mathcal{H}_n)$$

is a homotopy equivalence for $n > 0$, and induces the natural isomorphism

$$\mathrm{AS}: \ K^{\tau - j}_{Cl(n)}(X, Y) \longrightarrow K^{\tau - j - 1}_{Cl(n-1)}(X, Y).$$

The aim of this subsection is to introduce a counterpart of this construction to twisted $Cl(n)$-vectorial bundles: For any space U and $(E, h) \in \mathcal{HF}_{Cl(n)}(U)$, we can define an object $(\tilde{E}, \tilde{h}) \in \mathcal{HF}_{Cl(n-1)}(U \times I)$ by setting

$$\tilde{E} = E \times I, \qquad\qquad \tilde{h}(x, t) = h(x) + \sqrt{-1} t e_n,$$

where $I = [-1, 1]$. The assignment $(E, h) \mapsto (\tilde{E}, \tilde{h})$ gives rise to a functor

$$\mathrm{AS}: \ \mathcal{HF}_{Cl(n)}(U) \longrightarrow \mathcal{HF}_{Cl(n-1)}(U \times I).$$

It is easy to globalize this construction to get the following functor for any principal $PU(H)$-bundle τ over a space X and its subspace $Y \subset X$:

$$\mathrm{AS}: \ \mathcal{KF}^{\tau}_{Cl(n)}(X, Y) \longrightarrow \mathcal{KF}^{\tau}_{Cl(n-1)}(X \times I, Y \times I \cup X \times \partial I).$$

This then induces a natural homomorphism for any $j \geq 0$.

$$\mathrm{AS}: \ KF^{\tau - j}_{Cl(n)}(X, Y) \longrightarrow KF^{\tau - j - 1}_{Cl(n-1)}(X, Y).$$

LEMMA 6.1. *For any positive integer $n > 0$ and any principal $PU(H)$-bundle τ over a space X and its subspace $Y \subset X$, the following diagram is commutative:*

$$\begin{array}{ccc} K^{\tau}_{Cl(n)}(X, Y) & \xrightarrow{\mathrm{AS}} & K^{\tau - 1}_{Cl(n-1)}(X, Y) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ KF^{\tau}_{Cl(n)}(X, Y) & \xrightarrow{\mathrm{AS}} & KF^{\tau - 1}_{Cl(n-1)}(X, Y). \end{array}$$

PROOF. Let $\mathbb{A} \in \Gamma(X, Y, \mathcal{F}_n(\tau))$ represent an element in $K^{\tau}_{Cl(n)}(X, Y)$. Suppose that we apply the construction in Subsection 5.1 to $\mathbb{A}$ to have a vectorial bundle

$$\mathbb{E} = (\{U_\alpha\}_{\alpha \in \mathfrak{A}}, s_\alpha, g_{\alpha\beta}, (E_\alpha, h_\alpha), \phi_{\alpha\beta}) \in \mathcal{KF}^{\tau}_{Cl(n)}(X, Y).$$

Hence we have $E_\alpha = \bigcup_{x \in U_\alpha} (\mathcal{H}_n, (A_\alpha)_x)_{<\mu_\alpha}$ under a choice of a positive number μ_α. Without loss of generality, we can assume that there is $\varepsilon_\alpha > 0$ satisfying

$$\lambda_1(x) \le \lambda_2(x) \le \cdots \le \lambda_r(x) < \mu_\alpha - \varepsilon_\alpha < \mu_\alpha + \varepsilon_\alpha < \lambda_{r+1}(x)$$

for all $x \in U_\alpha$, where r is the rank of the vector bundle E_α, and $\lambda_i(x)$ is the ith eigenvalue of $(A_\alpha)^2_x$. Then the twisted $Cl(n-1)$-vectorial bundle

$$\begin{aligned} \mathrm{AS}(\mathbb{E}) &= (\{\tilde{U}_\alpha\}_{\alpha \in \mathfrak{A}}, \tilde{s}_\alpha, \tilde{g}_{\alpha\beta}, (\tilde{E}_\alpha, \tilde{h}_\alpha), \tilde{\phi}_{\alpha\beta}) \\ &\in \mathcal{KF}^{\tau \times I}_{Cl(n-1)}(X \times I, Y \times I \cup X \times \partial I) \end{aligned}$$

is given by setting $\tilde{U}_\alpha = \pi^{-1}(U_\alpha) = U_\alpha \times I$, $\tilde{s}_\alpha = \pi^* s_\alpha$, $\tilde{g}_{\alpha\beta} = \pi^* g_{\alpha\beta}$, $\tilde{E} = \pi^* E_\alpha$, $\tilde{h}_\alpha(x, t) = h_\alpha(x) + \sqrt{-1} t e_n$ and $\tilde{\phi}_{\alpha\beta} = \pi^* \phi_{\alpha\beta}$, where $\pi : X \times I \to X$ is the projection. Then $\mathrm{AS}(\mathbb{E})$ represents the image $\mathrm{AS}(\alpha([\mathbb{A}]))$.

Next, we describe the image $\alpha(\mathrm{AS}([\mathbb{A}]))$ applying the construction in Subsection 5.1 to $\mathrm{AS}(\mathbb{A}) \in \Gamma(X \times I, Y \times I \cup X \times \partial I, \pi^* \mathcal{F}_n(\tau))$. By means of the local trivialization $\pi^* s_\alpha$ of $\pi^* \tau = \tau \times I$, the section $\mathrm{AS}(\mathbb{A})$ defines a map $\tilde{A}_\alpha : \tilde{U}_\alpha \to \mathcal{F}_{n-1}(\mathcal{H}_n)$. By our definition of the Atiyah-Singer map, we have $(\tilde{A}_\alpha)_{(x,t)} = (A_\alpha)_x + \sqrt{-1} t e_n$. We here define an open cover $\{V(s; \varepsilon_\alpha)\}_{s \in I}$ of $I = [-1, 1]$ by

$$V(s; \varepsilon_\alpha) = \{t \in I |\ s - \varepsilon_\alpha < t^2 < s + \varepsilon_\alpha\}.$$

Then, for any $(x, t) \in U_\alpha \times V(s; \varepsilon_\alpha)$, the eigenvalues $\tilde{\lambda}_i(x, t)$ of $(\tilde{A}_\alpha)^2_{(x,t)}$ satisfy

$$\tilde{\lambda}_1(x, t) \le \tilde{\lambda}_2(x, t) \le \cdots \le \tilde{\lambda}_r(x, t) < \mu_\alpha + s < \tilde{\lambda}_{r+1}(x, t),$$

since $\tilde{\lambda}_i(x, t) = \lambda_i(x) + t^2$. This implies

$$\bigcup_{(x,t) \in U_\alpha \times V(s; \varepsilon_\alpha)} (\mathcal{H}_n, \tilde{A}_{(x,t)})_{<\mu_\alpha + s} = \tilde{E}_\alpha|_{U_\alpha \times V(s; \varepsilon_\alpha)}.$$

Thus, $\alpha(\mathrm{AS}([\mathbb{A}]))$ is represented by the twisted $Cl(n-1)$-vectorial bundle obtained from $\mathrm{AS}(\mathbb{E})$ through the refinement $\{U_\alpha \times V(s;\varepsilon_\alpha)\}$ of the open cover $\{\tilde{U}_\alpha\}$, which is isomorphic to $\mathrm{AS}(\mathbb{E})$ itself. Hence $\mathrm{AS}(\alpha([\mathbb{A}])) = \alpha(\mathrm{AS}([\mathbb{A}]))$. $\square$

THEOREM 6.2. *For any $(X, Y, \tau) \in \hat{\mathcal{C}}$, $j \in \mathbb{Z}$ and $n > 0$, the homomorphism*

$$\mathrm{AS}: \ KF^{\tau - j}_{Cl(n)}(X, Y) \longrightarrow KF^{\tau - j - 1}_{Cl(n-1)}(X, Y)$$

is bijective.

PROOF. Lemma 6.1 provides us the commutative diagram

$$\begin{array}{ccc} K^{\tau-j}_{Cl(n)}(X,Y) & \xrightarrow{\mathrm{AS}} & K^{\tau-j-1}_{Cl(n-1)}(X,Y) \\ \alpha \big\downarrow & & \big\downarrow \alpha \\ KF^{\tau-j}_{Cl(n)}(X,Y) & \xrightarrow{\mathrm{AS}} & KF^{\tau-j-1}_{Cl(n-1)}(X,Y). \end{array}$$

Since AS in the upper row is bijective by [6], Theorem 5.2 implies the conclusion. $\square$

Lemma 5.6 is generalized to the twisted case, so that we have a natural isomorphism $KF^{\tau-j}_{Cl(n)}(X, Y) \to KF^{\tau-j}_{Cl(n+2m)}(X, Y)$. The composition of maps

$$KF^{\tau-j}_{Cl(n)}(X, Y) \longrightarrow KF^{\tau-j}_{Cl(n+2)}(X, Y) \xrightarrow{\mathrm{AS}^2} KF^{\tau-j-2}_{Cl(n)}(X, Y)$$

is readily identified with the multiplication of a generator of $K(D^2, S^1)$. Thus, Theorem 6.2 reproduces Corollary 5.4 (b).

6.2. Twisted K-theory with coefficients $\mathbb{Z}/p$

Let p be a positive integer. The aim of this subsection is to provide a model of twisted K-theory with its coefficients $\mathbb{Z}/p$, or twisted mod p K-theory by using twisted vectorial bundles. For this aim, we begin with a formulation of twisted mod p K-theory based on an idea in [3].

DEFINITION 6.3. Let τ be a principal $PU(H)$-bundle over a space X.

(a) For a non-negative integer n, we define a *τ-twisted mod p K-cocycle of degree $-n-1$ on X* to be a pair (A, T) consisting of $A \in \Gamma(X, \tau \times_{PU(H)} \mathcal{F}_n(\mathcal{H}_n))$ and $T \in \Gamma(X \times [0,1], (\tau \times [0,1]) \times_{PU(H)} \mathcal{F}_n(\mathcal{H}_n^{\oplus p}))$ such that $T|_{t=0} = A^{\oplus p}$ and $\mathrm{Supp}(T|_{t=1}) = \emptyset$.

(b) We define a *homotopy* between τ-twisted mod p K-cocycles (A_0, T_0) and (A_1, T_1) of degree $-n-1$ on X to be a τ-twisted mod p K-cocycle $(\tilde{A}, \tilde{T})$ of degree $-n-1$ on $X \times [0,1]$ such that $(\tilde{A}, \tilde{T})|_{t=i} = (A_i, T_i)$ for $i = 0, 1$.

(c) We define $K^{\tau-1}_{Cl(n)}(X; \mathbb{Z}/p)$ to be the group of homotopy classes of mod p K-cocycles of degree $-n-1$ on X. (The group structure is defined in the same way as $K^{\tau}_{Cl(n)}(X)$.)

LEMMA 6.4. *There exists a natural exact sequence:*

$$K^{\tau-1}_{Cl(n)}(X) \xrightarrow{m_p} K^{\tau-1}_{Cl(n)}(X) \xrightarrow{\rho_p} K^{\tau-1}_{Cl(n)}(X; \mathbb{Z}/p) \xrightarrow{\delta_p} K^{\tau}_{Cl(n)}(X) \xrightarrow{m_p} K^{\tau}_{Cl(n)}(X).$$

PROOF. We define δ_p by $\delta_p([(A, T)]) = [A]$ and m_p by $m_p([A]) = [A^{\oplus p}] = p[A]$. To define ρ_p, we represent an element in $K^{\tau-1}_{Cl(n)}(X)$ by a section $B \in \Gamma(X \times I, X \times \partial I, (\tau \times I) \times_{PU(H)} \mathcal{F}_n(\mathcal{H}_n^{\oplus p}))$, where $I = [0,1]$. The section $B|_{t=0}$ takes values in the space of invertible operators in $\mathcal{F}_n(\mathcal{H}_n^{\oplus p})$. Hence we can assume $B|_{t=0} = J^{\oplus p}$ for some invertible operator $J \in \mathcal{F}_n^*(\mathcal{H}_n)$. If we put $\rho_p([B]) = [(J, B)]$, then ρ_p gives rise to a well-defined a homomorphism.

Now, if $[B] \in K^{\tau-1}_{Cl(n)}(X)$ is such that $\rho_p([B]) = 0$, then there exists a homotopy $(\tilde{A}, \tilde{T})$ between (J, B) and $(J, J^{\oplus p})$. By a reparametrization of $\tilde{T}$, we can construct a homotopy connecting B and $\tilde{A}^{\oplus p}$, so that the exactness at the second term $K^{\tau-1}_{Cl(n)}(X)$ holds. To see the exactness at the third term $K^{\tau-1}_{Cl(n)}(X; \mathbb{Z}/p)$, let (A, T) be such that $[A] = 0$ in $K^{\tau}(X)$. Then there is a homotopy H between $A \in \mathcal{F}_n(\mathcal{H}_n)$ and an invertible operator $J \in \mathcal{F}_n^*(\mathcal{H}_n)$. Concatenating $H^{\oplus p}$ and T, we have B such that $\rho_p([B]) = [(A, T)]$. The exactness at the forth term $K^{\tau}(X)$ directly follows from the definitions of δ_p and m_p. $\square$

Since $K^{\tau-1}_{Cl(n)}(X) \cong K^{\tau-n-1}(X)$, the group $K^{\tau-1}_{Cl(n)}(X;\mathbb{Z}/p)$ fits into

$$K^{\tau-n-1}(X) \xrightarrow{m_p} K^{\tau-n-1}(X) \xrightarrow{\rho_p} K^{\tau-1}_{Cl(n)}(X;\mathbb{Z}/p) \xrightarrow{\delta_p} K^{\tau-n}(X) \xrightarrow{m_p} K^{\tau-n}(X).$$

Thus, the τ-twisted mod p K-theory $K^{\tau-n-1}(X;\mathbb{Z}/p)$ of X of degree $-n-1$ can be defined as $K^{\tau-n-1}(X;\mathbb{Z}/p) = K^{\tau-1}_{Cl(n)}(X;\mathbb{Z}/p)$. (By the help of the Bott periodicity, we can actually give an isomorphism between $K^{\tau-1}_{Cl(n)}(X;\mathbb{Z}/p)$ and the group $K^{\tau-n-1}(X;\mathbb{Z}/p)$ constructed out of the so-called Moore space.)

Now, we introduce our finite-dimensional model of $K^{\tau-1}_{Cl(n)}(X;\mathbb{Z}/p)$.

DEFINITION 6.5. Let τ be a principal $PU(H)$-bundle over a space X.

(a) For a non-negative integer n, we define a *τ-twisted mod p $Cl(n)$-vectorial bundle on X* to be a pair $(\mathbb{E},\mathbb{H})$ consisting of $\mathbb{E} \in \mathcal{KF}^{\tau}_{Cl(n)}(X)$ and $\mathbb{H} \in \mathcal{KF}^{\tau\times I}_{Cl(n)}(X \times I)$ such that $\mathbb{H}|_{t=0}$ is isomorphic to $\mathbb{E}$ and $\mathrm{Supp}(\mathbb{H}|_{t=1}) = \emptyset$.

(b) We define a *homotopy* between τ-twisted mod p $Cl(n)$-vectorial bundles $(\mathbb{E}_0,\mathbb{H}_0)$ and $(\mathbb{E}_1,\mathbb{H}_1)$ on X to be a $(\tau \times I)$-twisted mod p $Cl(n)$-vectorial bundle $(\tilde{\mathbb{E}},\tilde{\mathbb{H}})$ on $X \times I$ such that $\tilde{\mathbb{E}}|_{t=i}$ and $\tilde{\mathbb{H}}|_{t=i}$ are isomorphic to $\mathbb{E}_i$ and $\mathbb{H}_i$ respectively, for $i = 0, 1$.

(c) We define $KF^{\tau-1}_{Cl(n)}(X)$ to be the group of homotopy classes of τ-twisted mod p $Cl(n)$-vectorial bundles on X.

LEMMA 6.6. *There exists a natural exact sequence:*

$$KF^{\tau-1}_{Cl(n)}(X) \stackrel{m_p}{\to} KF^{\tau-1}_{Cl(n)}(X) \stackrel{\rho_p}{\to} KF^{\tau-1}_{Cl(n)}(X;\mathbb{Z}/p) \stackrel{\delta_p}{\to} KF^{\tau}_{Cl(n)}(X) \stackrel{m_p}{\to} KF^{\tau}_{Cl(n)}(X).$$

PROOF. We define δ_p by $\delta_p([\mathbb{E},\mathbb{H}]) = [\mathbb{E}]$ and $m_p([\mathbb{F}]) = [\mathbb{F}^{\oplus p}] = p[\mathbb{F}]$. To define ρ_p, let $\mathbb{F} \in \mathcal{KF}^{\tau\times I}_{Cl(n)}(X \times I, X \times \partial I)$ represent an element in $KF^{\tau-1}_{Cl(n)}(X)$. Then $\mathrm{Supp}(\mathbb{F}|_{t=0}) = \emptyset$, so that $\mathbb{F}|_{t=0}$ is isomorphic to $\mathbb{O}^{\oplus p}$, where $\mathbb{O} \in \mathcal{KF}^{\tau}_{Cl(n)}(X)$ is such that $\mathrm{Supp}(\mathbb{O}) = \emptyset$, or equivalently $[\mathbb{O}] = 0$ in $KF^{\tau}_{Cl(n)}(X)$. If we put $\rho_p([\mathbb{F}]) = [(\mathbb{O},\mathbb{F})]$, then ρ_p is a well-defined homomorphism. Now, the exactness of the sequence can be shown by using the argument in the proof of Lemma 6.4: The only thing to notice is that we apply a Mayer-Vietoris construction (Lemma 4.2, [10]) to a "concatenation" of twisted $Cl(n)$-vectorial bundles. □

Lemma 6.7. *There exists a natural homomorphism*

$$\alpha : \ K^{\tau-1}_{Cl(n)}(X;\mathbb{Z}/p) \longrightarrow KF^{\tau-1}_{Cl(n)}(X;\mathbb{Z}/p)$$

making the following diagram commutative:

$$\begin{array}{ccccc} K^{\tau-1}_{Cl(n)}(X) & \xrightarrow{\rho_p} & K^{\tau-1}_{Cl(n)}(X;\mathbb{Z}/p) & \xrightarrow{\delta_p} & K^{\tau}_{Cl(n)}(X) \\ \downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow \\ KF^{\tau-1}_{Cl(n)}(X) & \xrightarrow{\rho_p} & KF^{\tau-1}_{Cl(n)}(X;\mathbb{Z}/p) & \xrightarrow{\delta_p} & KF^{\tau}_{Cl(n)}(X), \end{array}$$

where the vertical maps other than α are those constructed in Subsection 5.1.

Proof. We define α in question based on the construction in Subsection 5.1: Suppose that a τ-twisted mod p K-cocycle (A, T) of degree $-n-1$ on X is given. By definition, $A_x = T_{(x,0)}$ holds for all $x \in X$. To have a finite-dimensional approximation of A, we choose an open cover $\{U_\alpha\}$ of X, local trivializations s_α of τ, lifts of transition functions $g_{\alpha\beta}$ and positive numbers μ_α so that $\bigcup_{x \in U_\alpha}(\mathcal{H}_n, (A_\alpha)_x)_{<\mu_\alpha}$ gives rise to a vector bundle. Also, to have a finite-dimensional approximation of T, we choose an open cover $\{\tilde{U}_{\tilde{\alpha}}\}$ of $X \times I$, local trivializations $\tilde{s}_{\tilde{\alpha}}$ of $\tau \times I$, lifts of transition functions $\tilde{g}_{\tilde{\alpha}\tilde{\beta}}$, and positive numbers $\tilde{\mu}_{\tilde{\alpha}}$ so that $\bigcup_{(x,t) \in \tilde{U}_{\tilde{\alpha}}}(\mathcal{H}_n^{\oplus p}, (T_{\tilde{\alpha}})_{(x,t)})_{<\tilde{\mu}_{\tilde{\alpha}}}$ gives rise to a vector bundle. We can choose these data for T in a way compatible with the data for A, that is,

- the open cover $\{U_\alpha\}$ agrees with the open cover $\{\tilde{U}_{\tilde{\alpha}}|_{t=0}\}$ of $X \times \{0\}$;
- If $U_\alpha = \tilde{U}_{\tilde{\alpha}}|_{t=0}$, then $s_\alpha = \tilde{s}_{\tilde{\alpha}}|_{t=0}$, $g_{\alpha\beta} = \tilde{g}_{\tilde{\alpha}\tilde{\beta}}|_{t=0}$ and $\mu_\alpha = \tilde{\mu}_{\tilde{\alpha}}$.

Such a choice is possible because the eigenvalues of $(T_{\tilde{\alpha}})^2_{(x,t)}$ are continuous in (x, t). Under the choice above, we get a τ-twisted mod p $Cl(n)$-vectorial bundle $(\mathbb{E}, \mathbb{H})$ as a finite-dimensional approximation of (A, T). We put $\alpha([(A, T)]) = [(\mathbb{E}, \mathbb{H})]$ and define the homomorphism α. Once α is defined, the commutativity of the diagram is obvious from the construction. □

THEOREM 6.8. *For any $(X, \emptyset, \tau) \in \hat{\mathcal{C}}$, the homomorphism in Lemma 6.6*

$$\alpha : \ K^{\tau-1}_{Cl(n)}(X; \mathbb{Z}/p) \longrightarrow KF^{\tau-1}_{Cl(n)}(X; \mathbb{Z}/p)$$

is bijective, so that there is an isomorphism $K^{\tau-n-1}(X; \mathbb{Z}/p) \cong KF^{\tau-1}_{Cl(n)}(X; \mathbb{Z}/p)$.

PROOF. The theorem follows from Lemma 6.4, 6.6, 6.7 and Theorem 5.2. □

Though will not be detailed anymore, we can take into account additional support conditions to define the relative versions $K^{\tau-1}_{Cl(n)}(X, Y; \mathbb{Z}/p)$ as well as $KF^{\tau-1}_{Cl(n)}(X, Y; \mathbb{Z}/p)$ for any $(X, Y, \tau) \in \hat{\mathcal{C}}$. Then, in the same way as above, we get isomorphisms $K^{\tau-1}_{Cl(n)}(X, Y; \mathbb{Z}/p) \cong KF^{\tau-1}_{Cl(n)}(X, Y; \mathbb{Z}/p)$ and

$$K^{\tau-j-n-1}(X, Y; \mathbb{Z}/p) \cong KF^{\tau-1}_{Cl(n)}(X \times I^j, Y \times I^j \cup X \times \partial I^j; \mathbb{Z}/p).$$

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