

Totally Normal Cellular Stratified Spaces and Applications to the Configuration Space of Graphs

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Dedicated to Professor Yuli Rudyak on the occasion of his 65th birthday.

Abstract

The notion of regular cell complexes plays a central role in topological combinatorics because of its close relationship with posets. A generalization, called totally normal cellular stratified spaces, was introduced in [BGRT, Tama] by relaxing two conditions; face posets are replaced by acyclic categories and cells with incomplete boundaries are allowed. The aim of this article is to demonstrate the usefulness of totally normal cellular stratified spaces by constructing a combinatorial model for the configuration space of graphs. As an application, we obtain a simpler proof of Ghrist's theorem on the homotopy dimension of the configuration space of graphs. We also make sample calculations of the fundamental group of ordered and unordered configuration spaces of two points for small graphs.

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1 Introduction

Given a topological space X , the configuration space $\text{Conf}_k(X)$ of k distinct ordered points in X is defined by

$$\text{Conf}_k(X) = X^k \setminus \Delta_k(X),$$

where the discriminant set $\Delta_k(X)$ is given by

$$\Delta_k(X) = \{(x_1, \dots, x_k) \in X^k \mid x_i = x_j \text{ for some } i \neq j\}.$$

Configuration spaces have been studied by topologists when X is a manifold because of their appearance in geometry and homotopy theory. See [Fad62, FN62, Arn69, May72, Coh78], for example. When X is a manifold, we have a nice fibration of the following form

$$\text{Conf}_{k-1}(X \setminus \{x_0\}) \longrightarrow \text{Conf}_k(X) \xrightarrow{p_{k,1}} X, \quad (1)$$

which has been an indispensable tool for studying the homotopy type of configuration spaces of manifolds.

When X is not a manifold, however, we cannot expect the map $p_{k,1}$ to be a fibration and it is much harder to study its configuration spaces. It was Ghrist [Ghr01] who found an interpretation of $\text{Conf}_k(X)$ in terms of the problem of controlling automated guided vehicles (AGVs) in a factory and initiated the study of $\text{Conf}_k(X)$ when X is a 1-dimensional cell complex, i.e. a graph. It turns out that configuration spaces of graphs have many interesting properties and attracted much attention. For example, Ghrist proved that they are $K(\pi, 1)$ spaces. The fundamental group of the unordered configuration space $\text{Conf}_k(X)/\Sigma_k$ of a graph X is called the graph braid group of X and its relation to right-angled Artin groups has been studied by several people [Sab09, KKP12, FS12].

Because of the failure of the projection $p_{k,1}$ in (1) to be a fibration, we need to find a completely different method when X is not a manifold. One of successful and popular approach is to use Abrams' cellular model.

Definition 1.1 (Abrams Model). For a space X equipped with a cell decomposition $\pi : X = \bigcup_{\lambda \in \Lambda} e_\lambda$, define a subcomplex $C_k^{\text{Abrams}}(X, \pi)$ of X^k by

$$\begin{aligned} C_k^{\text{Abrams}}(X, \pi) &= \bigcup_{\substack{e_{\lambda_i} \cap e_{\lambda_j} = \emptyset \\ (i \neq j)}} e_{\lambda_1} \times \cdots \times e_{\lambda_k} \\ &= \bigcup_{e_{\lambda_1} \times \cdots \times e_{\lambda_k} \cap \Delta_k(X) = \emptyset} e_{\lambda_1} \times \cdots \times e_{\lambda_k}. \end{aligned}$$

Obviously $C_k^{\text{Abrams}}(X, \pi)$ is included in $\text{Conf}_k(X)$. Abrams proved that his model $C_k^{\text{Abrams}}(X)$ is a deformation retract of $\text{Conf}_k(X)$ under certain conditions.

Theorem 1.2 ([Abr00]). *For a 1-dimensional finite cell complex X , the inclusion*

$$C_k^{\text{Abrams}}(X, \pi) \hookrightarrow \text{Conf}_k(X)$$

is a homotopy equivalence as long as the cell decomposition on X satisfies the following two conditions:

1. *each path connecting vertices X of valency more than 2 has length at least $k + 1$, and*
2. *each homotopically essential path connecting a vertex to itself has length at least $k + 1$.*

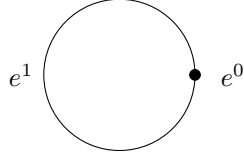
For precise definitions of terminologies used in the above theorem, see Abrams' thesis. Although Abrams' model has been used by several authors to study configuration spaces of graphs [Ghr01, FS05, Sab09, KKP12, FS12, Kur12] and higher dimensional cell complexes [CLW] successfully, there are some difficulties in using this model because of the following facts:

- Deformation retractions are not constructed explicitly.
- The action of the symmetric group Σ_k is not taken into account, either.
- The two conditions in Abrams' theorem require us to subdivide X finely. And taking subdivisions of X makes the model larger.

Let us take a look at a couple of examples to see the effect of subdivisions on Abrams model.

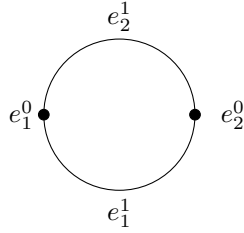
Example 1.3. Consider the case when $X = S^1$ and $k = 2$. It is well known that we have a Σ_2 -equivariant homotopy equivalence $\text{Conf}_2(S^1) \simeq_{\Sigma_2} S^1$.

Regard S^1 as a graph under the minimal cell decomposition $\pi_1: S^1 = e^0 \cup e^1$



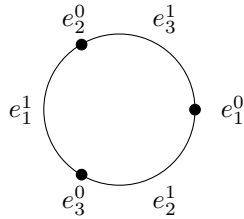
Abrams model $C_2^{\text{Abrams}}(S^1, \pi_1)$ for this graph is the empty set and is not homotopy equivalent to $\text{Conf}_2(S^1)$.

By taking a subdivision once, we obtain a cell decomposition π_2 of S^1 consisting of two 1-cells: $S^1 = e_1^0 \cup e_2^0 \cup e_1^1 \cup e_2^1$.



The model $C_2^{\text{Abrams}}(S^1, \pi_2)$ is still too small. It is merely a set of two points $\{e_1^0 \times e_2^0, e_2^0 \times e_1^0\}$.

In general, let π_n be the cell decomposition of S^1 as a cyclic graph with n edges.



When $n \geq 3$, the assumptions in Abrams' Theorem are satisfied and we have a homotopy equivalence $C_2^{\text{Abrams}}(S^1, \pi_n) \simeq \text{Conf}_2(S^1)$. When $n = 3$, it is easy to see that $C_2^{\text{Abrams}}(S^1, \pi_3)$ is the boundary of a hexagon and is homeomorphic to S^1 . When $n > 3$, however, $C_2^{\text{Abrams}}(S^1, \pi_n)$ is a cell complex of dimension 2. \square

The following is our wish list for a combinatorial model $C_k(\Gamma)$ for the configuration space $\text{Conf}_k(\Gamma)$ of k points of a graph Γ :

1. We would like our model to be as small as possible. It is desirable that its dimension coincides with the homotopy dimension of $\text{Conf}_k(\Gamma)$.
2. The model $C_k(\Gamma)$ should be a deformation retract of $\text{Conf}_k(\Gamma)$ under a reasonable condition. Furthermore we would like the deformation retraction to be equivariant with respect to the action of the symmetric group Σ_k .
3. Abrams' proof relies on Ghrist's $K(\pi, 1)$ theorem and the Whitehead theorem. A more direct proof is desirable.

In this paper, we propose a new model for configuration spaces of graphs based on the notion of totally normal cellular stratified spaces introduced in [BGRT] and developed in [Tama, Tamb]. In general, given a 1-dimensional cellular stratified space X , we define an appropriate cellular subdivision on X^k which contains $\text{Conf}_k(X)$ as a stratified subspace. And we obtain an acyclic category $C(\pi_{k,X}^{\text{comp}})$.

Theorem 1.4 (Corollary 3.9). *For any 1-dimensional finite cellular stratified space X and a positive integer k , there exists a finite acyclic category $C(\pi_{k,X}^{\text{comp}})$ whose classifying space $BC(\pi_{k,X}^{\text{comp}})$ can be embedded in $\text{Conf}_k(X)$ as a strong Σ_k -equivariant deformation retract.*

The space $C_k^{\text{comp}}(X) = BC(\pi_{k,X}^{\text{comp}})$ is one of our models for $\text{Conf}_k(X)$. The Σ_k -equivariance of the deformation retraction in the above theorem follows from the naturality of the construction. Our deformation retraction is also explicitly constructed. Thus the last two requirements in our wish list are satisfied.

The next question is how small our model is. It is easy to show that the number of vertices controls the dimension of our model.

Theorem 1.5 (Theorem 4.2). *Let X be a connected finite 1-dimensional cellular stratified space. Then*

$$\dim BC(\pi_{k,X}^{\text{comp}}) \leq v(X),$$

where $v(X)$ is the number of 0-cells in X .

Thus we obtain a smaller model if we could reduce the number of vertices. In other words, we obtain a small model by using the minimal cellular stratification of a given 1-dimensional cellular stratified space. We can reduce the dimension further by removing vertices of valency 1, as we will see in §4.1. As a consequence, we obtain an alternative proof of Ghrist's theorem [Ghr01] on the homotopy dimension of the configuration space of graphs. Recall that the *homotopy dimension* $\text{hodim } X$ of a space X is defined by

$$\text{hodim } X = \min_{Y \simeq X} \dim Y$$

where Y runs over all finite cell complexes that are homotopy equivalent to X .

Corollary 1.6. *Let X be a 1-dimensional connected finite cellular stratified space. Then we have*

$$\text{hodim } \text{Conf}_k(X) \leq \min\{k, v^{\text{ess}}(X)\},$$

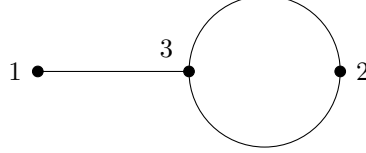
where $v^{\text{ess}}(X)$ is the number of essential vertices, i.e. 0-cells that are neither of valency 1 nor incident to exactly two regular 1-cells.

Here the valency of a 0-cell e^0 in a 1-dimensional cellular stratified space Γ is defined to be the cardinality of the set

$$\bigcup_{e^0 \subset \bar{e}} \{b : D^0 \rightarrow D \mid \varphi \circ b = \varphi_0\},$$

where $\varphi : D \rightarrow \bar{e}$ is the characteristic map for a cell e , $\varphi_0 : D^0 \rightarrow e^0$ is the characteristic map for the 0-cell e^0 , and e runs over all 1-cells containing e^0 in its closure.

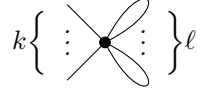
For example, the numbers in the following figure indicates the valencies of vertices in this 1-dimensional cell complex.



Thus the number of essential vertices in this cell complex is 1.

As a more concrete application, we compute the braid group $\pi_1(\text{Conf}_2(\Gamma)/\Sigma_2)$ of two strands of graphs with vertices ≤ 2 .

Theorem 1.7. *Let $W_{k,\ell}$ be a finite 1-dimensional cell complex of the following form.*

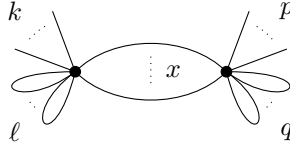


Then the fundamental groups of ordered and unordered configuration spaces of two points in $W_{k,\ell}$ are given by

$$\begin{aligned} \pi_1(\text{Conf}_2(W_{k,\ell})) &\cong F_{2n_{k,\ell}+1} \\ \pi_1(\text{Conf}_2(W_{k,\ell})/\Sigma_2) &\cong F_{n_{k,\ell}+1}, \end{aligned}$$

where $n_{k,\ell} = \frac{1}{2}(k+\ell)(k+3\ell-3)$ and F_n denotes the free group of rank n .

Theorem 1.8 (Theorem 4.8). *Let ${}_x B_{p,q}^{k,\ell}$ be the finite 1-dimensional cell complex obtained by gluing the essential vertices of $W_{k,\ell}$ and $W_{p,q}$ by x parallel edges.*



Then the fundamental groups of ordered and unordered configuration spaces of two points in ${}_x B_{p,q}^{k,\ell}$ are given by

$$\begin{aligned} \pi_1(\text{Conf}_2({}_x B_{p,q}^{k,\ell})) &\cong A_{\ell,q} * A_{q,\ell} * F_{2{}_x m_{p,q}^{k,\ell}-1} \\ \pi_1(\text{Conf}_2({}_x B_{p,q}^{k,\ell})/\Sigma_2) &\cong A_{\ell,q} * F_{{}_x m_{p,q}^{k,\ell}}, \end{aligned}$$

where

$$A_{\ell,q} = \langle a_1, \dots, a_\ell, b_1, \dots, b_q \mid [a_j, b_t] \ (1 \leq j \leq \ell, 1 \leq t \leq q) \rangle$$

and

$${}_x m_{p,q}^{k,\ell} = n_{k,\ell} + n_{p,q} + x(k+\ell+p+q) + \frac{x(x-1)}{2}.$$

Organization

Here is an outline of this paper.

- §2 is preliminary. We recall definitions and basic properties of our main tools, i.e. acyclic categories in §2.1 and cellular stratified spaces in §2.2.

Although a more general class of cellular stratified spaces, i.e. cylindrically normal cellular stratified spaces, is studied in [Tama, Tamb], we prove basic properties of totally normal cellular stratified spaces from scratch in §2.3 in order to be self-contained. Homotopy-theoretic properties of totally normal cellular stratified spaces used in this paper are stated and proved in §2.4 by appealing to homotopy theory of acyclic categories.

- A new combinatorial model for configuration spaces of graphs is constructed in §3 in two steps. After introducing a stratification on the configuration spaces of graphs in §3.1, we define an acyclic category model in §3.2, which is the first step.

In many cases, our acyclic category model can be collapsed further. This is done for the case of configuration spaces of two points in §3.4.

- We discuss two applications of our model in §4. Theorem 1.5 is proved in §4.1 and Theorem 1.7 and 1.8 are proved in §4.2.
- We include a proof of elementary fact on cellular stratified subspaces of spheres in an appendix §A, which plays an essential role in our proof of Theorem 2.50.

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2 Cellular Stratified Spaces

Homotopy theory of acyclic categories plays the fundamental role in this paper. The notion of cellular stratified spaces makes the connection between configuration spaces and acyclic categories. In this section, we first review homotopy-theoretic properties of acyclic categories in §2.1. And then the definition of cellular stratified spaces is recalled from [Tama]. Homotopy-theoretic properties of totally normal cellular stratified spaces are stated and proved based on the discussion in §2.1. .

2.1 Acyclic Categories

Acyclic categories are generalizations of posets, having possibly multiple “orders” (morphisms) between two objects. There is a popular way to define a quiver from a poset, i.e. the Hasse diagram. Any small category has an underlying quiver and can be regarded as a “quiver with relations”.

In this section, we review relations among these concepts. A good reference is Kozlov’s book [Koz08]. See also Appendix B of [Tama].

Definition 2.1. A *quiver* is a diagram of sets of the form

$$Q_1 \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} Q_0.$$

Elements of Q_0 and Q_1 are called *vertices* and *arrows*, respectively. For an arrow $u \in Q_1$, $s(u)$ and $t(u)$ are called the *source* and the *target* of u , respectively. For a pair of vertices x and y , we denote

$$Q(x, y) = \{u \in Q_1 \mid s(u) = x, t(u) = y\}.$$

For $n \geq 1$, define

$$N_n(Q) = \{(u_1, \dots, u_n) \mid s(u_1) = t(u_2), \dots, s(u_{n-1}) = t(u_n)\}.$$

We also use the notation $N_0(Q) = Q_0$. Elements of $N_n(Q)$ are called *n-chains*.

Definition 2.2. A *small category* is a quiver C equipped with maps

$$\begin{aligned} \circ & : N_2(C) \longrightarrow N_1(C) \\ i & : N_0(C) \longrightarrow N_1(C) \end{aligned}$$

satisfying the following conditions:

1. $(u \circ v) \circ w = u \circ (v \circ w)$ for $(u, v, w) \in N_3(C)$, and
2. $u \circ i(s(u)) = u = i(t(u)) \circ u$ for $u \in C_1$.

Elements of C_0 and C_1 are called *objects* and *morphisms*, respectively. For objects $x, y \in C_0$, the set of morphisms with source x and target y is denoted by $C(x, y)$. For an object $x \in C_0$, $i(x)$ is called the *identity morphism* on x and is denoted by 1_x .

Sometimes it is convenient to remove identity morphisms.

Definition 2.3. For a small category C , define

$$\overline{N}_n(C) = \{(u_n, \dots, u_1) \in N_n(C) \mid \text{none of } u_i \text{'s is identity}\}.$$

Elements of $\overline{N}_n(C)$ are called *nondegenerate n-chains*.

Definition 2.4. For a small category C , define $Q(C)_0 = C_0$ and $Q(C)_1 = \overline{N}_1(C) = C_1 \setminus i(C_0)$. And define maps

$$s, t : Q(C)_1 \longrightarrow Q(C)_0$$

by the restrictions of s and t of C . This quiver is called the *underlying quiver* of C .

Conversely, any quiver generates a small category.

Definition 2.5. Let Q be a quiver. Define a small category $\text{Path}(Q)$ as follows. Objects and morphisms are given by

$$\begin{aligned}\text{Path}(Q)_0 &= Q_0 \\ \text{Path}(Q)_1 &= \coprod_{n \geq 0} N_n(Q).\end{aligned}$$

The source and the target maps on $N_n(Q)$

$$s_n, t_n : N_n(Q) \longrightarrow Q_0$$

are given by

$$\begin{aligned}s_n(u_n, \dots, u_1) &= s(u_1), \\ t_n(u_n, \dots, u_1) &= t(u_n).\end{aligned}$$

The composition and the identity

$$\begin{aligned}\circ &: N_2(\text{Path}(Q)) = \coprod_{m, n \geq 0} \{(\mathbf{u}, \mathbf{v}) \in N_m(Q) \times N_n(Q) \mid s(\mathbf{u}) = t(\mathbf{v})\} \longrightarrow \text{Path}(Q) \\ i &: \text{Path}(Q)_0 = Q_0 \longrightarrow \text{Path}(Q)_1\end{aligned}$$

are given by the concatenation and the inclusion.

The resulting category $\text{Path}(Q)$ is called the *path category* of Q .

We are mainly interested in acyclic categories.

Definition 2.6. A quiver Q is said to be *acyclic* if either $Q(x, y)$ or $Q(y, x)$ is empty. A small category C is called *acyclic* if its underlying quiver $Q(C)$ is acyclic.

Remark 2.7. A small category C is acyclic if and only if

- $C(x, x) = \{1_x\}$ and,
- either $C(x, y)$ or $C(y, x)$ is empty for $x \neq y$.

Any poset P can be regarded as an acyclic category by $P(x, y) = \{*\}$ if $x \leq y$ and $P(x, y) = \emptyset$ otherwise. Conversely any acyclic category has an associated poset.

Definition 2.8. For an acyclic category C , define a poset $P(C)$ as follows. As sets, $P(C) = C_0$. For $x, y \in P(C)$, $x \leq y$ if and only if $C(x, y) \neq \emptyset$.

The canonical projection functor is denoted by

$$p : C \longrightarrow P(C).$$

We use the classifying space functor to translate combinatorial (category-theoretic) structures into the subject of homotopy theory.

Definition 2.9. Given a small category C , the collection of chains $N(C) = \{N_n(C)\}_{n \geq 0}$ forms a simplicial set, called the *nerve* of C . The geometric realization of this simplicial set is called the *classifying space* of C and is denoted by $BC = |N(C)|$.

Recall that the face operators $d_i : N_n(C) \rightarrow N_{n-1}(C)$ are defined by

$$d_i(u_n, \dots, u_1) = \begin{cases} (u_n, \dots, u_2), & \text{if } i = 0 \\ (u_n, \dots, u_{i+1} \circ u_i, \dots, u_1), & \text{if } 1 \leq i \leq n-1 \\ (u_{n-1}, \dots, u_1), & \text{if } i = n \end{cases}$$

for $i = 0, \dots, n$.

When C is acyclic, we only need these face operators to describe BC .

Lemma 2.10. *For an acyclic category C , face operators can be restricted to $d_i : \overline{N}_n(C) \rightarrow \overline{N}_{n-1}(C)$, and, when C is finite, we have a homeomorphism*

$$BC \cong \left(\prod_n \overline{N}_n(C) \times \Delta^n \right) / \sim,$$

where \sim is the equivalence relation generated by

$$(d_i(u_n, \dots, u_1), (t_0, \dots, t_{n-1})) \sim ((u_n, \dots, u_1), (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})). \quad (2)$$

Here our n -simplex Δ^n is given by

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^n \mid t_0 + \dots + t_n = 1, t_i \geq 0\}.$$

Proof. It is immediate to verify that d_i can be restricted to $\overline{N}(C)$. The inclusions $\overline{N}_n(C) \hookrightarrow N_n(C)$ induce a continuous bijective map

$$\left(\prod_n \overline{N}_n(C) \times \Delta^n \right) / \sim \longrightarrow BC.$$

By the finiteness assumption and acyclicity of C , $\overline{N}_n(C) = \emptyset$ for sufficiently large n and each $\overline{N}_n(C)$ is a finite set. Thus the above map is a continuous bijection from a compact space to a Hausdorff space, hence is a homeomorphism. \square

Remark 2.11. It is well-known that when P is a poset, the collection of nondegenerate chains $\overline{N}(P)$ has a structure of ordered simplicial complex and is often called the *order complex* of P .

The following definition is slightly different from but equivalent to the definition of graded posets in combinatorics.

Definition 2.12. A poset P is called *graded*, if for any element $x \in P$, $\dim BP_{\leq x}$ is finite, where $P_{\leq x} = \{y \in P \mid y \leq x\}$.

For a graded poset P , define the *rank function*

$$r : P \longrightarrow \mathbb{Z}_{\geq 0}$$

by

$$r(x) = \dim BP_{\leq x}.$$

When a poset P is regarded as a small category, the rank function $r : P \rightarrow \mathbb{Z}_{\geq 0}$ is a functor. We would like to define an analogous notion for small categories. Bessis [Bes] considered functions on the set of morphisms.

Definition 2.13. A *length function* on a small category C is a map

$$\ell : C_1 \longrightarrow \mathbb{Z}_{\geq 0}$$

with the properties that

1. $\ell(u \circ v) = \ell(u) + \ell(v)$;
2. $\ell(u) = 0$ if and only if $u = 1_x$ for some $x \in C_0$.

A small category equipped with a length function is called *homogeneous* by Bessis. We use the following terminology.

Definition 2.14. A triple (C, ℓ, B) of a small category C , a length function ℓ on C , and a set B of objects in C , is called a *category with length function* if the following conditions are satisfied:

1. For any object $x \in C_0$, there exists an object $b \in B$ with $C(b, x) \neq \emptyset$.
2. For any morphisms $u : b \rightarrow x$ and $u' : b' \rightarrow x$ with $b, b' \in B$, we have $\ell(u) = \ell(u')$.

Definition 2.15. For a category with length function (C, ℓ, B) , define a function

$$r : C_0 \longrightarrow \mathbb{Z}_{\geq 0}$$

by

$$r(x) = \ell(u),$$

where $x \in C_0$ and $u : b \rightarrow x$ is a morphism in C with $b \in B$. This is called a *rank functor* on C because of the following reason.

Lemma 2.16. *The above function r can be extended to a functor*

$$r : C \longrightarrow \mathbb{Z}_{\geq 0}.$$

Proof. We need to show that if there exists a morphism $u : x \rightarrow y$ in C , then $r(x) \leq r(y)$. Choose a morphism $v : b \rightarrow x$ with $b \in B$. Then

$$r(y) = \ell(u \circ v) = \ell(u) + \ell(v) = \ell(u) + r(x) \geq r(x).$$

□

Remark 2.17. For a category with length function (C, ℓ, B) , we can recover ℓ and B from the rank functor r by

$$\begin{aligned} \ell(u) &= r(t(u)) - r(s(u)) \\ B &= \{x \in C_0 \mid r(x) = 0\}. \end{aligned}$$

In the rest of this article, we denote a category with length function by the pair (C, r) .

Lemma 2.18. *Any category with length function is acyclic.*

Proof. For $u \in C(x, x)$, $\ell(u) = r(t(u)) - r(s(u)) = 0$. By the definition of length function $u = 1_x$. Suppose $C(x, y) \neq \emptyset$ and $C(y, x) \neq \emptyset$. By Lemma 2.16, $r(x) \leq r(y)$ and $r(y) \leq r(x)$, which imply that $x = y$ and that C is acyclic. □

Definition 2.19. A category with length function (C, ℓ, B) with rank functor r is called a *ranked category* if, for any morphism $u : x \rightarrow y$ with $\ell(u) = k$, there exists a factorization of u into a composition

$$u : x = x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_k = y$$

with $r(x_{i+1}) = r(x_i) + 1$ for $0 \leq i \leq k - 1$.

We mainly use functors of the following form.

Definition 2.20. A functor $f : C \rightarrow \mathbf{Top}$ from a small category C to the category \mathbf{Top} of topological spaces is said to be *continuous* if for each morphism $u : x \rightarrow y$ in C , the induced map $f(u) : f(x) \rightarrow f(y)$ is continuous.

Remark 2.21. This condition is equivalent to the continuity of the adjoint

$$C(x, y) \times f(x) \longrightarrow f(y)$$

of the map $f(x, y) : C(x, y) \rightarrow \text{Map}(f(x), f(y))$ when $C(x, y)$ is equipped with the discrete topology.

The following construction will be used later when we study boundaries of cells.

Definition 2.22. For an object $x \in C_0$ in a small category C , define a small category $C \downarrow x$ by

$$\begin{aligned} (C \downarrow x)_0 &= \{u \in C_1 \mid t(u) = x\} \\ (C \downarrow x)(u, v) &= \{w \in C_1 \mid u = v \circ w\}. \end{aligned}$$

The composition of morphisms is given by that of C . This is called the *comma category of C over x* .

When C is acyclic, we denote $C \downarrow x$ by $C_{\leq x}$ for simplicity. The subcategory of $C_{\leq x}$ consisting of $(C \downarrow x)_0 \setminus \{1_x\}$ is denoted by $C_{< x}$.

2.2 Cellular Stratified Spaces

Cellular stratified spaces are generalizations of cell complexes, having possibly non-closed cells. Let us begin with the definition of stratifications.

Definition 2.23. A *stratification* on a topological space X indexed by a poset Λ is a map

$$\pi : X \longrightarrow \Lambda$$

satisfying the following two conditions:

1. $\lambda \leq \mu$ in Λ if and only if $\pi^{-1}(\lambda) \subset \overline{\pi^{-1}(\mu)}$.
2. Each $\pi^{-1}(\lambda)$ is connected and locally closed.

The image of π as a full subposet is denoted by $P(X, \pi)$, $P(X)$, or $P(\pi)$, and is called the *face poset* of X . The space $e_\lambda = \pi^{-1}(\lambda)$ is called the *stratum* indexed by $\lambda \in \Lambda$.

Definition 2.24. Let $\pi : X \rightarrow \Lambda$ be a stratification on a Hausdorff space X .

- For a stratum e_λ , an *n -cell structure* on e_λ is a pair $(D_\lambda, \varphi_\lambda)$ of a subspace D_λ of the unit n -disk D^n with $\text{Int}(D^n) \subset D_\lambda$ and a quotient map

$$\varphi_\lambda : D_\lambda \longrightarrow X$$

satisfying the following conditions:

1. $\varphi(D_\lambda) = \overline{e_\lambda}$.
2. The restriction $\varphi_\lambda|_{\text{Int}(D^n)} : \text{Int}(D^n) \rightarrow e_\lambda$ is a homeomorphism.

For simplicity, we refer to an n -cell structure $(D_\lambda, \varphi_\lambda)$ on e_λ by φ_λ when there is no risk of confusion.

- A stratum equipped with an n -cell structure is called an n -cell.
- The map φ_λ is called the *characteristic map* of e_λ and D_λ is called the *domain* of e_λ . The dimension n of the disk D^n containing the domain D_λ is called the *dimension* of e_λ .
- A *cellular stratification* on X consists of
 - a stratification $\pi : X \rightarrow \Lambda$, and
 - a collection $\Phi = \{\varphi_\lambda : D_\lambda \rightarrow \overline{e_\lambda}\}_{\lambda \in P(X)}$ of cell structures on strata

satisfying the condition that, for each n -cell e_λ , $\partial e_\lambda = \overline{e_\lambda} \setminus e_\lambda$ is covered by a finite number of cells of dimension less than or equal to $n - 1$.

The triple (X, π, Φ) is called a *cellular stratified space*.

Remark 2.25. Note that we require that a cell structure map to be a quotient map. This condition is automatic in the classical definition of cell complex, since a cell complex is always assumed to be Hausdorff and D^n is compact.

Definition 2.26. Let (X, π_X, Φ_X) and (Y, π_Y, Φ_Y) be cellular stratified spaces.

- A *morphism of cellular stratified spaces* \mathbf{f} from (X, π_X, Φ_X) to (Y, π_Y, Φ_Y) consists of
 - a continuous map $f : X \rightarrow Y$,
 - a map of posets $\underline{f} : P(X) \rightarrow P(Y)$, and
 - a family of maps $f_\lambda : D_\lambda \rightarrow D_{f(\lambda)}$ indexed by cells $\varphi_\lambda : D_\lambda \rightarrow \overline{e_\lambda}$ in X

making the following diagrams commutative

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 \pi_X \downarrow & & \downarrow \pi_Y \\
 P(X) & \xrightarrow{\underline{f}} & P(Y)
 \end{array}
 \quad
 \begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 \varphi_\lambda \uparrow & & \uparrow \psi_{f(\lambda)} \\
 D_\lambda & \xrightarrow{f_\lambda} & D_{f(\lambda)}
 \end{array}$$

where $\psi_{f(\lambda)} : D_{f(\lambda)} \rightarrow \overline{e_{f(\lambda)}}$ is the characteristic map for $e_{f(\lambda)}$.

- When $X = Y$ and f is the identity, \mathbf{f} is called a *subdivision*.
- When $f(e_\lambda) = e_{\underline{f}(\lambda)}$ and $f_\lambda(0) = 0$ for each λ , \mathbf{f} is called a *strict morphism*.
- When \mathbf{f} is a strict morphism of cellular stratified spaces and f is an embedding of topological spaces, \mathbf{f} is said to be an *embedding* of cellular stratified spaces and X is said to be a *cellular stratified subspace* of Y .

We need to impose certain “niceness” conditions to our cellular stratified spaces. As is the case of cell complexes, we usually require CW conditions.

Definition 2.27. A cellular stratification on a space X is said to be *CW*, if the following two conditions are satisfied:

1. (Closure Finite) For each cell e_λ , ∂e_λ is covered by a finite number of cells.
2. (Weak Topology) X has the weak topology determined by the covering $\{\overline{e_\lambda} \mid \lambda \in P(X)\}$.

The CW condition allow us to express a cellular stratified space as a quotient space as follows.

Lemma 2.28. For a CW cellular stratified space X with cell structure $\Phi = \{\varphi_\lambda : D_\lambda \rightarrow \overline{e_\lambda} \mid \lambda \in P(X)\}$, define $D(X) = \coprod_{\lambda \in P(X)} D_\lambda$ and $\tilde{\Phi} : D(X) \rightarrow X$ by $\tilde{\Phi}(x) = \varphi_\lambda(x)$ if $x \in D_\lambda$. Then $\tilde{\Phi}$ is a quotient map. In particular, we have a homeomorphism

$$X \cong D(X) / \sim_\Phi,$$

where $x \sim_\Phi y$ if and only if $\varphi_\mu(x) = \varphi_\lambda(y)$ for $x \in D_\mu$ and $y \in D_\lambda$.

Proof. The map $\tilde{\Phi}$ factors as

$$\tilde{\Phi} : D(X) \xrightarrow{\coprod \varphi_\lambda} \coprod_{\lambda \in P(X)} \overline{e_\lambda} \xrightarrow{\rho} X.$$

All characteristic maps φ_λ are quotient maps. By the CW assumption, ρ is a quotient map. Hence $\tilde{\Phi}$ is a quotient map. \square

In order to study configuration spaces, we need to understand products, subdivisions, and taking complements of cellular stratified spaces. Let us first consider products. We need to impose the following condition.

Lemma 2.29. Let (X, π_X, Φ_X) and (Y, π_Y, Φ_Y) be cellular stratified spaces and consider the product map

$$\pi_X \times \pi_Y : X \times Y \longrightarrow P(X) \times P(Y).$$

For a pair of cells e_λ, e_μ in X and Y , define a continuous map

$$\varphi_{\lambda, \mu} : D_{\lambda, \mu} \cong D_\lambda \times D_\mu \xrightarrow{\varphi_\lambda \times \varphi_\mu} \overline{e_\lambda} \times \overline{e_\mu} = \overline{e_\lambda \times e_\mu} \subset X \times Y,$$

where $D_{\lambda, \mu}$ is the subspace of $D^{\dim e_\lambda + \dim e_\mu}$ defined by pulling back $D_\lambda \times D_\mu$ via the standard homeomorphism

$$D^{\dim e_\lambda + \dim e_\mu} \cong D^{\dim e_\lambda} \times D^{\dim e_\mu}.$$

If $\varphi_{\lambda, \mu}$ is a quotient map for any pair (λ, μ) , we have a cellular stratification on $X \times Y$.

Proof. Obvious from the definition. \square

The problem is when $\varphi_\lambda \times \varphi_\mu$ is a quotient map. When both X and Y are cell complexes, the compactness of $D_\lambda \times D_\mu$ implies that $\varphi_\lambda \times \varphi_\mu$ is a quotient map. In general, D_λ or D_μ is neither closed nor open. This problem is discussed in §3.2 of [Tamb] in detail. For configuration spaces of graphs, the following fact is enough.

Lemma 2.30. If $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$ are surjective closed maps between metrizable spaces, then the product $f_1 \times f_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ is a quotient map.

Proof. See Corollary right after Theorem 3 in [Him65]. \square

We are interested in 1-dimensional cellular stratified spaces.

Corollary 2.31. *When Γ_1 and Γ_2 are 1-dimensional cellular stratified spaces, $\Gamma_1 \times \Gamma_2$ is a cellular stratified space.*

Proof. When $\varphi : D \rightarrow \bar{e}$ is a characteristic map on a 1-cell e , the domain D is one of $(-1, 1)$, $(-1, 1]$, $[-1, 1)$, or $[-1, 1]$. In any of these cases, φ is a closed map. Furthermore, the closure \bar{e} of the cell is homeomorphic to one of $(-1, 1)$, $(-1, 1]$, $[-1, 1)$, $[-1, 1]$, or S^1 , and is metrizable. Thus products of characteristic maps are again quotient maps by Lemma 2.30.

Other requirements hold obviously and $\Gamma_1 \times \Gamma_2$ is a cellular stratified space. \square

Let us consider subdivisions next. We have already defined subdivisions of stratified spaces in Definition 2.26. Subdivisions of cell structures are defined as follows.

Definition 2.32. A *cellular subdivision* of a cellular stratified space (X, π, Φ) consists of

- a subdivision $\mathbf{s} = (1_{X, \mathbf{s}}) : (X, \pi') \rightarrow (X, \pi)$ of (X, π) as a stratified space, and
- a regular cellular stratification $(\pi_\lambda, \Phi_\lambda)$ on the domain D_λ for each cell $\varphi_\lambda : D_\lambda \rightarrow \bar{e}_\lambda$ containing $\text{Int}(D_\lambda)$ as a strict stratified subspace

satisfying the following conditions:

1. For each $\lambda \in P(X, \pi)$, the characteristic map

$$\varphi_\lambda : (D_\lambda, \pi_\lambda) \longrightarrow (X, \pi')$$

of e_λ is a strict morphism of stratified spaces.

2. The maps

$$P(\varphi_\lambda) : P(\text{Int}(D_\lambda)) \longrightarrow P(X, \pi')$$

induced by the characteristic maps φ_λ give rise to a bijective morphism of posets

$$\coprod_{\lambda \in P(X, \pi)} P(\varphi_\lambda) : \coprod_{\lambda \in P(X, \pi)} P(\text{Int}(D_\lambda), \pi_\lambda) \longrightarrow P(X, \pi').$$

3. For each $\lambda' \in P(X, \pi')$ with $s(\lambda') = \lambda \in P(X, \pi)$, let us denote the corresponding strata in (X, π') and (D_λ, π_λ) by $e_{\lambda'}$ and $E_{\lambda'}$, respectively. If $\psi_{\lambda'} : D_{\lambda'} \rightarrow \bar{E}_{\lambda'}$ is the characteristic map for $E_{\lambda'}$ in the regular cellular stratification $(D_\lambda, \pi_\lambda, \Phi_\lambda)$, then the composition

$$\varphi_\lambda \circ \psi_{\lambda'} : D_{\lambda'} \longrightarrow \bar{e}_{\lambda'}$$

is a quotient map.

Remark 2.33. The map $\coprod_{\lambda \in P(X, \pi)} P(\varphi_\lambda)$ may not be an isomorphism of posets, although it is assumed to be a bijection.

The composition $\varphi_\lambda \circ \psi_{\lambda'}$ is essentially the restriction of φ_λ to $E_{\lambda'}$, since the cellular stratification $(D_\lambda, \phi_\lambda, \Phi_\lambda)$ is assumed to be regular. In general, however, a restriction of a quotient map may not be a quotient map. This is the reason we need to impose the condition 3 in the above definition. In other words, the definition is designed to make the following proposition hold.

Proposition 2.34. *A cellular subdivision of a cellular stratified space is again a cellular stratified space.*

2.3 Totally Normal Cellular Stratified Spaces

Regularity and normality conditions are important in our analysis of cellular stratified spaces.

Definition 2.35. A cellular stratification on a space X is said to be

- *normal*, if $e_\mu \subset \overline{e_\lambda}$ whenever $e_\mu \cap \overline{e_\lambda} \neq \emptyset$, for any cell e_λ ,
- *regular*, if the characteristic map $\varphi : D_\lambda \rightarrow \overline{e_\lambda}$ of each cell e_λ is a homeomorphism onto $\overline{e_\lambda}$, and
- *totally normal*, if, for each n -cell e_λ ,
 1. there exist a structure of regular cell complex on S^{n-1} containing ∂D_λ as a cellular stratified subspace of S^{n-1} , and
 2. for each cell e in the cellular stratification on ∂D_λ , there exists a cell e_μ in X and a map $b : D_\mu \rightarrow \partial D_\lambda$ $b(\text{Int}(D_\mu)) = e$ and $\varphi_\lambda \circ b = \varphi_\mu$.

Remark 2.36. The regularity of the cellular stratification on ∂D_λ implies that b is an embedding.

There is a canonical way to associate a small category to any totally normal cellular stratified space.

Definition 2.37. For a totally normal cellular stratified space X , define a category $C(X)$ as follows. Objects are cells

$$C(X)_0 = \{e \mid \text{cells in } X\}.$$

A morphism from a cell $\varphi_\mu : D_\mu \rightarrow \overline{e_\mu}$ to another cell $\varphi_\lambda : D_\lambda \rightarrow \overline{e_\lambda}$ is a lift of the characteristic map φ_μ of e_μ , i.e. a map $b : D_\mu \rightarrow D_\lambda$ making the following diagram commutative

$$\begin{array}{ccc} D_\lambda & \xrightarrow{\varphi_\lambda} & \overline{e_\lambda} \hookrightarrow X \\ \uparrow b & & \nearrow \\ D_\mu & \xrightarrow{\varphi_\mu} & \overline{e_\mu} \end{array}$$

The composition is given by the composition of maps. This category $C(X)$ is called the *face category* of X .

Remark 2.38. In [Tama], the set of morphisms $C(X)(e_\mu, e_\lambda)$ from e_μ to e_λ is topologized by the compact open topology as a subspace of $\text{Map}(D_\mu, D_\lambda)$ and $C(X)$ is defined as a topological category.

In this paper, we only consider face categories of totally normal cellular stratified spaces, in which case the topology on $C(X)(e, e')$ is automatically discrete.

Example 2.39. All cellular stratified spaces of dimension 1 are totally normal, since possible domains of characteristic maps are $(-1, 1)$, $(-1, 1]$, $[-1, 1)$, or $[-1, 1]$. \square

Note that the existence of a morphism $b : e_\mu \rightarrow e_\lambda$ in $C(X)$ implies $\overline{e_\mu} \subset \overline{e_\lambda}$. Thus we obtain a functor

$$p_X : C(X) \longrightarrow P(X).$$

Lemma 2.40. For a totally normal cellular stratified space X , $C(X)$ is a category with length function, hence is acyclic. The associated poset $P(C(X))$ coincides with $P(X)$ and the canonical projection $p : C(X) \rightarrow P(C(X))$ can be identified with p_X .

Proof. Define $\ell : C(X)_1 \rightarrow \mathbb{Z}_{\geq 0}$ by

$$\ell(b) = \dim e_\lambda - \dim e_\mu \quad (3)$$

for a morphism $b \in C(X)(e_\mu, e_\lambda)$. This is obviously a length function in the sense of Definition 2.13.

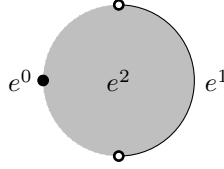
Define $B = \{e \in C(X)_0 \mid \dim e = 0\}$. Then the conditions of Definition 2.14 are satisfied and $(C(X), \ell, B)$ is a category with length function. The associated rank functor is obviously the dimension function \dim .

By definition, there exists a morphism $b \in C(X)(e_\mu, e_\lambda)$ if and only if $e_\mu \subset \overline{e_\lambda}$, i.e. $e_\mu \leq e_\lambda$. Thus $P(C(X)) = P(X)$. \square

Remark 2.41. In general $C(X)$ is not a ranked category if we use the function ℓ in (3) as a length function. For example, define

$$X = \text{Int}D^2 \cup \{(x, y) \in S^1 \mid x > 0\} \cup \{(-1, 0)\}.$$

X has a structure of regular cellular stratified space with cells $e^0 = \{(-1, 0)\}$, $e^1 = \{(x, y) \in S^1 \mid x > 0\}$, and $e^2 = \text{Int}D^2$.



$C(X)(e^0, e^2)$ contains a single element which cannot be factored into a composition of morphisms of codimension 1.

We use the following terminology for 1-dimensional cellular stratified spaces.

Definition 2.42.

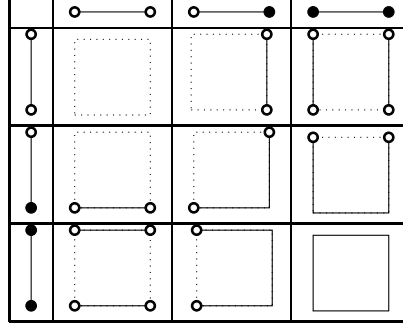
- A 1-dimensional cellular stratified space Γ is called a *graph*.
- 0-cells and 1-cells are called *vertices* and *edges*, respectively.
- Let $\varphi : D \rightarrow \overline{e}$ be a 1-cell in Γ .
 - An edge e is called a *loop* if $D = [-1, 1]$ and $\varphi(-1) = \varphi(1)$.
 - An edge e is called a *connection* if $D = [-1, 1]$, φ is an embedding, and both $\varphi(-1)$ and $\varphi(1)$ are contained in more than one edge.
 - An edge e is called a *branch* if it is not a loop nor a connection.
- For a vertex v , let b_v be the number of branches and bridges e with $v \in \overline{e}$ and ℓ_v be the number of loops with $v \in \overline{e}$. Then the number $b_v + 2\ell_v$ is called the *valency* at v .
- A vertex v is called a *leaf* if it has valency 1.
- A graph Γ is said to be *finite* if the numbers of vertices and edges are finite.

For graphs, Corollary 2.31 can be refined as follows.

Lemma 2.43. *For graphs Γ_1 and Γ_2 , the product $\Gamma_1 \times \Gamma_2$ is a totally normal cellular stratified space.*

Proof. We have verified in Corollary 2.31 that $\Gamma_1 \times \Gamma_2$ has a cellular stratification under the product stratification. Let us verify that this is totally normal. For simplicity, we regard domains for characteristic maps of 2-cells in $\Gamma_1 \times \Gamma_2$ as stratified subspaces of $[-1, 1]^2$ instead of D^2 .

There are three types of domains of 1-cells in Γ_1 or Γ_2 up to homeomorphisms, i.e. $(-1, 1)$, $(-1, 1]$, or $[-1, 1]$. The possible types of domains for 2-cells $e_\mu \times e_\lambda$ in $\Gamma_1 \times \Gamma_2$ are depicted as follows.



In any of these cases, the boundary is a stratified subspace of the standard cell decomposition of a square and characteristic maps for 0 and 1 dimensional cells lifts to maps into those boundaries. \square

Remark 2.44. More generally, a k -fold product $\Gamma_1 \times \cdots \times \Gamma_k$ of graphs is totally normal. For higher dimensional cellular stratified spaces, see §3.2 of [Tamb].

For totally normal cellular stratified spaces, Proposition 2.34 can be refined as follows.

Proposition 2.45. *Let (X, π, Φ) be a totally normal cellular stratified space and (X, π', Φ') a cellular subdivision. Suppose that each morphism $b \in C(X)(e_\mu, e_\lambda)$ in the face category of (X, π, Φ) is a strict morphism*

$$b : (D_\mu, \pi_\mu, \Phi_\mu) \longrightarrow (D_\lambda, \pi_\lambda, \Phi_\lambda)$$

of stratified spaces. Then (X, π', Φ') has a structure of totally normal cellular stratified space.

Proof. By the very definition of cellular subdivision, the boundary ∂D_λ of the domain of each cell e_λ in (X, π', Φ') is equipped with a regular cellular stratification. It remains to show that, for each $\lambda' \in P(X, \pi')$ and a cell e' in $\partial D_{\lambda'}$, there exist a cell $e_{\mu'}$ in (X, π, Φ) and a map

$$b' : D_{\mu'} \longrightarrow D_{\lambda'}$$

making the diagram

$$\begin{array}{ccc} D_{\lambda'} & \xrightarrow{\varphi_{\lambda'}} & \overline{e_{\lambda'}} \hookrightarrow X \\ \uparrow b' & & \nearrow \\ D_{\mu'} & \xrightarrow{\varphi_{\mu'}} & \overline{e_{\mu'}} \end{array}$$

commutative and satisfying $b'(\text{Int}(D_{\mu'})) = e'$.

Suppose $s(\lambda') = \lambda$ under the subdivision $s : P(X, \pi') \rightarrow P(X, \pi)$. Let e be a cell in D_λ containing e' . By the total normality of (X, π, Φ) , there exists a cell e_μ in (X, π, Φ) and a map $b : D_\mu \rightarrow D_\lambda$ with $b(\text{Int}(D_\mu)) = e$ and $\varphi_\lambda \circ b = \varphi_\mu$. Define $e_{\mu'} = (\varphi_\lambda \circ b)(e')$. Since both b

and φ_λ are strict morphisms of stratified spaces, $e_{\mu'}$ is a cell in (X, π', Φ') . By the definition of cellular subdivision, there exist cells $\psi_{\mu'} : D_{\mu'} \rightarrow \overline{E_{\mu'}}$ and $\psi_{\lambda'} : D_{\lambda'} \rightarrow \overline{E_{\lambda'}}$ in $(D_\mu, \pi_\mu, \Phi_\mu)$ and $(D_\lambda, \pi_\lambda, \Phi_\lambda)$, respectively, such that $\varphi_{\mu'} = \varphi_\mu \circ \psi_{\mu'}$ and $\varphi_{\lambda'} = \varphi_\lambda \circ \psi_{\lambda'}$.

When $\mu = \lambda$, both $E_{\mu'}$ and $E_{\lambda'}$ are cells in the regular cellular stratification of D_λ and b is the identity map. Hence there exists a unique map $b' : D_{\mu'} \rightarrow D_{\lambda'}$ satisfying the required conditions, since $\psi_{\mu'}$ and $\psi_{\lambda'}$ are embeddings. When $\mu < \lambda$, we have the following diagram

$$\begin{array}{ccccc}
 & & D_\lambda & \xrightarrow{\varphi_\lambda} & \overline{e_\lambda} \\
 & \nearrow \psi_{\lambda'} & \uparrow \varphi_{\lambda'} & & \nearrow \\
 D_{\lambda'} & \xrightarrow{\varphi_{\lambda'}} & \overline{e_{\lambda'}} & & \\
 & \uparrow b & \uparrow & & \uparrow \\
 & D_\mu & \xrightarrow{\varphi_\mu} & \overline{e_{\mu'}} & \\
 & \uparrow \psi_{\mu'} & \uparrow & & \uparrow \\
 D_{\mu'} & \xrightarrow{\varphi_{\mu'}} & \overline{e_{\mu'}} & & \\
 & \uparrow b' & & & \\
 & D_{\mu'} & & &
 \end{array}$$

The regularity of cellular stratifications on D_μ and D_λ and the fact that b is an embedding implies that there exists a map $b' : D_{\mu'} \rightarrow D_{\lambda'}$ making the above diagram commutative. And thus (X, π', Φ') is totally normal. \square

2.4 Face Categories of Totally Normal Cellular Stratified Spaces

In this section, we show that the homotopy-theoretic informations of a totally normal cellular stratified spaces X are encoded in its face category $C(X)$. A main tool is the classifying space functor

$$B : \mathbf{Cats} \longrightarrow \mathbf{Top}$$

from the category of small categories to the category of topological spaces defined in Definition 2.9.

Let us begin with the following description.

Definition 2.46. For a totally normal cellular stratified space X , define a functor

$$D^X : C(X) \longrightarrow \mathbf{Top}$$

by assigning D_λ to each cell $\varphi_\lambda : D_\lambda \rightarrow \overline{e_\lambda}$. For a morphism $b \in C(X)(e_\mu, e_\lambda)$, define $D^X(b) = b$.

Proposition 2.47. *When X is a CW totally normal cellular stratified space, we have a natural homeomorphism*

$$\operatorname{colim}_{C(X)} D^X \xrightarrow{\cong} X.$$

Proof. Let \sim_c be the defining relation of the colimit $\operatorname{colim}_{C(X)} D^X$, i.e.

$$\operatorname{colim}_{C(X)} D^X = \left(\coprod_{\lambda \in P(X)} D_\lambda \right) / \sim_c = D(X) / \sim_c.$$

On the other hand, we have

$$X \cong D(X) / \sim_\Phi$$

by Lemma 2.28. Let us verify that these two equivalence relations coincide.

Suppose $x \sim_c y$ for $x \in D_\mu$ and $y \in D_\lambda$. Without loss of generality, we may assume $b(x) = y$ for some $b \in C(X)(e_\mu, e_\lambda)$. We have $\varphi_\mu(x) = \varphi_\lambda(y)$ since $\varphi_\mu = \varphi_\lambda \circ b$.

Suppose $\varphi_\mu(x) = \varphi_\lambda(y)$. There are three cases:

1. $x \in \text{Int}D_\mu$ and $y \in \text{Int}D_\lambda$.
2. $x \in \text{Int}D_\mu$ and $y \in \partial D_\lambda$ (or $x \in \partial D_\mu$ and $y \in \text{Int}D_\lambda$).
3. $x \in \partial D_\mu$ and $y \in \partial D_\lambda$.

In the first case, $x = y$ and thus $x \sim_c y$.

In the second case, $\varphi_\mu(x) \in e_\mu$, $\varphi_\lambda(y) \in \partial e_\lambda$, and $\varphi_\mu(x) = \varphi_\lambda(y)$. Thus we have $e_\mu \subset \partial e_\lambda$ and $\varphi_\lambda(y) \in e_\mu$. Choose a cell e in ∂D_λ with $y \in e$. By the total normality, there exists a cell e_ν in X and a map $b : D_\nu \rightarrow \bar{e}$ with $\varphi_\nu = \varphi_\lambda \circ b$. Since b is a characteristic map, there exists a unique $z \in \text{Int}(D_\nu)$ such that $b(z) = y$. Then $\varphi_\mu(x) = \varphi_\lambda(y) = \varphi_\nu(z)$. Since both x and z lie in the interiors of domains of characteristic maps, $x = z$. We have $z \sim_c y$ by $b(z) = y$. Thus $x \sim_c y$.

In the third case, the normality of the stratification of X implies that there exists a cell e_ν such that $\varphi_\mu(x) = \varphi_\lambda(y) \in e_\nu$. By the second case, we obtain $x \sim_c y$. \square

One of the most important features of totally normal cellular stratified spaces is that the cellular stratification of each domain D_λ can be described by using the comma category (Definition 2.22) $C(X)_{\leq e_\lambda} = C(X) \downarrow e_\lambda$. For simplicity, let us denote $C(X)_{\leq \lambda} = C(X)_{\leq e_\lambda}$ and $C(X)_{< \lambda} = C(X)_{< e_\lambda}$. We obtain the following description of D_λ and ∂D_λ as a corollary to Proposition 2.47.

Corollary 2.48. *Let X be a totally normal cellular stratified space. For a cell $\varphi_\lambda : D_\lambda \rightarrow \bar{e}_\lambda$ in X , define a functor*

$$D_{\leq \lambda}^X : C(X)_{\leq \lambda} \longrightarrow \mathbf{Top}$$

by $D_{\leq \lambda}^X(u) = D_\mu$ for an object $u : D_\mu \rightarrow D_\lambda$ in $C(X)_{\leq \lambda}$. Then we have a natural homeomorphism

$$\text{colim}_{C(X)_{\leq \lambda}} D_{\leq \lambda}^X \xrightarrow{\cong} D_\lambda.$$

Define $D_{< \lambda}^X = D_{\leq \lambda}^X|_{C(X)_{< \lambda}}$. Then the above homeomorphism induces a homeomorphism

$$\text{colim}_{C(X)_{< \lambda}} D_{< \lambda}^X \xrightarrow{\cong} \partial D_\lambda.$$

Proof. By the definition of totally normal cellular stratified spaces, there is a one to one correspondence between cells in D_λ and morphisms $e_\mu \rightarrow e_\lambda$ in $C(X)$. Thus the comma category $C(X)_{\leq \lambda}$ is isomorphic to the face category $C(D_\lambda)$ and the functor $D_{\leq \lambda}^X$ can be identified with D^{D_λ} . And the result follows from Proposition 2.47. By removing 1_{D_λ} , we obtain a homeomorphism $\text{colim}_{C(X)_{< \lambda}} D_{< \lambda}^X \cong \partial D_\lambda$. \square

We use the following notation and terminology for the classifying space of the face category of a cellular stratified space.

Definition 2.49. For a cellular stratified space X , the classifying space $BC(X)$ of the face category $C(X)$ is called the *barycentric subdivision* of X and is denoted by $\text{Sd}(X)$.

When X is a regular cell complex, $\text{Sd}(X)$ coincides with the usual barycentric subdivision of X , hence is homeomorphic to X . In general, however, $\text{Sd}(X)$ is smaller than X . A precise relation between X and $\text{Sd}(X)$ is given by the following theorem, which is the main tool in this paper.

Theorem 2.50. *For a CW totally normal cellular stratified space X , the barycentric subdivision $\text{Sd}(X)$ of X can be embedded into X as a strong deformation retract. When X is a CW complex, the embedding is a homeomorphism.*

Furthermore the embeddings and homotopies can be chosen to be natural with respect to morphisms of cellular stratified spaces.

The rest of this section is devoted to a proof of this Theorem. We first need to construct an embedding $i_X : \text{Sd}(X) \hookrightarrow X$.

Proposition 2.51. *For a CW totally normal cellular stratified space X , there exists an embedding*

$$i_X : \text{Sd}(X) \hookrightarrow X$$

which is natural with respect to strict morphisms of cellular stratified spaces.

Proof. By Lemma 2.10, $\text{Sd}(X) = BC(X)$ is the quotient of $\coprod_{n \geq 0} \overline{N}_n(C(X)) \times \Delta^n$ under the equivalence relation generated by (2). Thus it suffices to construct maps

$$i_n : \overline{N}_n(C(X)) \times \Delta^n \longrightarrow X$$

making the following diagram commutative

$$\begin{array}{ccc} \overline{N}_n(C(X)) \times \Delta^{n-1} & \xrightarrow{1 \times d^i} & \overline{N}_n(C(X)) \times \Delta^n \\ d_i \times 1 \downarrow & & \downarrow i_n \\ \overline{N}_{n-1}(C(X)) \times \Delta^{n-1} & \xrightarrow{i_{n-1}} & X, \end{array} \quad (4)$$

where $d^i(t_0, \dots, t_{n-1}) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$. Consider the space $D(X) = \coprod_{\lambda} \{e_{\lambda}\} \times D_{\lambda}$ and a map $\tilde{\Phi} : D(X) \rightarrow X$ in Proposition 2.28. We construct an embedding

$$z_n : \overline{N}_n(C(X)) \times \Delta^n \longrightarrow D(X)$$

and define $i_n = \tilde{\Phi} \circ z_n$. Note that we have a decomposition

$$\overline{N}_n(C(X)) = \coprod_{e \in \overline{N}_n(P(X))} \{e\} \times \overline{N}(\pi)_n^{-1}(e),$$

where $\overline{N}(\pi)_n : \overline{N}_n(C(X)) \rightarrow \overline{N}_n(P(X))$ is the map induced by the canonical projection $\pi : C(X) \rightarrow P(X)$. Thus it suffices to construct an embedding

$$z_e : \overline{N}(\pi)_n^{-1}(e) \longrightarrow D_{\lambda_n}$$

for each $e = (\lambda_n > \lambda_{n-1} > \dots > \lambda_0) \in \overline{N}_n(P(X))$.

The embedding z_e is constructed by induction on n . When $n = 0$, $\overline{N}_0(C(X)) = C(X)_0 \cong P(X)$. For each $\lambda \in P(X)$, define $z_{e_{\lambda}}(e_{\lambda}, *) = \varphi_{\lambda}(0)$, where $\Delta^0 = \{*\}$ and $0 \in D_{\lambda}$ is the origin.

Suppose we have constructed z_e for $e \in \overline{N}_n(P(X))$ with $n \leq k-1$. For $e = (\lambda_k > \dots > \lambda_0) \in \overline{N}_k(P(X))$, we have

$$z_{d_k(e)} : \overline{N}(\pi)_{k-1}^{-1}(d_k(e)) \times \Delta^{k-1} \longrightarrow D_{\lambda_{k-1}}$$

by the inductive assumption. Note that we have a decomposition

$$\overline{N}(\pi)_k^{-1}(e) = C(X)(e_{\lambda_{k-1}}, e_{\lambda_k}) \times \overline{N}(\pi)_{k-1}^{-1}(d_k(e)). \quad (5)$$

We have a map $\tilde{z}_e : \overline{N}_k^{-1}(e) \times \Delta^{k-1} \rightarrow \partial D_{\lambda_k}$ defined by the composition

$$\begin{aligned} \overline{N}(\pi)_k^{-1}(e) \times \Delta^{k-1} &= C(X)(e_{\lambda_{k-1}}, e_{\lambda_k}) \times \overline{N}(\pi)_{k-1}^{-1}(d_k(e)) \times \Delta^k \\ &\xrightarrow{1 \times z_{d_k(e)}} C(X)(e_{\lambda_{k-1}}, e_{\lambda_k}) \times D_{\lambda_{k-1}} \\ &\xrightarrow{b^{\lambda_{k-1}, \lambda_k}} \partial D_{\lambda_k}, \end{aligned}$$

where

$$b_{\lambda_{k-1}, \lambda_k} : C(X)(e_{\lambda_{k-1}}, e_{\lambda_k}) \times D_{\lambda_{k-1}} \longrightarrow \partial D_{\lambda_k} \subset D_{\lambda_k}$$

is the adjoint of the inclusion $C(X)(e_{\lambda_{k-1}}, e_{\lambda_k}) \subset \text{Map}(D_{\lambda_{k-1}}, D_{\lambda_k})$. Since Δ^k is the join of Δ^{k-1} and the k -th vertex $\mathbf{v}_k = (0, \dots, 0, 1)$, the above map extends to

$$z_e : \overline{N}(\pi)_k^{-1}(e) \times \Delta^k = \overline{N}(\pi)_k^{-1}(e) \times \Delta^{k-1} * \mathbf{v}_k \xrightarrow{\tilde{z}_e * 0} \partial D_{\lambda_k} * \{0\} \subset D_{\lambda_k}.$$

This completes the induction and we obtain maps z_k for all k .

Let us verify that these maps make the diagram (4) commutative. Under the decomposition (5), it suffices to show the commutativity of the diagram

$$\begin{array}{ccccc} \overline{N}(\pi)_k^{-1}(e) \times \Delta^{k-1} & \xrightarrow{1 \times d^i} & \overline{N}(\pi)_k^{-1}(e) \times \Delta^k & \xrightarrow{z_e} & D(X) \\ \downarrow d_i \times 1 & & & & \downarrow \tilde{\Phi} \\ \overline{N}(\pi)_{k-1}^{-1}(d_i(e)) & \xrightarrow{z_{d_k(e)}} & D(X) & \xrightarrow{\tilde{\Phi}} & X \end{array}$$

for each $e = (\lambda_k > \dots > \lambda_0) \in \overline{N}_k(P(X))$.

When $0 \leq i < k$, the last element in $d_i(e)$ is also λ_k and the diagram reduces to

$$\begin{array}{ccc} \overline{N}(\pi)_k^{-1}(e) \times \Delta^{k-1} & \xrightarrow{1 \times d^i} & \overline{N}(\pi)_k^{-1}(e) \times \Delta^k \\ \downarrow d_i \times 1 & & \downarrow z_e \\ \overline{N}(\pi)_{k-1}^{-1}(d_i(e)) \times D^{k-1} & \xrightarrow{z_{d_i(e)}} & D_{\lambda_k}. \end{array}$$

The commutativity of this diagram follows from an easy diagram chasing based on the inductive

definition of z_e . The case $i = k$ follows from the commutativity of the diagram

$$\begin{array}{ccc}
\overline{N}(\pi)_k^{-1}(e) \times \Delta^{k-1} & \xrightarrow{1 \times d^k} & \overline{N}(\pi)_k^{-1}(e) \times \Delta^k \\
\parallel & & \downarrow z_e \\
C(X)(e_{\lambda_{k-1}}, e_{\lambda_k}) \times \overline{N}_{k-1}(d_k(e)) \times \Delta^{k-1} & & \\
\downarrow 1 \times z_{d_k(e)} & & \downarrow \\
C(X)(e_{\lambda_{k-1}}, e_{\lambda_k}) \times D_{\lambda_{k-1}} & \xrightarrow{b_{\lambda_{k-1}, \lambda_k}} & D_{\lambda_k}.
\end{array}$$

Now we have constructed a sequence of maps $i_n : \overline{N}_n(C(X)) \times \Delta^n \rightarrow X$ compatible with the face relation. Let us denote the resulting continuous map by

$$i_X : BC(X) \longrightarrow X.$$

We only used structure maps of cellular stratified spaces and the origin in each D_λ in the construction of i_X . Hence it is natural with respect to strict morphisms of cellular stratified spaces.

Let us show that $i_X : BC(X) \rightarrow \text{Im}(i_X)$ is a bijective closed map, hence is a homeomorphism. The surjectivity is obvious. The injectivity of i_X can be proved inductively by using the definition of z_e . It remains to show that i_X is a closed map. By definition, we have the following commutative diagram

$$\begin{array}{ccc}
\coprod_{n, e \in \overline{N}_n(P(X))} \{e\} \times \overline{N}(\pi)_n^{-1}(e) \times \Delta^n & \xrightarrow{\coprod z_e} & D(X) \\
\downarrow p & & \downarrow \tilde{\Phi} \\
\text{Sd}(X) & \xrightarrow{i_X} & X,
\end{array}$$

where p is the canonical projection. For each closed set $A \subset \text{Sd}(X)$, it suffices to show that $\tilde{\Phi}^{-1}(i_X(A))$ is closed in $D(X)$, since $\tilde{\Phi}$ is a quotient map by Lemma 2.28. Note that $\coprod z_e$ is a closed map, since it is a disjoint union of

$$\coprod_{t(e)=\lambda} \{e\} \times \overline{N}_n^{-1}(e) \times \Delta^n \longrightarrow \{e_\lambda\} \times D_\lambda \quad (6)$$

that are continuous maps from compact sets to Hausdorff spaces. The compactness of the domain of the above map comes from the finiteness of the number of cells in ∂D_λ .

We claim that

$$\tilde{\Phi}^{-1}(i_X(A)) = \left(\coprod z_e \right) (f^{-1}(A)). \quad (7)$$

Once this is shown, the proof is complete. The commutativity of the above diagram implies that $\tilde{\Phi}^{-1}(i_X(A)) \subset (\coprod z_e)(f^{-1}(A))$. On the other hand, for $x \in \tilde{\Phi}^{-1}(i_X(A))$, there exists $a \in A$ such that $\tilde{\Phi}(x) = i_X(a)$.

If $x \in \text{Int}(D_\lambda)$ and $a = [\mathbf{p}, \mathbf{t}]$,

$$\varphi_\lambda(x) = i_X(a) = i_X(p(\mathbf{p}, \mathbf{t})) = \tilde{\Phi}(z_{\pi(\mathbf{p})}(\mathbf{p}, \mathbf{t})).$$

Since $x \in \text{Int}(D_\lambda)$, \mathbf{t} is of the form $\mathbf{t} = (1-t)\mathbf{s} + t\mathbf{v}_k$ for some $0 < t \leq 1$ and $\mathbf{s} \in \Delta^{k-1}$. This implies that $z_{\pi(\mathbf{p})}(\mathbf{p}, \mathbf{t}) \in \text{Int}(D_\lambda)$ and $x = z_{\pi(\mathbf{p})}(\mathbf{p}, \mathbf{t})$. And we have $x \in \tilde{\Phi}^{-1}(i_X(A))$.

When $x \in \partial D_\lambda$, let e be a cell in ∂D_λ with $x \in e$. By the total normality, there exists a cell e_μ in X and a map b making the following diagram commutative

$$\begin{array}{ccccc} \bar{e} & \hookrightarrow & \partial D_\lambda & \hookrightarrow & D_\lambda & \xrightarrow{\varphi_\lambda} & X \\ & & & & & \nearrow \varphi_\mu & \\ & & & & D & \xlongequal{\quad} & D_\mu \end{array}$$

Let $y \in \text{Int}(D)$ be the unique element with $b(y) = x$. By the commutativity of the above diagram, $\varphi_\mu(y) = \varphi_\lambda(x) \in i_X(A)$. Since $y \in \text{Int}(D_\mu)$, the previous argument implies that there exists $(\mathbf{p}', \mathbf{t}') \in p^{-1}(A)$ with $y = z_{\pi(\mathbf{p}')}(\mathbf{p}', \mathbf{t}')$. Thus we have $x = b(z_{\pi(\mathbf{p}')}(\mathbf{p}', \mathbf{t}'))$. Now define \mathbf{p} to be the element of $\overline{N}(C(X))$ obtained by adjoining b to \mathbf{p}' as follows

$$\mathbf{p} : \underbrace{D_{\lambda_0} \rightarrow \cdots \rightarrow D_{\lambda_k}}_{\mathbf{p}'} \xrightarrow{b} D_\lambda$$

and define $\mathbf{t} = d^k(\mathbf{t}')$. Then

$$\begin{aligned} p(\mathbf{p}, \mathbf{t}) &= [\mathbf{p}, d^k(\mathbf{t}')] \\ &= [d_k(\mathbf{p}), \mathbf{t}'] \\ &= [\mathbf{p}', \mathbf{t}'] \\ &= p(\mathbf{p}', \mathbf{t}') \in A. \end{aligned}$$

Thus $(\mathbf{p}, \mathbf{t}) \in p^{-1}(A)$. This completes the proof.

In the above proof, the image of $\coprod_{t(e)=\lambda} \{e\} \times \overline{N}_n^{-1}(e) \times \Delta^n$ in $\text{Sd}(X)$ via p can be identified with $\text{Sd}(D_\lambda)$ and the map (6) induces an embedding

$$i_\lambda : \text{Sd}(D_\lambda) \hookrightarrow D_\lambda.$$

Thus the embedding i_X can be constructed by gluing embeddings i_λ together. When X is a cell complex, all D_λ are regular cell complexes and the embedding i_λ is a homeomorphism. Thus i_X is a homeomorphism. \square

Remark 2.52. The above observation that i_X is obtained by gluing i_λ can be verified by using Proposition 2.47. In particular, define a functor

$$\text{Sd}(D^X) : C(X) \longrightarrow \mathbf{Top}$$

by $\text{Sd}(D^X)(e_\lambda) = \text{Sd}(D_\lambda)$ on objects. Then, the embeddings i_λ give rise to a natural transformation

$$i_X : \text{Sd}(D^X) \Longrightarrow D^X$$

which induces an embedding

$$i_X : \text{Sd}(X) = \text{colim}_{C(X)} \text{Sd}(D^X) \xrightarrow{\cong} \text{colim}_{C(X)} D^X \cong X.$$

Now we are ready to prove Theorem 2.50.

Proof of Theorem 2.50. Let us show that the image of the embedding constructed in Proposition 2.51 is a strong deformation retract of X . The idea is essentially the same as the proof of Proposition 2.51. Following Remark 2.52, we construct a homotopy

$$H_\lambda : D_\lambda \times [0, 1] \longrightarrow D_\lambda$$

by gluing deformation retractions on the domain of each cell. This can be done by appealing to Proposition 2.47. More precisely, define a functor

$$D^{X \times [0, 1]} : C(X) \longrightarrow \mathbf{Top}$$

by $D^{X \times [0, 1]}(e_\lambda) = D_\lambda \times [0, 1]$. By applying Proposition 2.47 to $X \times [0, 1]$, we also have $X \times [0, 1] \cong \operatorname{colim}_{C(X)} D^{X \times [0, 1]}$. If we have constructed a natural transformation

$$H : D^{X \times [0, 1]} \Longrightarrow D^X,$$

we would have a homotopy

$$X \times [0, 1] \cong \operatorname{colim}_{C(X)} D^{X \times [0, 1]} \xrightarrow{\operatorname{colim} H} \operatorname{colim}_{C(X)} D^X \cong X.$$

Thus all we have to do is to construct H_λ for each $\lambda \in P(X)$ satisfying the following conditions:

1. H_λ is a strong deformation retraction of D_λ onto $i_{D_\lambda}(\operatorname{Sd}(D_\lambda))$.
2. For each $\lambda \in P(X)$ and a cell e in ∂D_λ , let e_μ be the cell in X corresponding to e and $b : D_\mu \rightarrow D_\lambda$ be the left of φ_μ . Then the following diagram is commutative

$$\begin{array}{ccc} D_\lambda \times [0, 1] & \xrightarrow{H_\lambda} & D_\lambda \\ \uparrow b & & \uparrow b \\ D_\mu \times [0, 1] & \xrightarrow{H_\mu} & D_\mu \end{array}$$

This is done by induction on the dimension of cells. When e_λ is a bottom cell, $\partial D_\lambda = \emptyset$ and $\operatorname{Sd}(D_\lambda)$ is a single point and can be identified with the origin of D_λ . Define

$$H_\lambda(x, t) = (1 - t)x.$$

Suppose that we have defined H_μ for all cells e_μ of dimension less than k . Let e_λ be a k -dimensional cell. Define

$$h_\lambda = \operatorname{colim}_{C(X)_{<\lambda}} H_\mu : \partial D_\lambda \times [0, 1] \cong \operatorname{colim}_{C(X)_{<\lambda}} D_{<\lambda}^{X \times [0, 1]} \longrightarrow \operatorname{colim}_{C(X)_{<\lambda}} D_{<\lambda}^X \cong \partial D_\lambda.$$

Then this is a strong deformation retraction of ∂D_λ onto $i_{\partial D_\lambda}(\operatorname{Sd}(\partial D_\lambda))$. This strong deformation retraction can be extended to

$$H_\lambda : D_\lambda \times [0, 1] \longrightarrow D_\lambda$$

by using Theorem 2.53 below. The construction by colimit implies that the following diagram is commutative

$$\begin{array}{ccc}
D_\mu \times [0, 1] & \xrightarrow{H_\mu} & D_\mu \\
\downarrow b \times 1_{[0,1]} & & \downarrow b \\
\partial D_\lambda \times [0, 1] & \xrightarrow{h_\lambda} & \partial D_\lambda \\
\downarrow & & \downarrow \\
D_\lambda \times [0, 1] & \xrightarrow{H_\lambda} & D_\lambda.
\end{array}$$

□

Theorem 2.53. *Let π be a regular cell decomposition of S^{n-1} and $L \subset S^{n-1}$ be a stratified subspace. Let $\tilde{\pi}$ be the cellular stratification on $K = \text{Int}D^n \cup L$ obtained by adding $\text{Int}D^n$ as an n -cell. Then there exists a deformation retraction H of K to $i_K(\text{Sd}(K, \tilde{\pi}))$. Furthermore if a deformation retraction h of L onto $i_L(\text{Sd}(L))$ is given, H can be taken to be an extension of h .*

A proof of this fact is given in Appendix A.

3 Acyclic Category Models for Configuration Spaces of Graphs

In this section, we construct a combinatorial model $C_k^{\text{comp}}(X)$ of the configuration space $\text{Conf}_k(X)$ of a graph X , by using the braid arrangements. Our model has the following advantages compared to Abrams' model:

- $C_k^{\text{comp}}(X)$ is always homotopy equivalent to the configuration space $\text{Conf}_k(X)$. And the homotopy is explicitly given.
- The construction of $C_k^{\text{comp}}(X)$, the embedding $i : C_k^{\text{comp}}(X) \hookrightarrow \text{Conf}_k(X)$, and the deformation retraction, are functorial with respect to strict morphisms of cellular stratified spaces. Hence $C_k^{\text{comp}}(X)$ inherits the action of Σ_k .
- $C_k^{\text{comp}}(X)$ is often much smaller than Abrams' model.

A couple of sample applications will be given in §4.

3.1 The Braid Stratification

Given a graph X , our strategy to construct a combinatorial model for $\text{Conf}_k(X)$ is to define a suitable Σ_k -equivariant cellular stratification on the k -fold product X^k including the discriminant $\Delta_k(X)$ as a stratified subspace. Then Theorem 2.50 gives us a model when applied to the induced stratification on $\text{Conf}_k(X)$. In order to obtain a model with the lowest possible dimension, we would like the subdivision to be as coarse as possible, which suggests us to use the braid arrangements.

Let us first recall the stratification on \mathbb{R}^n induced by a hyperplane arrangement.

Definition 3.1. Let $\mathcal{A} = \{H_1, \dots, H_k\}$ be a hyperplane arrangement in \mathbb{R}^n defined by a collection of affine 1-forms $L = \{\ell_i : \mathbb{R}^n \rightarrow \mathbb{R}\}_{i=1, \dots, k}$. The evaluation on points in \mathbb{R}^n defines map

$$\text{ev} : \mathbb{R}^n \longrightarrow \text{Map}(L, \mathbb{R}) \cong \mathbb{R}^k.$$

Composed with the sign function

$$\text{sign} : \mathbb{R} \longrightarrow \{-1, 0, +1\} = S_1,$$

we obtain a map

$$s_{\mathcal{A}} : \mathbb{R}^n \xrightarrow{\text{ev}} \mathbb{R}^k \xrightarrow{\text{sign}^k} S_1^k,$$

where S_1 is regarded as a poset with ordering $0 < \pm 1$. This is called the *stratification on \mathbb{R}^n determined by \mathcal{A}* .

Lemma 3.2. *The stratification $s_{\mathcal{A}}$ is a regular totally normal cellular stratification on \mathbb{R}^n for any hyperplane arrangement \mathcal{A} in \mathbb{R}^n .*

In this paper, we only make use of the braid arrangements.

Definition 3.3. For $1 \leq i < j \leq n$, define a hyperplane in \mathbb{R}^n by

$$H_{i,j} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i = x_j\}.$$

The hyperplane arrangement $\{H_{i,j} \mid 1 \leq i < j \leq n\}$ is called the *braid arrangement* of rank $n-1$ and is denoted by \mathcal{A}_{n-1} .

The structure of cellular stratification $s_{\mathcal{A}_{n-1}}$ on \mathbb{R}^n defined by the braid arrangement is well-known.

Lemma 3.4. *Cells in the cellular stratification $s_{\mathcal{A}_{n-1}}$ are in one-to-one correspondence with ordered partitions Π_n of $\{1, \dots, n\}$.*

Proof. An n -cell in the stratification $s_{\mathcal{A}_{n-1}}$ is given by a system of inequalities

$$x_{\sigma(1)} < x_{\sigma(2)} < \dots < x_{\sigma(n)}$$

for a permutation $\sigma \in \Sigma_n$. Lower dimensional cells are given by replacing $<$ by $=$, hence they corresponds to partitions of the set $\{1, \dots, n\}$. \square

Now we are ready to introduce our stratification on X^k . The starting point is the following observation, which is an immediate generalization of Lemma 2.43.

Lemma 3.5. *For any 1-dimensional finite cellular stratified space X , the product X^k is totally normal.*

Definition 3.6. Let X be a 1-dimensional finite cellular stratified space. Define a subdivision $\pi_{k,X}^{\text{braid}}$ of the product stratification on X^k as follows: Let $\{e_{\lambda}^0\}_{\lambda \in \Lambda_0}$ and $\{e_{\lambda}^1\}_{\lambda \in \Lambda_1}$ be 0-cells and 1-cells of X , respectively. We choose linear orders of Λ_0 and Λ_1 . For a cell $e_{\lambda_1}^{\varepsilon_1} \times \dots \times e_{\lambda_k}^{\varepsilon_k}$ in X^k , choose a permutation $\sigma \in \Sigma_k$ with

$$(e_{\lambda_1}^{\varepsilon_1} \times \dots \times e_{\lambda_k}^{\varepsilon_k})\sigma = (\text{a product of 0-cells}) \times (e_{\mu_1}^1)^{m_1} \times \dots \times (e_{\mu_\ell}^1)^{m_\ell} \quad (8)$$

and $\mu_1 < \dots < \mu_\ell$.

By using the characteristic map

$$\mathbb{R} \xrightarrow{\cong} \text{Int}(D^1) \xrightarrow{\varphi_{\mu_j}} e_{\mu_j},$$

we obtain a canonical homeomorphism

$$(e_{\mu_j}^1)^{m_j} \cong \mathbb{R}^{m_j}.$$

Subdivide each $(e_{\mu_j}^1)^{m_j}$ by the braid arrangement \mathcal{A}_{m_j-1} under this identification. The resulting stratification on X^k is denoted by $\pi_{k,X}^{\text{braid}}$ and is called the *braid stratification* on X^k .

Proposition 2.45 gives us the following important fact.

Proposition 3.7. *The braid stratification $\pi_{k,X}^{\text{braid}}$ on X^k for a graph X is totally normal and contains the discriminant $\Delta_k(X)$ as a stratified subspace. Hence the configuration space $\text{Conf}_k(X)$ is also a totally normal cellular stratified subspace of X^k .*

Proof. In the product stratification on X^k , we use (stratified subspaces of) cubes as domains of characteristic maps. The braid stratification on each domain cube induces a stratification on each face of codimension 1. The restriction of the braid arrangement \mathcal{A}_{n-1} to the hyperplane $x_n = 1$ or $x_n = -1$ is the arrangement \mathcal{A}_{n-2} in \mathbb{R}^{n-1} . The same is true for other codimension 1 faces $x_i = \pm 1$.

Since any morphism in the face category of the product stratification is a composition of inclusions of codimension 1 faces, the condition of Proposition 2.45 is satisfied. \square

3.2 A Combinatorial Model for Configuration Spaces of Graphs

By Proposition 3.7, the restriction of the braid stratification $\pi_{k,X}^{\text{braid}}$ to the configuration space $\text{Conf}_k(X)$ gives rise to an acyclic category $C(\pi_{k,X}^{\text{braid}}|_{\text{Conf}_k(X)})$. Its classifying space $BC(\pi_{k,X}^{\text{braid}}|_{\text{Conf}_k(X)})$ is our first model for $\text{Conf}_k(X)$.

Definition 3.8. For a 1-dimensional finite cellular stratified space X , let us abbreviate the induced stratification $\pi_{k,X}^{\text{braid}}|_{\text{Conf}_k(X)}$ on $\text{Conf}_k(X)$ by $\pi_{k,X}^{\text{comp}}$. Define a cell complex $C_k^{\text{comp}}(X)$ by

$$C_k^{\text{comp}}(X) = BC(\pi_{k,X}^{\text{comp}}).$$

By Theorem 2.50, $C_k^{\text{comp}}(X)$ is a cell complex model and $C(\pi_{k,X}^{\text{comp}})$ is an acyclic category model of the homotopy type of $\text{Conf}_k(X)$.

Corollary 3.9. *For a finite graph X , $C_k^{\text{comp}}(X)$ is embedded in $\text{Conf}_k(X)$ as a Σ_k -equivariant strong deformation retract.*

There is an alternative description. We may define $C_k^{\text{comp}}(X)$ as a ‘‘cellular complement’’ of the discriminant set in the barycentric subdivision $\text{Sd}(X^k, \pi_{k,X})$ of the braid stratification on X^k . More generally, we use the following notation.

Definition 3.10. Let X be a cellular stratified space and $A \subset X$ be a subset. Define a cellular stratified subspace $M(A; X)$ by

$$M(A; X) = \bigcup_{e \in P(X), \bar{e} \cap A = \emptyset} e$$

This is called the *cellular complement* of A in X .

In the case of totally normal cell complexes, we have the following description.

Lemma 3.11. *Let X be a totally normal cell complex and $A \subset X$ be a cellular stratified subspace. Then we have*

$$i_X(\text{Sd}(A)) = M(X \setminus A; i_X(\text{Sd}(X)))$$

under the inclusion $\text{Sd}(A) \hookrightarrow \text{Sd}(X)$ induced by $A \hookrightarrow X$.

Proof. Since X is a cell complex, $i_X(\text{Sd}(X)) = X$ and $X \setminus A$ can be regarded as a subset of $i_X(\text{Sd}(X))$. By the definition of classifying space, cells in $\text{Sd}(A)$ are in one-to-one correspondence with elements in $\overline{N}_*(C(A))$. For $e \in \overline{N}_*(C(A))$, let us denote the corresponding cell by $\sigma(e)$. By the construction of i_X , the image $i_X(\sigma(e))$ is given by choosing interior points in each cell in $e : e_0 \rightarrow \cdots \rightarrow e_n$ and then by ‘‘connecting’’ them. Thus

$$i_X(\sigma(e)) \subset e_0 \cup \cdots \cup e_n \subset A,$$

or $i_X(\sigma(e)) \cap (X \setminus A) = \emptyset$.

Conversely, if $i_X(\sigma(e))$ is a cell in $i_X(\text{Sd}(X))$ with $i_X(\sigma(e)) \cap (X \setminus A) = \emptyset$ and $e : e_0 \rightarrow \cdots \rightarrow e_n$, all vertices of $i_X(\sigma(e))$ should belong to A . Since vertices of $i_X(\sigma(e))$ are in the interiors of cells e_0, \dots, e_n . Thus these cells must be cells in A and we have $i_X(\sigma(e)) \subset i(\text{Sd}(A))$. \square

Thus the image of the embedding of $C_k^{\text{comp}}(X)$ can be described as a cellular complement.

Corollary 3.12. *Let X be a finite graph. Under the braid stratification $\pi_{k,X}^{\text{braid}}$ of X^k , we have*

$$i_{X^k}(C_k^{\text{comp}}(X)) = M(\Delta_k(X); \text{Sd}(\pi_{k,X}^{\text{braid}})).$$

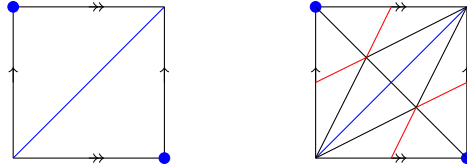
In other words, $i_{X^k}(C_k^{\text{comp}}(X))$ consists of those cells in $\text{Sd}(\pi_{k,X}^{\text{braid}})$ whose closures do not touch the discriminant $\Delta_k(X)$.

Let us take a look at a couple of examples.

Example 3.13. Let us consider the minimal cell decomposition $S^1 = e^0 \cup e^1$ of the circle, regarded as a graph. The product stratification on $(S^1)^2$ is given by

$$(S^1)^2 = e^0 \times e^0 \cup e^0 \times e^1 \cup e^1 \times e^0 \cup e^1 \times e^1.$$

The 2-cell $e^1 \times e^1$ is divided into a 1-cell e_Δ^1 and two 2-cells e_-^2 and e_+^2 . The braid stratification and its barycentric subdivision can be depicted as follows:



The discriminant $\Delta_2(S^1)$ is colored by blue.

By Corollary 3.12, we see that our model $C_2^{\text{comp}}(S^1)$ consists of those cells whose closures do not touch the blue part, namely the four red segments. It follows that $C_2^{\text{comp}}(S^1)$ is isomorphic to the boundary of a square.

It is left to the reader to verify that our model $C_2^{\text{comp}}(\text{Sd}(S^1))$ for the subdivision $\text{Sd}(S^1)$ is a 2-dimensional cell complex, which is homotopy equivalent to S^1 . \square

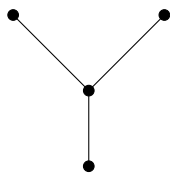
As we have seen in the above example, our model (as well as Abrams' model) gets fat as the cell decomposition becomes finer. In order to obtain a good model, therefore, we need to reduce the number of cells.

Definition 3.14. We say a graph is *reduced* if the only vertices with valency 2 are those contained in loops.

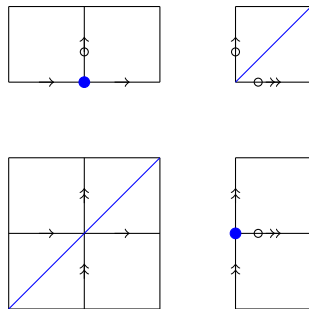
By replacing a pair of 1-cells e and e' sharing a vertex v by a 1-cell $e \cup v \cup e'$, we can always make a graph into reduced without changing the homeomorphism type. For example, $\text{Sd}(S^1)$ is not reduced but S^1 is.

The following example shows, however, the reducedness is not enough to obtain the lowest possible dimension.

Example 3.15. Consider a graph Y of the following shape.



By cutting out an edge, the product $Y \times Y$ can be developed into the following diagram:



The discriminant $\Delta_2(Y)$ is drawn by blue. The barycentric subdivision of the braid stratification is obtained by taking the barycentric subdivisions of squares and triangles in the above figure. It is easy to see, by Corollary 3.12, that our model $C_2^{\text{comp}}(Y)$ is a 2-dimensional cell complex which is homotopy equivalent to S^1 . The model $C_2^{\text{comp}}(Y)$ is not optimal from the viewpoint of homotopy dimension.

It is possible, however, to construct a 1-dimensional subcomplex in $C_2^{\text{comp}}(Y)$ which is a strong deformation retraction. This is the subject of the next section. \square

3.3 Removing Leaves

The first step to simplify the acyclic category model $C_k^{\text{comp}}(X)$ is to remove leaves.

Definition 3.16. For a graph X , the strict cellular stratified subspace obtained by removing all leaves from X is denoted by X° .

Lemma 3.17. *The inclusion $X^\circ \hookrightarrow X$ induces a Σ_k -equivariant homotopy equivalence $\text{Conf}_k(X^\circ) \simeq \text{Conf}_k(X)$.*

Proof. Choose an embedding $X \hookrightarrow X^\circ$ by squeezing the lengths of edges having leaves to the halves. Then the composition $X \hookrightarrow X^\circ \hookrightarrow X$ is isotopic to the identity. The same is true for the composition $X^\circ \hookrightarrow X \hookrightarrow X^\circ$. Since Conf_k is functorial with respect to embeddings, both compositions

$$\begin{aligned} \text{Conf}_k(X) &\hookrightarrow \text{Conf}_k(X^\circ) \hookrightarrow \text{Conf}_k(X) \\ \text{Conf}_k(X^\circ) &\hookrightarrow \text{Conf}_k(X) \hookrightarrow \text{Conf}_k(X^\circ) \end{aligned}$$

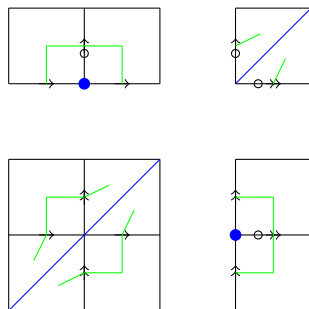
are isotopic to identities. In particular, we have a homotopy equivalence $\text{Conf}_k(X^\circ) \simeq \text{Conf}_k(X)$. \square

Corollary 3.18. *For a finite graph X , $C_k^{\text{comp}}(X^\circ)$ can be embedded in $\text{Conf}_k(X)$ as a Σ_k -equivariant strong deformation retraction.*

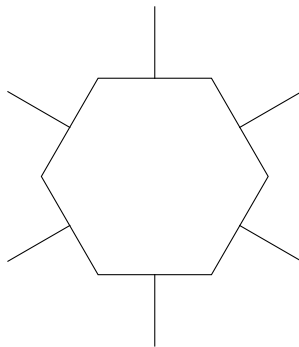
Thus the complex $C_k^{\text{comp}}(X^\circ)$ is a smaller model for $\text{Conf}_k(X)$. This model will be used in §4.1.

Example 3.19. Consider the graph Y in Example 3.15. It has three leaves. Then Y° is a strict stratified subspace of Y having only one vertex.

Then $C_2^{\text{comp}}(Y^\circ)$ is a 1-dimensional complex given by the green lines in the following figure:



By gluing these green parts, $C_2^{\text{comp}}(Y^\circ)$ is a 1-dimensional simplicial complex depicted below.



Thus $C_2^{\text{comp}}(Y^\circ)$ is a model which realizes the homotopy dimension of $\text{Conf}_2(Y)$.

As we have seen from this example, $C_k^{\text{comp}}(X^\circ)$ is much smaller than $C_k^{\text{comp}}(X)$ in general. It often realizes the homotopy dimension of $\text{Conf}_k(X)$.

The above example also says that $C_k^{\text{comp}}(X^\circ)$ still has some room to be simplified. By removing six spines from $C_2^{\text{comp}}(Y^\circ)$, it can be collapsed to a dodecagon. \square

3.4 A Simplified Model for Graphs

Although our model $C_k^{\text{comp}}(X^\circ)$ often realizes the homotopy dimension of $\text{Conf}_k(X)$, it still contains collapsible parts, as we have seen in Example 3.19. In this section, we concentrate on the case $k = 2$ and construct a minimal model by collapsing inessential parts in $C_2^{\text{comp}}(X^\circ)$.

Recall from Proposition 2.47 and the proof of Theorem 2.50 that both $\text{Conf}_k(X^\circ)$ and $C_k^{\text{comp}}(X^\circ)$ have colimit decompositions

$$\begin{aligned}\text{Conf}_k(X^\circ) &= \text{colim}_{C(\pi_{k, X^\circ}^{\text{comp}})} D^{\text{Conf}_k(X^\circ)} \\ C_k^{\text{comp}}(X^\circ) &= \text{colim}_{C(\pi_{k, X^\circ}^{\text{comp}})} \text{Sd}\left(D^{\text{Conf}_k(X^\circ)}\right),\end{aligned}$$

under which the embedding of $C_k^{\text{comp}}(X^\circ)$ into $\text{Conf}_k(X^\circ)$ decomposes into a colimit

$$\begin{array}{ccc} C_k^{\text{comp}}(X^\circ) & \xrightarrow{i_{\text{Conf}_k(X^\circ)}} & \text{Conf}_k(X^\circ) \\ \parallel & & \parallel \\ \text{colim}_{C(\pi_{k, X^\circ}^{\text{comp}})} \text{Sd}\left(D^{\text{Conf}_k(X^\circ)}\right) & \xrightarrow{\text{colim}_{C(\pi_{k, X^\circ}^{\text{comp}})} i_\lambda} & \text{colim}_{C(\pi_{k, X^\circ}^{\text{comp}})} D^{\text{Conf}_k(X^\circ)}. \end{array}$$

In other words, $C_k^{\text{comp}}(X^\circ)$ is obtained by gluing $\text{Sd}(D_\lambda)$'s for all cells e_λ in the braid stratification of $\text{Conf}_k(X^\circ)$. By simplifying each $\text{Sd}(D_\lambda)$, we may collapse $C_k^{\text{comp}}(X^\circ)$ further.

When $k = 2$, we use the following notation for D_λ and $\text{Sd}(D_\lambda)$.

Definition 3.20. Let X be a connected finite graph. The sets of vertices and edges are denoted by

$$\begin{aligned}V(X) &= \{e_\lambda^0 \mid \lambda \in \Lambda_0\} \\ E(X) &= \{e_\lambda^1 \mid \lambda \in \Lambda_1\},\end{aligned}$$

respectively. The sets of loops, branches, and connections (Definition 2.42) are denoted by

$$\begin{aligned}L(X) &= \{e_\lambda^1 \mid \lambda \in \Lambda_L\} \\ B(X) &= \{e_\lambda^1 \mid \lambda \in \Lambda_B\} \\ C(X) &= \{e_\lambda^1 \mid \lambda \in \Lambda_C\},\end{aligned}$$

respectively.

For 1-cells $\varphi_\lambda : D_\lambda \rightarrow \overline{e_\lambda^1}$ and $\varphi_\mu : D_\mu \rightarrow \overline{e_\mu^1}$ in X , we use

$$\varphi_\lambda \times \varphi_\mu : D_{\lambda, \mu} = D_\lambda \times D_\mu \rightarrow \overline{e_\lambda^1 \times e_\mu^1}$$

as the characteristic map for $e_\lambda^1 \times e_\mu^1$.

Up to an action of Σ_2 , we classify 2-cells in $X \times X$ into the following nine types:

1. $e_\lambda^1 \times e_\lambda^1$ for $\lambda \in \Lambda_L$
2. $e_\lambda^1 \times e_{\lambda'}^1$ for $\lambda, \lambda' \in \Lambda_L$ ($\lambda \neq \lambda'$)

3. $e_\lambda^1 \times e_\mu^1$ for $\lambda \in \Lambda_L, \mu \in \Lambda_B$
4. $e_\lambda^1 \times e_\nu^1$ for $\lambda \in \Lambda_L, \nu \in \Lambda_C$
5. $e_\mu^1 \times e_\nu^1$ for $\mu \in \Lambda_B$
6. $e_\mu^1 \times e_{\mu'}^1$ for $\mu, \mu' \in \Lambda_B$ ($\mu \neq \mu'$)
7. $e_\mu^1 \times e_\nu^1$ for $\mu \in \Lambda_B, \nu \in \Lambda_C$
8. $e_\nu^1 \times e_{\nu'}^1$ for $\nu \in \Lambda_C$
9. $e_\nu^1 \times e_{\nu'}^1$ for $\nu, \nu' \in \Lambda_C$ ($\nu \neq \nu'$)

Domains for 2-cells of type 1, 2, 4, 8, and 9 are $[-1, 1]^2$. In $(X^\circ)^2$, domains for 2-cells of type 3, 5, 6, and 7 are $[-1, 1] \times [-1, 1)$, $[-1, 1)^2$, $[-1, 1)^2$, and $[-1, 1) \times [-1, 1]$, respectively.

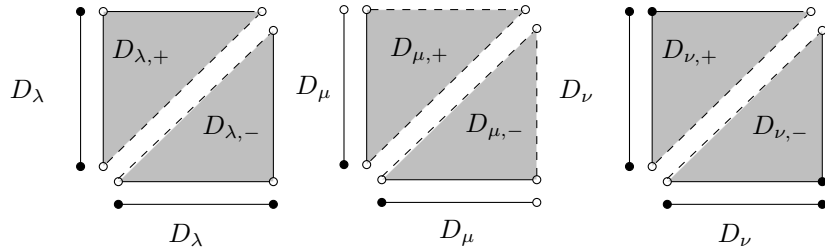
Under the subdivision via the braid arrangement \mathcal{A}_1 , 2-cells of type 1, 5, and 8 are subdivided and so are their domains. For a 2-cell $e_\lambda^1 \times e_\lambda^1$ of type 1, 5, or 8, we denote the subdivision by

$$e_\lambda^1 \times e_\lambda^1 = e_{\lambda,+}^2 \cup e_{\lambda,\Delta}^1 \cup e_{\lambda,-}^2.$$

Then we have the following stratification of $\text{Conf}_2(X)$

$$\begin{aligned} \text{Conf}_2(X) = & \bigcup_{\alpha, \beta \in \Lambda_0, \alpha \neq \beta} e_\alpha^0 \times e_\beta^0 \\ & \bigcup_{(\alpha, \lambda) \in \Lambda_0 \times \Lambda_1} e_\alpha^0 \times e_\lambda^1 \cup \bigcup_{(\lambda, \alpha) \in \Lambda_1 \times \Lambda_0} e_\lambda^1 \times e_\alpha^0 \\ & \bigcup_{\lambda \in \Lambda_L} e_{\lambda,+}^2 \cup e_{\lambda,-}^2 \cup \bigcup_{\lambda, \lambda' \in \Lambda_L, \lambda \neq \lambda'} e_\lambda^1 \times e_{\lambda'}^1 \cup \bigcup_{\lambda \in \Lambda_L, \mu \in \Lambda_B} e_\lambda^1 \times e_\mu^1 \cup \bigcup_{\lambda \in \Lambda_L, \nu \in \Lambda_C} e_\lambda^1 \times e_\nu^1 \\ & \bigcup_{\mu \in \Lambda_B} e_{\mu,+}^2 \cup e_{\mu,-}^2 \cup \bigcup_{\mu, \mu' \in \Lambda_B, \mu \neq \mu'} e_\mu^1 \times e_{\mu'}^1 \cup \bigcup_{\mu \in \Lambda_B, \nu \in \Lambda_C} e_\mu^1 \times e_\nu^1 \\ & \bigcup_{\nu \in \Lambda_C} e_{\nu,+}^2 \cup e_{\nu,-}^2 \cup \bigcup_{\nu, \nu' \in \Lambda_C} e_\nu^1 \times e_{\nu'}^1. \end{aligned}$$

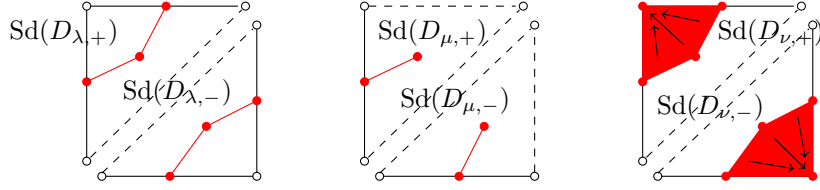
For $\lambda \in \Lambda_L$, $\mu \in \Lambda_B$, and $\nu \in \Lambda_C$, domains $D_{\lambda,\pm}$, $D_{\mu,\pm}$, and $D_{\nu,\pm}$ of the characteristic map of the cells $e_{\lambda,\pm}^2$, $e_{\mu,\pm}^2$, and $e_{\nu,\pm}^2$ are defined as follows:



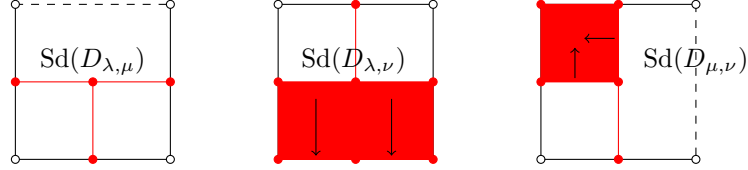
Let us consider Sd of these domains.

Proposition 3.21. *The barycentric subdivisions of domains of 2-cells in the braid stratification on $\text{Conf}_2(X)$ are given by the red regions in the following figures:*

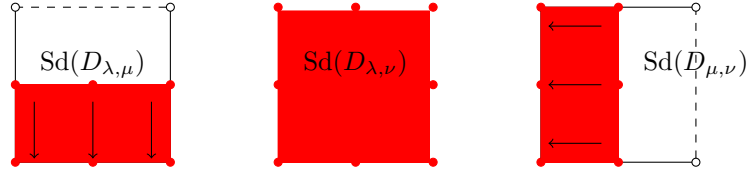
1. For $\lambda \in \Lambda_L$, $\mu \in \Lambda_B$, and $\nu \in \Lambda_C$, $\text{Sd}(D_{\lambda,\pm})$, $\text{Sd}(D_{\mu,\pm})$, and $\text{Sd}(D_{\nu,\pm})$ are given by



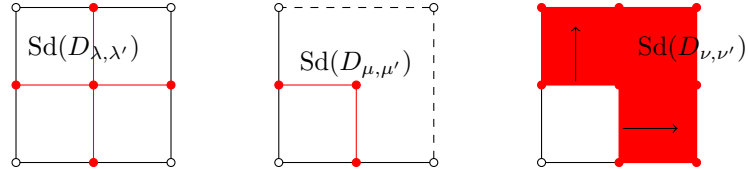
2. For $\lambda \in \Lambda_L$, $\mu \in \Lambda_B$, and $\nu \in \Lambda_C$, if $\overline{e_\lambda} \cap \overline{e_\mu} = \{v\}$, $\overline{e_\lambda} \cap \overline{e_\nu} = \{v\}$, and $\overline{e_\mu} \cap \overline{e_\nu} = \{v\}$ for a vertex v , then $\text{Sd}(D_{\lambda,\mu})$, $\text{Sd}(D_{\lambda,\nu})$, and $\text{Sd}(D_{\mu,\nu})$ are given by



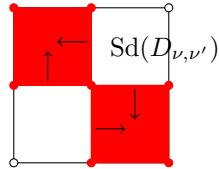
3. For $\lambda \in \Lambda_L$, $\mu \in \Lambda_B$, and $\nu \in \Lambda_C$, if the pairs (e_λ, e_μ) , (e_λ, e_ν) , and (e_μ, e_ν) do not share common vertices, then $\text{Sd}(D_{\lambda,\mu})$, $\text{Sd}(D_{\lambda,\nu})$, and $\text{Sd}(D_{\mu,\nu})$ are given by



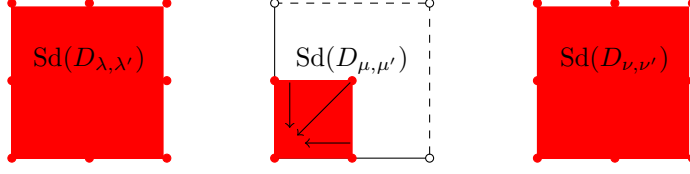
4. For $\lambda, \lambda' \in \Lambda_L$ ($\lambda \neq \lambda'$), $\mu, \mu' \in \Lambda_B$ ($\mu \neq \mu'$), and $\nu, \nu' \in \Lambda_C$ ($\nu \neq \nu'$), if $\overline{e_\lambda} \cap \overline{e_{\lambda'}} = \{v\}$, $\overline{e_\mu} \cap \overline{e_{\mu'}} = \{v\}$, and $\overline{e_\nu} \cap \overline{e_{\nu'}} = \{v\}$ for a vertex v , then $\text{Sd}(D_{\lambda,\lambda'})$, $\text{Sd}(D_{\mu,\mu'})$, and $\text{Sd}(D_{\nu,\nu'})$ are given by



5. For $\nu, \nu' \in \Lambda_C$ ($\nu \neq \nu'$), if $\overline{e_\nu} \cap \overline{e_{\nu'}} = \{v, w\}$ for vertices v and w , then $\text{Sd}(D_{\nu,\nu'})$ is given by



6. For $\lambda, \lambda' \in \Lambda_L$ ($\lambda \neq \lambda'$), $\mu, \mu' \in \Lambda_B$ ($\mu \neq \mu'$), and $\nu, \nu' \in \Lambda_C$ ($\nu \neq \nu'$), if the pairs $(e_\lambda, e_{\lambda'})$, $(e_\mu, e_{\mu'})$, and $(e_\nu, e_{\nu'})$ do not share common vertices, then $\text{Sd}(D_{\lambda,\lambda'})$, $\text{Sd}(D_{\mu,\mu'})$, and $\text{Sd}(D_{\nu,\nu'})$ are given by



Obviously $\text{Sd}(D_{\mu,+})$ and $\text{Sd}(D_{\mu,-})$ for $\mu \in \Lambda_B$ can be collapsed to $(-1, 0)$ and $(0, -1)$, respectively. There are two dimensional cells whose Sd can be collapsed to boundaries. They are indicated by black arrows in the above figure. This is because the boundaries of these 2-dimensional parts are glued to each other and we can define deformation retractions which moves the boundaries at the same speed.

Corollary 3.22. *Define a functor*

$$\text{Sd}^{\text{red}} \left(D^{\text{Conf}_2(X^\circ)} \right) \longrightarrow \mathbf{Top}$$

by modifying the $\text{Sd} \left(D^{\text{Conf}_2(X^\circ)} \right)$ by the following replacement of its values:

- for $\mu \in \Lambda_B$, replace $\text{Sd}(D_{\mu,+})$ by $\{(-1, 0)\}$, and $\text{Sd}(D_{\mu,-})$ by $\{(0, -1)\}$,
- for $\nu \in \Lambda_C$, replace $\text{Sd}(D_{\nu,+})$ by $\{(-1, 1)\}$, and $\text{Sd}(D_{\nu,-})$ by $\{(1, -1)\}$,
- for $\lambda \in \Lambda_L$ and $\nu \in \Lambda_C$ with $\overline{e_\lambda} \cap \overline{e_\nu} = \{v\}$ for a vertex v , replace $\text{Sd}(D_{\lambda,\nu})$ by $L_{\lambda,\nu} = \{0\} \times D_\nu \cup D_\lambda \times \{-1\}$ and $\text{Sd}(D_{\nu,\lambda})$ by $L_{\nu,\lambda} = D_\nu \times \{0\} \cup \{-1\} \times D_\lambda$,
- for $\mu \in \Lambda_B$ and $\nu \in \Lambda_C$ with $\overline{e_\mu} \cap \overline{e_\nu} = \{v\}$ for a vertex v , replace $\text{Sd}(D_{\mu,\nu})$ by $L_{\mu,\nu} = \{(s, -s) \mid -1 \leq s \leq 0\} \cup \{(0, t) \mid -1 \leq t \leq 0\}$ and $\text{Sd}(D_{\nu,\mu})$ by $L_{\nu,\mu} = \{(s, -s) \mid 0 \leq s \leq 1\} \cup \{(t, 0) \mid -1 \leq t \leq 0\}$,
- for $\lambda \in \Lambda_L$ and $\mu \in \Lambda_B$ with $\overline{e_\lambda} \cap \overline{e_\mu} = \emptyset$, replace $\text{Sd}(D_{\lambda,\mu})$ by $D_\lambda \times \{-1\}$ and $\text{Sd}(D_{\mu,\lambda})$ by $\{-1\} \times D_\lambda$,
- for $\mu \in \Lambda_B$ and $\nu \in \Lambda_C$ with $\overline{e_\mu} \cap \overline{e_\nu} = \emptyset$, replace $\text{Sd}(D_{\mu,\nu})$ by $\{-1\} \times D_\nu$, and $\text{Sd}(D_{\nu,\mu})$ by $D_\nu \times \{-1\}$,
- for $\nu, \nu' \in \Lambda_C$ with $\nu \neq \nu'$ and $\overline{e_\nu} \cap \overline{e_{\nu'}} = \{v\}$ for a vertex v , replace $\text{Sd}(D_{\nu,\nu'})$ by $D_\nu \times \{1\} \cup \{1\} \times D_{\nu'}$,
- for $\nu, \nu' \in \Lambda_C$ with $\nu \neq \nu'$ and $\overline{e_\nu} \cap \overline{e_{\nu'}} = \{v, w\}$ for vertices v and w , replace $\text{Sd}(D_{\nu,\nu'})$ by $L_{\nu,\nu'} = \{(s, -s) \mid -1 \leq s \leq 1\}$, and
- for $\mu, \mu' \in \Lambda_B$ with $\mu \neq \mu'$ and $\overline{e_\mu} \cap \overline{e_{\mu'}} = \emptyset$, replace $\text{Sd}(D_{\mu,\mu'})$ by $\{(-1, -1)\}$.

And define

$$C_2^{\text{comp},r}(X^\circ) = \text{colim}_{C(\pi_2, X^\circ)} \text{Sd}^{\text{red}} \left(D^{\text{Conf}_2(X^\circ)} \right)$$

Then $C_2^{\text{comp},r}(X^\circ)$ is a strong deformation retract of $C_2^{\text{comp}}(X^\circ)$.

This space $C_2^{\text{comp},r}(X^\circ)$ is our combinatorial (cell-complex) model for $\text{Conf}_2(X)$.

4 Sample Applications

In this final section, we present a couple of applications of our acyclic category model for the configuration space of 1-dimensional cellular stratified spaces.

4.1 The Homotopy Dimension

Given a finite graph X , we have a Σ_k -equivariant homotopy equivalence $\text{Conf}_k(X) \simeq_{\Sigma_k} C_k^{\text{comp}}(X^\circ)$ by Corollary 3.18, and hence a homotopy equivalence $\text{Conf}_k(X)/\Sigma_k \simeq C_k^{\text{comp}}(X^\circ)/\Sigma_k$. Let us consider the dimension of $C_k^{\text{comp}}(X^\circ)$, which is the classifying space of the finite acyclic category $C(\pi_{k,X^\circ}^{\text{comp}})$. In general, it is easy to count the dimension of the classifying space of a finite acyclic category.

Lemma 4.1. *Let C be a finite acyclic category. Then*

$$\dim BC = \max \{k \mid \overline{N}_k(C) \neq \emptyset\} = \dim BP(C).$$

Proof. Let $d = \max \{k \mid \overline{N}_k(C) \neq \emptyset\}$. Then the cell decomposition of BC is given by

$$BC = \left(\prod_{k=0}^d \overline{N}_k(C) \times \Delta^k \right) / \sim = \prod_{k=0}^d \overline{N}_k(C) \times \text{Int}(\Delta^k).$$

Interior points in cells of dimension d can not be equivalent to points in lower dimensional cells. Thus $\dim BC = d$. By the definition of the associated poset $P(C)$, d is also the rank of this poset. \square

Lemma 4.1 says that, for a finite totally normal cellular stratified space X , $\dim BC(X)$ is the length of maximal chains in the face poset $P(X)$.

Theorem 4.2. *For a finite connected graph X , we have*

$$\dim C_k^{\text{comp}}(X) \leq \min\{k, v(X)\},$$

where $v(X)$ is the number of 0-cells in X .

Proof. Since $\text{Conf}_k(X^\circ)$ is a k -dimensional cellular stratified space, $\dim C_k^{\text{comp}}(X) \leq k$. It remains to prove that the length of maximal chains is at most $v(X)$. By the symmetry under the action of Σ_k , it suffices to consider subdivisions of cells in X^k of the form (8).

First of all, by the definition of the subdivision, any cell of dimension less than k in the stratification $\pi_{k,X}^{\text{comp}}$ is a face of a k -dimensional face. Thus it is enough to count how many times we can take a boundary face of a k -dimensional face in $\text{Conf}_k(X)$ under the stratification $\pi_{k,X}^{\text{comp}}$.

Any k -dimensional cell in the stratification $\pi_{k,X}^{\text{comp}}$ is of the form

$$(e_{\lambda_1}^1 \times \cdots \times e_{\lambda_s}^1 \times e_{\mu_1, \tau_1}^{m_1} \times \cdots \times e_{\mu_t, \tau_t}^{m_t}) \sigma \quad (9)$$

with $s + m_1 + \cdots + m_t = k$ for some $\sigma \in \Sigma_k$, where $e_{\lambda_i}^1$ is a 1-cell in X and $e_{\mu_j, \tau_j}^{m_j}$ is an m_j -dimensional cell in the braid stratification of $(e_{\mu_j}^1)^{m_j}$ corresponding to a permutation $\tau_j \in \Pi_{m_j}$ under the correspondence in Lemma 3.4.

An $(m_i - 1)$ -dimensional face of $e_{\mu_i, \tau_i}^{m_i}$ is of the form $(e_{\lambda}^0 \times e_{\mu_i, \tau_i'}^{m_i-1}) \sigma'$ for a vertex e_{λ}^0 in $e_{\mu_i}^1$. By taking the boundary of $e_{\lambda_i}^1$, it is replaced by one of its vertices. Thus we increase the number of 0-cells by one if we take a boundary face of codimension 1.

Iterate the process of taking codimension 1 faces starting from one of the highest dimensional cells (9) in $\text{Conf}_k(X)$. When the same vertices appear twice as product factors, the boundary face cannot belong to $\text{Conf}_k(X)$ and the game is over. Thus the inequality is proved. \square

Corollary 4.3. *Let X be a finite connected graph and n be the number of essential vertices. Then we have*

$$\text{hdim } \text{Conf}_k(X)/\Sigma_k \leq \min\{n, k\}.$$

Remark 4.4. The above Corollary is not new. It has been already proved by Ghrist in [Ghr01]. We included a proof of this fact in order to show an optimality of our model.

4.2 Graph Braid Groups

Farley and Sabalka [FS05, FS12] used Abrams' model and discrete Morse theory to find presentations of graph braid groups.

Definition 4.5. For a topological space X , the fundamental groupoids $\pi_1(\text{Conf}_n(X)/\Sigma_n)$ and $\pi_1(\text{Conf}_n(X))$ are called the *braid groupoid* and the *pure braid groupoid* of n strands in X .

Fix a base point $\mathbf{x} \in \text{Conf}_n(X)$. Then the fundamental groups based on \mathbf{x} are denoted by

$$\begin{aligned} \text{Br}_n(X) &= \pi_1(\text{Conf}_n(X)/\Sigma_n)([\mathbf{x}], [\mathbf{x}]) = \pi_1(\text{Conf}_n(X)/\Sigma_n, [\mathbf{x}]) \\ \text{PBr}_n(X) &= \pi_1(\text{Conf}_n(X))(\mathbf{x}, \mathbf{x}) = \pi_1(\text{Conf}_n(X), \mathbf{x}), \end{aligned}$$

and called the *braid group* and the *pure braid group* of n strands in X , respectively. When X is a 1-dimensional CW complex, they are called the n -th *graph braid group* and the *pure graph braid group* of X , respectively.

As a sample application of our model introduced in §3, we give presentations of these groups for graphs with at most two essential vertices in this section.

Let us begin with the case of a graph with a single vertex.

Theorem 4.6 (Theorem 1.7). *Let $W_{k,\ell}$ be the finite graph with a single vertex v , k branches, and ℓ loops with leaves removed. (See Definition 2.42 for our terminology and Theorem 1.7 for a figure of this graph.)*

Then the fundamental groups of ordered and unordered configuration spaces of two points in $W_{k,\ell}$ are given by

$$\begin{aligned} \pi_1(\text{Conf}_2(W_{k,\ell})) &\cong F_{2n_{k,\ell}+1} \\ \pi_1(\text{Conf}_2(W_{k,\ell})/\Sigma_2) &\cong F_{n_{k,\ell}+1}, \end{aligned}$$

where $n_{k,\ell} = \frac{1}{2}(k + \ell)(k + 3\ell - 3)$ and F_n denotes the free group of rank n .

Proof. Since there is only one 0-cell in X , both $C_2^{\text{comp}}(W_{k,\ell})$ and $C_2^{\text{comp}}(W_{k,\ell})/\Sigma_2$ are 1-dimensional cell complexes by Theorem 4.2. Thus the fundamental groups $\text{PBr}_2(W_{k,\ell})$ and $\text{Br}_2(W_{k,\ell})$ are free groups. Their ranks as free groups coincide with the ranks of $H_1(C_2^{\text{comp}}(W_{k,\ell}))$ and $H_1(C_2^{\text{comp}}(W_{k,\ell})/\Sigma_2)$ as free Abelian groups.

Let us consider $H_1(C_2^{\text{comp}}(W_{k,\ell}))$. Since $W_{k,\ell}$ is connected, it suffices to compute the Euler characteristic

$$\text{rank } H_1(C_2^{\text{comp}}(W_{k,\ell})) = 1 - \chi(C_2^{\text{comp}}(W_{k,\ell})).$$

Let v be the vertex of $W_{k,\ell}$ and write

$$W_{k,\ell} = \{v\} \cup \left(\bigcup_{\mu \in \Lambda_B} e_\mu \right) \cup \left(\bigcup_{\lambda \in \Lambda_L} e_\lambda \right).$$

By Theorem 3.22, $C_2^{\text{comp},r}(W_{k,\ell})$ is obtained as a quotient space of

$$\begin{aligned} T_2^{\text{comp},r}(W_{k,\ell}) &= \left(\prod_{\lambda \in \Lambda_L} \text{Sd}(D_{\lambda,+}) \right) \amalg \left(\prod_{\lambda \in \Lambda_L} \text{Sd}(D_{\lambda,-}) \right) \\ &\amalg \left(\prod_{\mu \in \Lambda_B, \lambda \in \Lambda_L} \text{Sd}(D_{\mu,\lambda}) \right) \amalg \left(\prod_{\mu \in \Lambda_B, \lambda \in \Lambda_L} \text{Sd}(D_{\lambda,\mu}) \right) \\ &\amalg \left(\prod_{\mu, \mu' \in \Lambda_B, \mu \neq \mu'} \text{Sd}(D_{\mu,\mu'}) \right) \\ &\amalg \left(\prod_{\lambda, \lambda' \in \Lambda_L, \lambda \neq \lambda'} \text{Sd}(D_{\lambda,\lambda'}) \right). \end{aligned}$$

The numbers of subcomplexes of type $\text{Sd}(D_{\lambda,+})$, $\text{Sd}(D_{\lambda,-})$, $\text{Sd}(D_{\mu,\lambda})$, $\text{Sd}(D_{\lambda,\mu})$, $\text{Sd}(D_{\mu,\mu'})$, and $\text{Sd}(D_{\lambda,\lambda'})$ are ℓ , ℓ , $k\ell$, $k\ell$, $k(k-1)$, and $\ell(\ell-1)$, respectively.

These subcomplexes are glued together along their boundary vertices. Thus the number of edges in $C^{\text{comp},r}(W_{k,\ell})$ is the same as that of $T^{\text{comp},r}(W_{k,\ell})$, which is give by

$$2\ell + 2\ell + 3k\ell + 3k\ell + 2k(k-1) + 4\ell(\ell-1) = 6k\ell + 2k^2 - 2k + 4\ell^2.$$

The vertices in $C^{\text{comp},r}(W_{k,\ell})$ are in one-to-one correspondence with cells in the braid stratification on $\text{Conf}_2(W_{k,\ell})$ with cells of the form $e_{\mu,+}$ and $e_{\mu,-}$ removed. Besides the 2-cells described above, 1-cells are $\{v\} \times e_\mu$, $\{v\} \times e_\lambda$, $e_\mu \times \{v\}$, and $e_\lambda \times \{v\}$ for $\mu \in \Lambda_B$ and $\lambda \in \Lambda_L$. Thus the number of vertices in $C^{\text{comp},r}(W_{k,\ell})$ is given by

$$2k + 2\ell + 2\ell + 2k\ell + k(k-1) + \ell(\ell-1) = 2k\ell + k^2 + k + \ell^2 + 3\ell.$$

Thus the Euler characteristic is

$$\begin{aligned} \chi(C_2^{\text{comp},r}(W_{k,\ell})) &= 2k\ell + k^2 + k + \ell^2 + 3\ell - (6k\ell + 2k^2 - 2k + 4\ell^2) \\ &= -4k\ell - k^2 - 3\ell^2 + 3k + 3\ell \\ &= -(k+\ell)(k+3\ell-3). \end{aligned}$$

And the rank of $\text{PBr}(W_{k,\ell})$ is given by

$$1 - \chi(C_2^{\text{comp},r}(W_{k,\ell})) = 1 + (k+\ell)(k+3\ell-3).$$

By identifying cells under the action of Σ_2 , we see that the rank of $\text{Br}(W_{k,\ell})$ is given by $1 + \frac{1}{2}(k+\ell)(k+3\ell-3)$. \square

We need to deal with 2-cells to determine the graph braid groups of graphs with two vertices. We use the following elementary fact.

Proposition 4.7. *Let X be a CW complex. Suppose that it contains connected subcomplexes A and B and that there are regular 1-cells $\varphi_i : [-1, 1] \rightarrow \overline{e_i^1}$ ($i = 1, \dots, n$) with $\varphi_i(-1) \in A$, $\varphi_i(1) \in B$, and $X = A \cup B \cup \bigcup_{i=1}^n e_i^1$. Then we have a homotopy equivalence*

$$X \simeq A \vee B \vee \bigvee_{i=1}^{n-1} S^1.$$

Proof. Since A and B are path-connected, we may move the end points of $\overline{e_i^1}$ freely in A and B . Move $\varphi_i(-1)$ for $1 \leq i \leq n-1$ to $\varphi_n(-1)$ and then move those $n-1$ points to $\varphi_n(1)$ along $\overline{e_n^1}$.

Then we obtain a CW complex consisting of A and $B \vee \bigvee_{i=1}^{n-1} S^1$ connected by an edge $\overline{e_n^1}$. By

shrinking $\overline{e_n^1}$, we obtain $A \vee B \vee \bigvee_{i=1}^{n-1} S^1$. \square

Theorem 4.8 (Theorem 1.8). *Let ${}_x B_{p,q}^{k,\ell}$ be the finite graph obtained by gluing the essential vertices of $W_{k,\ell}$ and $W_{p,q}$ by x parallel edges. (See the figure in Theorem 1.8.)*

Then the fundamental groups of ordered and unordered configuration spaces of two points in ${}_x B_{p,q}^{k,\ell}$ are given by

$$\begin{aligned} \pi_1(\text{Conf}_2({}_x B_{p,q}^{k,\ell})) &\cong A_{\ell,q} * A_{q,\ell} * F_{2{}_x m_{p,q}^{k,\ell} - 1} \\ \pi_1(\text{Conf}_2({}_x B_{p,q}^{k,\ell}) / \Sigma_2) &\cong A_{\ell,q} * F_{{}_x m_{p,q}^{k,\ell}}, \end{aligned}$$

where

$$A_{\ell,q} = \langle a_1, \dots, a_\ell, b_1, \dots, b_q \mid [a_j, b_t] \ (1 \leq j \leq \ell, 1 \leq t \leq q) \rangle$$

and

$${}_x m_{p,q}^{k,\ell} = n_{k,\ell} + n_{p,q} + x(k + \ell + p + q) + \frac{x(x-1)}{2}.$$

Proof. Let v_1, v_2 be vertices of ${}_x B_{p,q}^{k,\ell}$ and write

$${}_x B_{p,q}^{k,\ell} = \{v_1, v_2\} \cup \left(\bigcup_{\lambda \in \Lambda_{L_1}} e_\lambda \right) \cup \left(\bigcup_{\lambda \in \Lambda_{L_2}} e_\lambda \right) \left(\bigcup_{\mu \in \Lambda_{B_1}} e_\mu \right) \cup \left(\bigcup_{\mu \in \Lambda_{B_2}} e_\mu \right) \cup \left(\bigcup_{\nu \in \Lambda_C} e_\nu \right),$$

where $\{e_\lambda \mid \lambda \in \Lambda_{L_i}\}$ is the set of loops adjacent to the vertex v_i for $i = 1, 2$ and $\{e_\mu \mid \mu \in \Lambda_{B_i}\}$ is the set of branches adjacent to v_i for $i = 1, 2$.

There are obvious embeddings

$$\begin{aligned} W_{k,\ell} &\hookrightarrow {}_x B_{p,q}^{k,\ell} \\ W_{p,q} &\hookrightarrow {}_x B_{p,q}^{k,\ell}, \end{aligned}$$

which induce embeddings of cell complexes

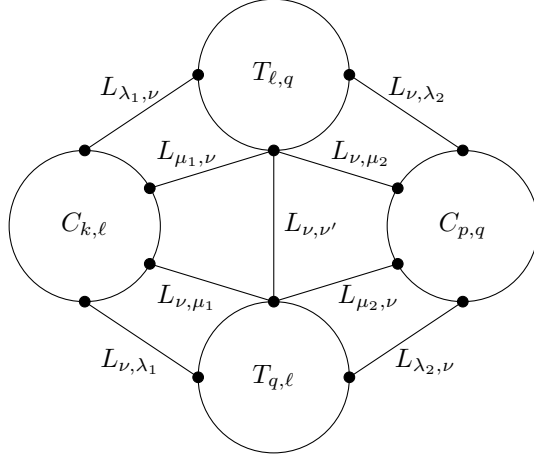
$$\begin{aligned} C^{\text{comp},r}(W_{k,\ell}) &\hookrightarrow C^{\text{comp},r}({}_x B_{p,q}^{k,\ell}) \\ C^{\text{comp},r}(W_{p,q}) &\hookrightarrow C^{\text{comp},r}({}_x B_{p,q}^{k,\ell}). \end{aligned}$$

Define

$$T_{q,\ell} = [0, q] \times [0, \ell] / \sim_{q,\ell}$$

where the relation $\sim_{q,\ell}$ is defined by $(s, t) \simeq_{q,\ell} (s', t')$ if and only if $s, s' \in \{0, 1, \dots, q\}$ and $t = t'$ or $s = s'$ and $t, t' \in \{0, 1, \dots, \ell\}$. The $T_{q,\ell}$ and $T_{\ell,q}$ can be embedded in $C^{\text{comp},r}({}_x B_{p,q}^{k,\ell})$ as subcomplexes consisting of $\text{Sd}(D_{\lambda_1, \lambda_2})$ for $\lambda_i \in \Lambda_{L_i}$ ($i = 1, 2$).

By Corollary 3.22, $C_2^{\text{comp},r}({}_x B_{p,q}^{k,\ell})$ is obtained by sewing $T_{\ell,q}$, $T_{q,\ell}$, $C_2^{\text{comp},r}(W_{k,\ell})$, and $C_2^{\text{comp},r}(W_{p,q})$ by using $L_{\lambda,\nu}$, $L_{\mu,\lambda}$, $L_{\mu,\nu}$, $L_{\nu,\mu}$, and $L_{\nu,\nu'}$ as strings, where $\lambda \in \Lambda_L$, $\mu \in \Lambda_B$, $\nu, \nu' \in \Lambda_C$ ($\nu \neq \nu'$). Note that all these strings are homeomorphic to a closed interval $[-1, 1]$. The following figure shows a rough idea of connections, in which $C_2^{\text{comp},r}(W_{k,\ell})$ is abbreviated by $C_{k,\ell}$ and indices runs over all $\lambda_1 \in \Lambda_{L_1}$, $\lambda_2 \in \Lambda_{L_2}$, $\mu_1 \in \Lambda_{B_1}$, $\mu_2 \in \Lambda_{B_2}$, and $\nu, \nu' \in \Lambda_C$ ($\nu \neq \nu'$).



We first apply Proposition 4.7 to the union of $C_{k,\ell}$, $T_{\ell,k}$, $L_{\lambda_1,\nu}$'s, and $L_{\mu_1,\nu}$'s. There are $x\ell$ $L_{\lambda_1,\nu}$'s and xk $L_{\mu_1,\nu}$'s. Thus we obtain $T_{\ell,k} \vee C_{k,\ell} \vee \bigvee^{x(k+\ell)-1} S^1$. Apply Proposition 4.7 to the union of $T_{\ell,k} \vee C_{k,\ell} \vee \bigvee^{x(k+\ell)-1} S^1$, $C_{p,q}$, L_{ν,λ_2} 's, and L_{ν,μ_2} 's and obtain $T_{\ell,k} \vee C_{k,\ell} \vee C_{p,q} \vee \bigvee^{x(k+\ell+p+q)-2} S^1$. Edges connecting this complex to $T_{q,\ell}$ are L_{ν,λ_1} 's, L_{ν,μ_1} 's, $L_{\lambda_2,\nu}$'s, $L_{\mu_2,\nu}$'s, and $L_{\nu,\nu'}$'s. Thus there are $x(k+\ell+p+q) + x(x-1)$ edges. By Proposition 4.7 again, we have a homotopy equivalence

$$\begin{aligned}
C_2^{\text{comp},r}(xB_{p,q}^{k,\ell}) &\simeq T_{\ell,q} \vee T_{q,\ell} \vee C_2^{\text{comp},r}(W_{k,\ell}) \vee C_2^{\text{comp},r}(W_{p,q}) \\
&\vee \left(\bigvee^{(k+p+\ell+q)x-2} S^1 \right) \vee \left(\bigvee^{(k+p+\ell+q)x+x(x-1)-1} S^1 \right) \\
&\simeq T_{\ell,q} \vee T_{q,\ell} \vee \left(\bigvee^{2n_{k,\ell}+1} S^1 \right) \vee \left(\bigvee^{2n_{p,q}+1} S^1 \right) \\
&\vee \left(\bigvee^{2(k+p+\ell+q)x+x(x-1)-3} S^1 \right) \\
&\simeq T_{\ell,q} \vee T_{q,\ell} \vee \left(\bigvee^{2n_{k,\ell}+2n_{p,q}+2(k+\ell+p+q)x+x(x-1)-1} S^1 \right)
\end{aligned}$$

Since $\pi_1(T_{\ell,q}) \cong A_{\ell,q}$, van Kampen Theorem tells us that

$$\text{PBr}(xB_{p,q}^{k,\ell}) \cong A_{\ell,q} * A_{q,\ell} * F_{2_x m_{p,q}^{k,\ell} - 1},$$

where

$${}_x m_{p,q}^{k,\ell} = n_{k,\ell} + n_{p,q} + x(k+\ell+p+q) + \frac{x(x-1)}{2}.$$

The action of the generator of Σ_2 on $C_2^{\text{comp},r}(xB_{p,q}^{k,\ell})$ is given as follows:

- $T_{\ell,q}$ and $T_{q,\ell}$ are identified,
- $C_{k,\ell}$ is mapped to itself,

- $C_{p,q}$ is mapped to itself,
- $L_{\lambda_i,\nu}$ is identified with L_{ν,λ_i} for $i = 1, 2$,
- $L_{\mu_i,\nu}$ is identified with L_{ν,μ_i} for $i = 1, 2$, and
- $L_{\nu,\nu'}$ and $L_{\nu',\nu}$ are identified in such a way that the end points of $L_{\nu,\nu'}$ are identified.

Thus we obtain

$$\begin{aligned}
C_2^{\text{comp},r}(x B_{p,q}^{k,\ell})/\Sigma_2 &\simeq T_{\ell,q} \vee C_{k,\ell}/\Sigma_2 \vee C_{p,q}/\Sigma_2 \\
&\vee \left(\bigvee^{(k+\ell)x-1} S^1 \right) \vee \left(\bigvee^{(p+q)x-1} S^1 \right) \vee \left(\bigvee^{\frac{x(x-1)}{2}} S^1 \right) \\
&\simeq T_{\ell,q} \vee \left(\bigvee^{n_{k,\ell}+1} S^1 \right) \vee \left(\bigvee^{n_{p,q}+1} S^1 \right) \vee \left(\bigvee^{(k+\ell+p+q)x-2+\frac{x(x-1)}{2}} S^1 \right) \\
&= T_{\ell,q} \vee \left(\bigvee^{x m_{p,q}^{k,\ell}} S^1 \right),
\end{aligned}$$

where

$$x m_{p,q}^{k,\ell} = n_{k,\ell} + n_{p,q} + (k + \ell + p + q)x + \frac{x(x-1)}{2}.$$

This completes the proof. \square

A Extending a Deformation Retraction on the Boundary

The aim of this appendix is to prove Theorem 2.53. A statement and a proof of this fact first appeared in [BGR1], but the paper was split into two parts and now they are not included in the paper.

Let us begin by recalling the definition of regular neighborhoods.

Definition A.1. Let K be a cell complex. For $x \in K$, define

$$\text{St}(x; K) = \bigcup_{x \in \bar{e}} e.$$

This is called the *open star* around x in K . For a subset $A \subset K$, define

$$\text{St}(A; K) = \bigcup_{x \in A} \text{St}(x; K).$$

When K is a simplicial complex and A is a subcomplex, $\text{St}(A; K)$ is called the *regular neighborhood* of A in K .

The regular neighborhood of a subcomplex is often defined in terms of vertices.

Lemma A.2. Let A be a subcomplex of a simplicial complex K . Then

$$\text{St}(A; K) = \bigcup_{v \in \text{sk}_0(A)} \text{St}(v; K).$$

Definition A.3. Let K be a simplicial complex. We say a subcomplex L is a *full* subcomplex if, for any collection of vertices v_0, \dots, v_k in L which form a simplex σ in K , the simplex σ belongs to L .

The following fact is fundamental.

Lemma A.4. *If K is a simplicial complex and A is a full subcomplex, then A is a strong deformation retract of the regular neighborhood $\text{St}(A; K)$.*

Proof. The retraction

$$r_A : \text{St}(A; K) \longrightarrow A$$

is given by

$$r_A(x) = \frac{1}{\sum_{v \in A \cap \sigma} t(v)} \sum_{v \in A \cap \sigma} t(v)v,$$

if $x = \sum_{v \in \sigma} t(v)v$ belongs to a simplex σ . A homotopy between $i \circ r_A$ and the identity map is given by a “linear homotopy”. See Lemma 9.3 in [ES52], for more details. \square

The following modification of this fact was first proved in [BGRT].

Lemma A.5. *Let K be a finite simplicial complex and K' a subcomplex. Given a full subcomplex A of K , let $A' = A \cap K'$. Suppose we are given a strong deformation retraction H of $\text{St}(A'; K')$ onto A' . Then there exists a deformation retraction \tilde{H} of $\text{St}(A; K)$ onto A extending H .*

Proof. We regard K as a subcomplex of a large simplex S . Then every point $x \in \|K\|$ can be expressed as a formal convex combination

$$x = \sum_{v \in V(K)} a_v v$$

with $\sum_{v \in V(K)} a_v = 1$ and $a_v \geq 0$, where $V(K)$ is the vertex set of K .

Let H be a strong deformation retraction of $\text{St}(A'; K')$ onto A' . Define

$$\tilde{H} : \text{St}(A; K) \times [0, 1] \longrightarrow \text{St}(A; K)$$

by

$$\begin{aligned} \tilde{H}(x, s) &= \frac{\alpha + (1-s)\beta}{(1-s) + s(\alpha + \gamma)} H' \left(\sum_i \frac{a_i}{\alpha + \beta} u'_i + \sum_j \frac{b_j}{\alpha + \beta} v'_j, s \right) \\ &\quad + \sum_k \frac{c_k}{(1-s) + s(\alpha + \gamma)} u_k + \sum_\ell \frac{(1-s)d_\ell}{(1-s) + s(\alpha + \gamma)} v_\ell, \end{aligned}$$

where $s \in [0, 1]$, $\alpha = \sum_i a_i$, $\beta = \sum_j b_j$, $\gamma = \sum_k c_k$, and $x \in \text{St}(A; K)$ has the form

$$x = \sum_i a_i u'_i + \sum_j b_j v'_j + \sum_k c_k u_k + \sum_\ell d_\ell v_\ell$$

with $u'_i \in V(A')$, $v'_j \in V(K') \setminus V(A')$, $u_k \in V(A) \setminus V(A')$, and $v_\ell \in V(K) \setminus (V(K') \cup V(A))$. Let us verify that this homotopy \tilde{H} satisfies our requirements.

$$\begin{aligned}
\tilde{H}(x, 0) &= (\alpha + \beta) \left(\sum_i \frac{a_i}{\alpha + \beta} u'_i + \sum_j \frac{b_j}{\alpha + \beta} v'_j \right) + \sum_k c_k u_k + \sum_\ell d_\ell v_\ell \\
&= \sum_i a_i u'_i + \sum_j b_j v'_j + \sum_k c_k u_k + \sum_\ell d_\ell v_\ell \\
&= x \\
\tilde{H}(x, 1) &= \frac{\alpha}{\alpha + \gamma} r_{A'} \left(\sum_i \frac{a_i}{\alpha + \beta} u'_i + \sum_j \frac{b_j}{\alpha + \beta} v'_j \right) + \sum_k \frac{c_k}{\alpha + \gamma} u_k \\
&= \frac{\alpha}{\alpha + \gamma} \sum_i \frac{\frac{a_i}{\alpha + \beta}}{\frac{\alpha}{\alpha + \beta}} u'_i + \sum_k \frac{c_k}{\alpha + \gamma} u_k \\
&= \sum_i \frac{a_i}{\alpha + \gamma} u'_i + \sum_k \frac{c_k}{\alpha + \gamma} u_k \\
&= r_A(x).
\end{aligned}$$

Furthermore, when $x \in K'$, we have $c_k = d_\ell = 0$ and $x = \sum_i a_i u'_i + \sum_j b_j v'_j$. Since $\alpha + \beta = 1$, we have

$$\begin{aligned}
\tilde{H}(x, s) &= \frac{\alpha + (1-s)\beta}{(1-s) + s\alpha} H \left(\sum_i a_i u'_i + \sum_j b_j v'_j, s \right) \\
&= \frac{1-s\beta}{1-s(1-\alpha)} H(x, s) \\
&= H(x, s).
\end{aligned}$$

Finally when $x \in A$, we have $b_j = d_\ell = 0$ and $x = \sum_i a_i u'_i + \sum_k c_k u_k$. Since $\alpha + \gamma = 1$, we have

$$\begin{aligned}
\tilde{H}(x, s) &= \alpha H \left(\sum_i \frac{a_i}{\alpha} u'_i, s \right) + \sum_k c_k u_k \\
&= \sum_i a_i u'_i + \sum_k c_k u_k \\
&= x.
\end{aligned}$$

□

In order to apply Lemma A.5 to prove Theorem 2.53, the following observation is crucial.

Lemma A.6. *Let K be a regular cell complex. For any stratified subspace L of K , the image of the regular neighborhood $\text{St}(\text{Sd}(L); \text{Sd}(\bar{L}))$ of $\text{Sd}(L)$ in $\text{Sd}(\bar{L})$ under the embedding*

$$i_K : \text{Sd}(K) \hookrightarrow K$$

contains L .

Proof. For a point $x \in L$, there exists a cell e in L with $x \in e$. Under the barycentric subdivision of \bar{L} , e is triangulated, namely there exists a sequence

$$e : e_0 < e_1 < \cdots < e_n = e$$

of cells in \bar{L} such that

$$x \in i_e(\text{Int}\Delta^n)$$

and

$$v(e) \in \overline{i_e(\text{Int}\Delta^n)},$$

where $v(e)$ is the vertex in $\text{Sd}(\bar{L})$ corresponding to e . By definition of St , we have

$$i_e(\text{Int}\Delta^n) \subset \text{St}(v(e); \text{Sd}(\bar{L})) = \text{St}(i_L(\text{Sd}(L)); \text{Sd}(\bar{L}))$$

and we have

$$L \subset \text{St}(i_L(\text{Sd}(L)); \text{Sd}(\bar{L})).$$

Conversely take an element $y \in \text{St}(i_L(\text{Sd}(L)); \text{Sd}(\bar{L}))$. There exists a simplex σ in $\text{Sd}(\bar{L})$ and a point $a \in \text{Sd}(L)$ with $a \in \sigma$ and $y \in \text{Int}(\sigma)$. a can be take to be a vertex. Thus there is a cell e in L with $a = v(e)$. By the definition of $\text{Sd}(L)$, there exists a chain

$$e : e_0 < \cdots < e_n$$

in L containing e with $\sigma = i_e(\Delta^n)$.

Since

$$\text{Int}(\sigma) \subset e_n \subset L,$$

we have $y \in L$. Thus we have proved

$$L = \text{St}(i_L(\text{Sd}(L)); \text{Sd}(\bar{L})).$$

It follows from the construction of the barycentric subdivision that $i_L(\text{Sd}(L))$ is a full sub-complex of \bar{L} . \square

Proof of Theorem 2.53. Let $L = \text{Int}(D^n) \cup K$. This is a stratified subspace of the regular cell decomposition on D^n . By Lemma A.6, L is a regular neighborhood of $i_L(\text{Sd}(L))$ in $\text{Sd}(\bar{L})$. By Lemma A.4, there is a standard “linear” homotopy which contracts L on to $i_L(\text{Sd}(L))$.

By the construction of the homotopy, it can be taken to be an extension of a given homotopy on K , under the identification $i_L(\text{Sd}(L)) = 0 * i_L(\text{Sd}(K))$. \square

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