

HIGHER TOPOLOGICAL COMPLEXITY AND ITS SYMMETRIZATION

IBAI BASABE, JESÚS GONZÁLEZ, YULI B. RUDYAK, AND DAI TAMAKI

ABSTRACT. We develop the properties of the n -th sequential topological complexity TC_n , a homotopy invariant introduced by the third author as an extension of Farber's topological model for studying the complexity of motion planning algorithms in robotics. We exhibit close connections of $\mathrm{TC}_n(X)$ to the Lusternik-Schnirelmann category of cartesian powers of X , to the cup-length of the diagonal embedding $X \hookrightarrow X^n$, and to the ratio between homotopy dimension and connectivity of X . We fully compute the numerical value of TC_n for products of spheres, closed 1-connected symplectic manifolds, and quaternionic projective spaces. Our study includes two symmetrized versions of $\mathrm{TC}_n(X)$. The first one, unlike Farber-Grant's symmetric topological complexity, turns out to be a homotopy invariant of X ; the second one is closely tied to the homotopical properties of the configuration space of cardinality- n subsets of X . Special attention is given to the case of spheres.

2010 Mathematics Subject Classification: Primary 55M30. Secondary 55R80, 55R05, 57Q40, 68T40.

Key words and phrases: Lusternik-Schnirelmann category; Schwarz genus; topological complexity; motion planning; configuration spaces.

CONTENTS

1. Introduction, main results, and organization	1
2. Preliminaries on notation	3
3. Properties of higher topological complexity	5
4. Symmetric topological complexity	10
5. Schwarz genus of ε_n and configuration spaces of spheres	15
References	17

1. INTRODUCTION, MAIN RESULTS, AND ORGANIZATION

A *motion planning algorithm* (mpa) for an autonomous system (robot) S is a rule assigning, to each pair (A, B) of initial-final positions of S , a (continuous) motion from A to B [La91, LV06]. If X stands for the space of all possible states of S , and $P(X)$ is the space of all paths $\gamma : [0, 1] \rightarrow X$, then a mpa for S is a (non-necessarily continuous) section for the end-points evaluation map $e : P(X) \rightarrow X \times X$ defined as $e(\gamma) = (\gamma(0), \gamma(1))$.

For practical applications one is interested in continuous mpa's. However it is easy to see that the end-points evaluation map e admits a continuous section if and only if the space

of states X is contractible. The alternative is to look at the Schwarz genus of the map e , which leads to Farber's concept of topological complexity. This gives a way of recognizing mpa's with the least possible order of instability (see [Fa04, Section 4]). The recognition is done directly from the homotopical properties of the space of states of the robot.

Definition (Farber). Given a path-connected topological space X , the *topological complexity* of X , $\text{TC}(X)$, is the least positive integer k such that the cartesian product $X \times X$ can be covered by k open subsets U_1, U_2, \dots, U_k on each of which e admits a continuous section $s_i: U_i \rightarrow P(X)$. Each pair (U_i, s_i) is called a *local motion planner* with domain U_i . We set $\text{TC}(X) = \infty$ if no such k exists.

A symmetrized version of topological complexity arises when attention is restricted to local planners for which the motion from A to B is the reverse of the motion from B to A , [FG07]. A number of properties of topological complexity and symmetric topological complexity can be found in [Fa03, Fa06, Fa08, FG07, FG08, FY04]. The papers [FTY03, GL09] identify these concepts in the case of real projective spaces as their immersion and embedding dimensions, respectively.

This paper is concerned with the third author's generalization of the above concepts. In such a view, the motion planning does not only depend on a couple of initial-final states of a robot, but in a sequence of prescribed intermediate stages that the robot should reach through the motion. Such a setting is standard in industrial production processes in which the manufacture of a given good goes through a series of production steps. The corresponding need to identify best possible sequential motion planning algorithms leads to a homotopy invariant $\text{TC}_n(X)$, the *n-th topological complexity* of X , introduced in [Ru10] and reviewed in Section 2 (where we use normalized notation, i.e. in such a way that contractible spaces have $\text{TC}_n = 0$.)

In Section 3 we discuss basic properties of TC_n , including methods for calculating this homotopy invariant. In Theorem 3.9 we describe optimal bounds for $\text{TC}_n(X)$: lower bounds are given in terms of the cup-length of elements in the kernel of the iterated diagonal, whereas connectivity and homotopy dimension of X lead to upper bounds. The subadditivity of TC_n is settled in Proposition 3.11. As an application, we obtain the full determination of the numerical value of $\text{TC}_n(X)$ when X is either a product of spheres (Corollary 3.12), a closed simply connected symplectic manifold (Corollary 3.15), or a quaternionic projective space (Corollary 3.16).

Many of our results generalize existing properties for Farber's TC. For instance, in Corollary 3.3 we show the following close connection between higher topological complexity and the Lusternik-Schnirelmann category of cartesian powers of spaces:

Theorem. *For a path-connected space X , $\text{cat}(X^{n-1}) \leq \text{TC}_n(X) \leq \text{cat}(X^n)$.*

Theorem 3.5 below gives $\text{TC}_n(G) = \text{cat}(G^{n-1})$ for a path-connected topological group G , which extends the $n = 2$ property proved by Farber in [Fa04, Lemma 8.2]. Lupton and Scherer have recently proved that this property extends to not-necessarily homotopy-associative Hopf spaces (see [LS13]).

Section 4 deals with symmetric versions of higher topological complexity. We begin by introducing $\mathrm{TC}^\Sigma(X)$, a variation of the symmetric topological complexity $\mathrm{TC}^S(X)$ introduced in [FG07]. We prove that the numerical values of the two invariants differ at most by a unit (Proposition 4.2). Such a fact should be prised by noticing that, although Farber and Grant observe that $\mathrm{TC}^S(X)$ is not a homotopy invariant, $\mathrm{TC}^\Sigma(X)$ depends only on the homotopy type of X . It should be noted that the homotopy invariance also fails in general for the *monoidal topological complexity* introduced by Iwase and Sakai (see [IS10, Definition 1.3 and Remark 1.4]), where the stasis property is imposed on the motion planning problem, instead of the symmetry condition we impose on TC^Σ . We construct the corresponding higher analogues TC_n^S and TC_n^Σ , and prove the homotopy invariance of the latter (Proposition 4.7).

The calculation of TC_n^S can turn out to be an extremely difficult task, mainly due to what seems to be a limited current knowledge of precise homotopy information about braid spaces (even braided manifolds, for that matter). In Section 5 we exhibit evidence leading to conjecture that

$$(1) \quad \mathrm{TC}_n^S(S^k) \leq [(n+2)(k-1) + 4] (n-1)/2k$$

holds for integers $k \geq 1$ and $n \geq 2$. In particular, we observe in Corollary 5.5 that the equality $\mathrm{TC}_n^S(S^k) = 2(n-1)$ holds provided $n = 2$ or $k = 1$.

Acknowledgments. The first author wishes to thank Michael Farber, Jesús González, Dirk Schütz and the Mathematisches Forschungsinstitut Oberwolfach for organizing a wonderful Arbeitsgemeinschaft mit aktuellem Thema in Topological Robotics. The third author is grateful for the support during a visit at the Max-Planck Institute for Mathematics in Bonn, Germany. The fourth author would like to thank the Centro di Ricerca Matematica Ennio De Giorgi, Scuola Normale Superiore di Pisa, for supporting his participation in the research program “Configuration Spaces: Geometry, Combinatorics and Topology”, during which a part of his work on this paper was done. The second, third and fourth authors were partially supported, respectively, by Conacyt Research Grant 102783, Simons Foundation Grant 209424, and Grants-in-Aid for Scientific Research, Ministry of Education, Culture, Sports, Science and Technology, Japan: 23540082. The authors wish to express their most sincere gratitude to Peter Landweber for valuable suggestions on earlier versions of this paper, and for pointing out an important extension of the authors’ original evidence for the conjectural assertion in (13).

2. PRELIMINARIES ON NOTATION

We use the normalized version of Schwarz’s concept of the genus of a map [Sv66].

Definition 2.1. The *Schwarz genus* (also known as *sectional category*) of a map $p : E \rightarrow B$ is the least number k such that there is an open covering U_0, U_1, \dots, U_k of B for which the restriction of p to each U_i ($i = 0, 1, \dots, k$) admits a homotopy section, i.e. a (continuous) map $s_i : U_i \rightarrow E$ such that ps_i is homotopic to the inclusion $U_i \hookrightarrow B$. We agree to set $\mathbf{genus}(f) = -1$ for $f : X \rightarrow Y$ with $X = \emptyset = Y$.

The following result, proved in [Sv66, Proposition 22 on page 84] (see also the comments in Section 1 on page 54 of [Sv66]), will be used in the proof of Proposition 3.11. Here we agree that a normal space is, by definition, required to be Hausdorff. This convention will also be in force throughout Section 3.

Proposition 2.2. *Let $f \times f' : X \times X' \rightarrow Y \times Y'$ be the product of two maps $f : X \rightarrow Y$ and $f' : X' \rightarrow Y'$. If $Y \times Y'$ is normal, then $\mathbf{genus}(f \times f') \leq \mathbf{genus}(f) + \mathbf{genus}(f')$.*

Definition 2.3. Let X be a path-connected space. The n -th topological complexity of X , $\mathrm{TC}_n(X)$, is the Schwarz genus of the fibration

$$(2) \quad e_n^X = e_n : X^{J_n} \rightarrow X^n, \quad e_n(\gamma) = (\gamma(1_1), \dots, \gamma(1_n))$$

where J_n is the wedge of n closed intervals $[0, 1]$ (each with $0 \in [0, 1]$ as the base point), and 1_i stands for 1 in the i^{th} interval.

Note that (2) is the standard fibrational substitute for the iterated diagonal map $d_n = d_n^X : X \rightarrow X^n$, so $\mathrm{TC}_n(X) = \mathbf{genus}(d_n^X)$. More generally, for a contractible space Y_n with n distinct distinguished points $v_1, \dots, v_n \in Y_n$, consider the evaluation map $e_{Y_n} : X^{Y_n} \rightarrow X^n$, $e_{Y_n}(f) = (f(y_1), \dots, f(y_n))$. Because of the contractibility of Y_n , the genus of e_{Y_n} is equal to $\mathrm{TC}_n(X)$, the proof is just as the one in [Ru10, Remark 3.2.5]. In particular, we can take Y_n to be a tree with n leaves, or the unit interval I_n , say with distinguished points $v_i = (i-1)/(n-1)$, $i = 1 \dots, n$. In the latter case we see that the n -th higher topological complexity gives a topological measure of the complexity of the motion planning problem where the robot is required to visit n ordered prescribed stages. For this reason, we also refer to TC_n as the n -th sequential topological complexity. Farber's TC is $\mathrm{TC}_2 + 1$.

Other fibrations (which not necessarily give fibrational substitutes of the iterated diagonal) can be used to define TC_n . Indeed, let G_n be any connected graph where n ordered distinct vertices v_1, \dots, v_n have been selected. We assert that the evaluation map $e_{G_n} : X^{G_n} \rightarrow X^n$ at the chosen vertices has $\mathbf{genus}(e_{G_n}) = \mathrm{TC}_n(X)$. To see this, choose maps $I_n \rightarrow G_n \rightarrow J_n$ preserving the selected vertices. For instance, the latter map can be taken so to collapse most of G_n to the base point in J_n , except that the first half of each directed edge (v_i, v) in G_n is mapped linearly onto the directed edge $(1_i, 0)$ in J_n (in particular vertices v_i are mapped to vertices 1_i). Since the induced maps $X^{J_n} \rightarrow X^{G_n} \rightarrow X^{I_n}$ are compatible with the three evaluation maps, we get $\mathbf{genus}(e_{I_n}) \leq \mathbf{genus}(e_{G_n}) \leq \mathbf{genus}(e_{J_n})$. But, as explained in the paragraph above, the extremes in the previous chain of inequalities agree with $\mathrm{TC}_n(X)$.

We close this section setting notation relevant to the construction (in Section 5) of our two symmetric versions of higher topological complexity.

The action of the symmetric group Σ_n on $\{1_1, \dots, 1_n\}$ extends to one on J_n . This yields corresponding Σ_n -actions on X^n and X^{J_n} in such a way that (2) is an equivariant map. The action is free on the configuration space $\mathrm{Conf}_n(X)$ of n ordered distinct points in X and, consequently, on $e_n^{-1}(\mathrm{Conf}_n(X))$. Thus, at the level of orbit spaces we get a fibration

$$\varepsilon_n^X = \varepsilon_n : Y_n(X) \rightarrow \mathrm{Braid}_n(X)$$

where $Y_n(X) = e_n^{-1}(\text{Conf}_n(X))/\Sigma_n$ and $\text{Braid}_n(X) = \text{Conf}_n(X)/\Sigma_n$ —the latter being the usual “braid” configuration space of cardinality- n subsets of X .

We think of $\mathbf{genus}(\varepsilon_n^X)$ as giving a measure for the topological complexity of the n -th ubiquitous motion planning problem on X . This concept serves in Section 4 as the building block relating our two symmetrized forms of TC_n , see Theorem 4.8 and Definition 4.13. Section 5 will be devoted to exploring $\mathbf{genus}(\varepsilon_n^{S^k})$.

Note that the commutative diagram (where horizontal arrows are canonical projections)

$$(3) \quad \begin{array}{ccc} e_n^{-1}(\text{Conf}_n(X)) & \longrightarrow & Y_n(X) \\ e_n \downarrow & & \downarrow \varepsilon_n \\ \text{Conf}_n(X) & \longrightarrow & \text{Braid}_n(X) \end{array}$$

is a pull-back square, so that (local) sections of ε_n correspond to Σ_n -equivariant (local) section of e_n . In particular, the homotopy fiber of ε_n is $(\Omega X)^{n-1}$, just as for e_n ([Ru10, Remark 3.2.3]). For instance, a copy of $(\Omega X)^{n-1}$ sits inside the fiber of e_n over an n -tuple (x_1, x_2, \dots, x_n) as the strong deformation retract consisting of *multipaths* $\{\gamma_j\}_{j=1}^n$ for which γ_1 is the constant path at x_1 . Here and below the term “multipath” refers to an element $\gamma \in X^{J_n}$, and we will use the notation $\gamma = \{\gamma_j\}_{j=1}^n$ where γ_j is the restriction of γ to the j -th wedge summand of J_n .

3. PROPERTIES OF HIGHER TOPOLOGICAL COMPLEXITY

The higher topological complexities of a space X are closely related to the category of cartesian powers of X . The first indication of such a property comes from the inequality

$$(4) \quad \text{TC}_n(X) \leq \text{cat}(X^n)$$

which is an immediate consequence of the well known fact that the Schwarz genus of a fibration does not exceed the category of the base space. On the other hand, the inequality $\text{cat}(X) \leq \text{TC}_2(X)$ is well known, and can be generalized to:

Proposition 3.1. *For any path-connected space X ,*

$$\text{cat}(X^{n-1}) \leq \text{TC}_n(X).$$

Proof. Let $\text{TC}_n(X) = k$ and choose a covering $B_0 \cup B_1 \cup \dots \cup B_k = X^n$ such that there is a continuous section s_i for e_n^X over B_i for $i = 0, \dots, k$. Let $p : X^n \rightarrow X$ be the projection onto the first factor, choose $x_1 \in X$, and put $A_i = p^{-1}(x_1) \cap B_i$. Note that $\{A_i\}_{i=0}^k$ is an open cover for $p^{-1}(x_1)$. Since $p^{-1}(x_1)$ is homeomorphic to X^{n-1} , it suffices to show that each A_i is contractible within $p^{-1}(x_1)$.

For a point $(x_1, x_2, \dots, x_n) \in A_i$ consider the n paths $\gamma_1, \dots, \gamma_n$ making up the multipath $s_i(x_1, x_2, \dots, x_n) = \{\gamma_j\}_{j=1}^n$. Then $\gamma_j(1) = x_j$ and $\gamma_j(0) = x_0$ for some $x_0 \in X$ which is independent of $j \in \{1, \dots, n\}$. Then, the constant path δ_1 at x_1 , and the paths δ_j ($j = 2, \dots, n$)—formed by using the time reversed path γ_j^{-1} the first half of the time, and γ_1 the second half—are the components of a path $\delta = (\delta_1, \dots, \delta_n)$ in $p^{-1}(x_1)$ from

$\delta(0) = (x_1, x_2, \dots, x_n)$ to $\delta(1) = (x_1, x_1, \dots, x_1)$. The continuity of s_i implies that δ depends continuously on (x_1, x_2, \dots, x_n) , so we have constructed a contraction of A_i to (x_1, x_1, \dots, x_1) in $p^{-1}(x_1)$. Thus, $\text{cat}(X^{n-1}) \leq \text{TC}_n(X)$. \square

Remark 3.2. Using the fact that $\text{cat}(X^n) \geq n$ if X is not contractible ([CLOT03, Theorem 1.47]), we see that Proposition 3.1 recovers [Ru10, Proposition 3.5].

Proposition 3.1 and (4) yield:

Corollary 3.3. *For any path-connected space X ,*

$$\text{cat}(X^{n-1}) \leq \text{TC}_n(X) \leq \text{cat}(X^n).$$

We next show that the lower bound in Corollary 3.3 is optimal for topological groups.

Proposition 3.4. *For any path-connected topological group G ,*

$$\text{TC}_n(G) \leq \text{cat}(G^{n-1}).$$

Proof. Let ϵ denote the neutral element of G . Let $k = \text{cat}(G^{n-1})$ and choose an open covering $A_0 \cup \dots \cup A_k = G^{n-1}$ where each A_i ($i \in \{0, \dots, k\}$) contracts in G^{n-1} to an $(n-1)$ -tuple p_i . Since G is path-connected, each contracting homotopy can be extended as to arrange that $p_i = (\epsilon, \dots, \epsilon) = \epsilon^{(n-1)}$ for all $i = 0, \dots, k$.

Then, for $i \in \{0, \dots, k\}$ set

$$B_i = \{(g, ga_2, \dots, ga_n) \mid (a_2, \dots, a_n) \in A_i, g \in G\},$$

which is open in G^n . We assert that e_n^G admits a (continuous) section over each B_i . Indeed, for each i the contractibility of A_i in G^{n-1} yields a path γ_a in G^{n-1} joining $\epsilon^{(n-1)}$ to each $a = (a_2, \dots, a_n) \in A_i \subset G^{n-1}$ and depending continuously on $a \in A_i$. Augment γ_a to a path γ'_a from $\epsilon^{(n)}$ to $(\epsilon, a_2, \dots, a_n) \in B_i$ with the first coordinate remaining constant. Then, for any $g \in G$, $g\gamma'_a$ is a path joining $(g, \dots, g) = g\epsilon^{(n)} \in G^n$ to $(g, ga_2, \dots, ga_n) \in B_i$ and depending continuously on n -tuples in B_i . Then, we get the required section

$$s_i : B_i \rightarrow G^{J_n}$$

where, on the j^{th} interval of J_n , $s_i(g, ga_2, \dots, ga_n)$ is the j^{th} coordinate of $g\gamma'_a$.

The proof will be complete once we check that $B_0 \cup \dots \cup B_k = G^n$. Take $(b_1, \dots, b_n) \in G^n$ and put $g = b_1$ and $a_i = g^{-1}b_i$. Then there exists j such that $(a_2, \dots, a_n) \in A_j$. So, $(b_1, \dots, b_n) \in B_j$. \square

Corollary 3.3 and Proposition 3.4 combined yield:

Theorem 3.5. *For any path-connected topological group G ,*

$$\text{TC}_n(G) = \text{cat}(G^{n-1}).$$

Alternatively, we can look at the growth of TC_n in terms of the difference of any two consecutive values of n .

Corollary 3.6. *Let G be a path-connected topological group all of whose finite cartesian powers G^k are normal¹. Then for $n \geq 3$,*

$$\mathrm{TC}_n(G) - \mathrm{TC}_{n-1}(G) \leq \mathrm{cat}(G).$$

Proof. This is a consequence of Theorem 3.5 and the product inequality for the category—valid under the current normality assumptions in view of Proposition 2.2. \square

Unlike with topological groups, higher topological complexities of an arbitrary path-connected space X do not appear to be completely determined by the category of cartesian powers of X . Nonetheless, we can directly obtain the following bound on the difference of two consecutive higher topological complexities of X .

Proposition 3.7. *Let X be a path-connected space all of whose finite cartesian powers X^k are normal. Then for $n \geq 3$,*

$$\mathrm{TC}_n(X) - \mathrm{TC}_{n-1}(X) \leq \mathrm{cat}(X^2).$$

Proof. Use the argument in the proof of Corollary 3.6, replacing Theorem 3.5 by the inequalities in Corollary 3.3. \square

In particular $\mathrm{TC}_n(X)$ is bounded from above by a linear function on n with slope $\mathrm{cat}(X^2)$. According to [Ru10, (5.1)], this slope can be improved to $\mathrm{TC}_2(X)$.

Next we consider the higher analogue of the usual cup-length lower bound for TC. Recall $d_n = d_n^X: X \rightarrow X^n$ stands for the iterated diagonal map. In the following definition we allow cohomology with local coefficients.

Definition 3.8. Given a space X and a positive integer n , $\mathrm{cl}(X, n)$ denotes the cup-length of elements in the kernel of the map induced in cohomology by d_n^X . Thus, $\mathrm{cl}(X, n)$ is the largest integer m for which there exist cohomology classes $u_i \in H^*(X^n; A_i)$ such that $d_n^* u_i = 0$ for $i = 1, \dots, m$ and

$$u_1 \smile \dots \smile u_m \neq 0 \in H^*(X^n; A_1 \otimes \dots \otimes A_m).$$

The following result, which follows directly from [Sv66, Theorems 4 and 5], bounds $\mathrm{TC}_n(X)$ from below by $\mathrm{cl}(X, n)$, and from above by a ratio between the connectivity $\mathrm{conn}(X)$ and homotopy dimension $\mathrm{hdim}(X)$ of X —the latter being the smallest dimension of CW complexes having the homotopy type of X .

Theorem 3.9. *For any path-connected space X we have the inequalities*

$$\mathrm{cl}(X, n) \leq \mathrm{TC}_n(X) \leq \frac{n \mathrm{hdim}(X)}{\mathrm{conn}(X) + 1}.$$

We will also need the following bound on $\mathrm{cl}(X \times S^k, n)$ in terms of $\mathrm{cl}(X, n)$.

¹As noted in Section 2, we assume that a normal space is, by definition, Hausdorff. Thus, in view of the classical Birkhoff-Kakutani theorem, the normality hypothesis in Corollary 3.6 holds when G satisfies the first axiom of countability—i.e. provided G is metrizable.

Theorem 3.10. *For any path-connected space X and positive integers n and k we have $\text{cl}(X \times S^k, n) \geq \text{cl}(X, n) + n - 1$. This inequality can be improved to $\text{cl}(X \times S^k, n) \geq \text{cl}(X, n) + n$ provided k is even and $H^*(X)$ is torsion-free.*

Proof. Let v be a generator of $H^k(S^k) = \mathbb{Z}$. Let $p_i : (S^k)^n \rightarrow S^k$ be the projection onto the i^{th} factor and put $v_i = p_i^*(v)$ for $i = 1, \dots, n$. Assume that $\text{cl}(X, n) = m$ and take u_1, \dots, u_m such that $d_n^*(u_j) = 0$ for $j = 1, \dots, m$ and $u_1 \smile \dots \smile u_m \neq 0$.

To prove the first assertion note that $d_n^*(v_i - v_1) = 0$ for $i > 1$, while the basis element $v_2 \smile \dots \smile v_n \in H^*((S^k)^n)$ appears in the reduced expansion (using distributivity) of $(v_2 - v_1) \smile \dots \smile (v_n - v_1)$. Hence,

$$u_1 \smile \dots \smile u_m \smile (v_2 - v_1) \smile \dots \smile (v_n - v_1) \neq 0.$$

Thus $\text{cl}(X \times S^k, n) \geq \text{cl}(X, n) + n - 1$.

Assume now that k is even and that $H^*(X)$ is torsion-free. The element $v_1 + v_2 + \dots + v_{n-1} - (n-1)v_n$ lies in the kernel of d_n^* and has cup n^{th} power equal to a non-zero multiple of $v_1 \smile v_2 \smile \dots \smile v_n$. Hence,

$$u_1 \smile \dots \smile u_m \smile (v_1 + v_2 + \dots + v_{n-1} - (n-1)v_n)^n \neq 0.$$

Thus $\text{cl}(X \times S^k, n) \geq \text{cl}(X, n) + n$. □

In [Fa03] Farber obtained the subadditivity of TC_2 under suitable topological hypothesis. The corresponding property for higher topological complexity is given next.

Proposition 3.11. *Let X and Y be path-connected spaces. If $(X \times Y)^n$ is normal, then $\text{TC}_n(X \times Y) \leq \text{TC}_n(X) + \text{TC}_n(Y)$.*

Proof. The natural homeomorphisms

$$(X \times Y)^n \rightarrow X^n \times Y^n,$$

$$((x_1, y_1), \dots, (x_n, y_n)) \mapsto (x_1, \dots, x_n, y_1, \dots, y_n), \quad x_i \in X, \quad y_j \in Y$$

and

$$(X \times Y)^{J_n} \rightarrow X^{J_n} \times Y^{J_n},$$

$$(\varphi : J_n \rightarrow X \times Y) \mapsto ((p_X \circ \varphi : J_n \rightarrow X), (p_Y \circ \varphi : J_n \rightarrow Y))$$

fit into the commutative diagram

$$\begin{array}{ccc} (X \times Y)^{J_n} & \longrightarrow & X^{J_n} \times Y^{J_n} \\ e_n^{X \times Y} \downarrow & & \downarrow e_n^X \times e_n^Y \\ (X \times Y)^n & \longrightarrow & X^n \times Y^n. \end{array}$$

So, the desired conclusion follows directly from Proposition 2.2. □

As revealed in the case of spheres (next), Proposition 3.11 is optimal in general.

Corollary 3.12. $\text{TC}_n(S^{k_1} \times S^{k_2} \times \dots \times S^{k_m}) = m(n-1) + l$ where l is the number of even dimensional spheres.

Proof. Note that $\text{TC}_n(S^k) = \text{cl}(S^k, n)$ for all k , [Ru10, Section 4]. Then the inequality $\text{cl}(S^{k_1} \times \cdots \times S^{k_m}, n) \geq m(n-1) + l$ follows from Theorem 3.10 by induction, so $\text{TC}_n(S^{k_1} \times \cdots \times S^{k_m}) \geq m(n-1) + l$ by Theorem 3.9. The opposite estimate follows from Proposition 3.11. \square

The calculation of the higher topological complexity of the k -dimensional torus $T^k = (S^1)^k$, partially solved for $k = 2$ in [Ru10, Proposition 5.1], is a consequence of either Corollary 3.12 or Theorem 3.5.

Corollary 3.13. $\text{TC}_n(T^k) = k(n-1)$.

Theorem 3.14. *Let X be a CW complex of finite type, and R a principal ideal domain. Take $u \in H^d(X; R)$ with $d > 0$, d even, and assume that the n -fold iterated self R -tensor product $u^m \otimes \cdots \otimes u^m \in (H^{md}(X; R))^{\otimes n}$ is an element of infinite additive order. Then $\text{TC}_n(X) \geq mn$.*

Proof. For $i = 1, \dots, n$, let $p_i : X^n \rightarrow X$ be the projection onto the i^{th} factor and put $u_i = p_i^*(u) \in H^d(X^n; R)$. In view of Theorem 3.9, the required inequality follows from

$$(5) \quad v := (u_2 - u_1)^{2m} (u_3 - u_1)^m \cdots (u_n - u_1)^m \neq 0.$$

In order to check (5), note that v comes from the tensor product, which injects into the cohomology of the cartesian product by the Künneth Theorem (this is where the finiteness hypotheses are used). So, calculations can be performed in the former R -module. Now, assuming that $\dim(X) \leq dm + 1$, we have

$$\begin{aligned} v &= (u_2 - u_1)^{2m} (u_3 - u_1)^m \cdots (u_n - u_1)^m \\ &= (-1)^m \binom{2m}{m} u_1^m u_2^m (u_3 - u_1)^m \cdots (u_n - u_1)^m \\ &= (-1)^m \binom{2m}{m} u_1^m u_2^m u_3^m (u_4 - u_1)^m \cdots (u_n - u_1)^m \\ &= \cdots \\ &= (-1)^m \binom{2m}{m} u_1^m u_2^m \cdots u_n^m, \end{aligned}$$

which is non-zero by hypothesis. On the other hand, for $\dim(X)$ arbitrary, consider the skeletal inclusion $j : X^{(dm+1)} \rightarrow X$ and note that $v \neq 0$ since $j^*(v) \neq 0$. \square

Corollary 3.15. *For every closed simply connected symplectic manifold M^{2m} we have $\text{TC}_n(M) = nm$.*

Proof. This follows from Theorem 3.14 (taking u to be the cohomology class given by the symplectic 2-form on M , and noting that the hypothesis on $u^m \otimes \cdots \otimes u^m$ holds since the coefficients are taken over the reals), inequality (4), the product inequality for category, and the inequality $\text{cat}(M^{2m}) \leq m$ which follows from [Sv66, Theorem 5, page 75]. (This argument also yields $\text{cat}(M^{2m}) = m$, a well known fact.) \square

Of course, Corollary 3.15 applies to complex projective spaces. In the quaternionic case essentially the same proof gives:

Corollary 3.16. *The quaternionic projective space of real dimension $4m$, $\mathbb{H}P^m$, has $\mathrm{TC}_n(\mathbb{H}P^m) = nm$.*

Note that Corollaries 3.15 and 3.16 imply that the upper bound in Corollary 3.3 as well as both bounds in Theorem 3.9 are optimal in general.

4. SYMMETRIC TOPOLOGICAL COMPLEXITY

In this section we introduce two symmetric versions of TC_n . One of them, TC_n^Σ , has the advantage of being a homotopy invariant. The other, TC_n^S , gives (up to our normalization convention) the natural generalization of the symmetric topological complexity studied by Farber and Grant in [FG07]. We begin with the $n = 2$ case of the homotopically well-behaved version.

Consider the involutions $\tau : X \times X \rightarrow X \times X$ and $\bar{\tau} : P(X) \rightarrow P(X)$ defined by $\tau(x, y) = (y, x)$ and $\bar{\tau}(\gamma)(t) = \gamma(1 - t)$, for $(x, y) \in X \times X$ and $\gamma \in P(X)$. We work with symmetric subsets $A \subseteq X \times X$ (i.e. those for which $\tau A = A$), and equivariant maps $s : A \rightarrow P(X)$ (i.e. those satisfying $\bar{\tau}(s(a)) = s(\tau(a))$ for all $a \in A$).

Definition 4.1. $\mathrm{TC}^\Sigma(X)$ is the least integer k such that $X \times X = A_0 \cup A_1 \cup \cdots \cup A_k$ where each A_i is open, symmetric, and admits a continuous equivariant section $s_i : A_i \rightarrow P(X)$ of the map e_2 in (2).

Before proving (in Proposition 4.7 below) that $\mathrm{TC}^\Sigma(X)$ is a homotopy invariant of X , we show that its numerical value differs by at most one from the numerical value of Fraber-Grant's symmetric topological complexity. In accordance with the normalization hypothesis in this paper, we must compare $\mathrm{TC}^\Sigma(X)$ with

$$(6) \quad \mathrm{TC}_2^S(X) = \mathbf{genus}(\varepsilon_2) + 1$$

where ε_2 is the map on the right hand side of (3). Note that, under the perspective of [FG07], the “+1” summand in (6) is meant to take into account the obvious equivariant section of e_2 on the diagonal.

Proposition 4.2. *For each ENR X we have*

$$\mathrm{TC}_2^S(X) - 1 \leq \mathrm{TC}^\Sigma(X) \leq \mathrm{TC}_2^S(X).$$

Remark 4.3. We will prove a more general version of Proposition 4.2 (Theorem 4.8 below). The proof of the general version is considerably more elaborate as it requires a rather involved use of *equivariant* euclidean neighborhood retracts. For the sake of clarity, we offer first the much simpler argument proving Proposition 4.2.

Proof of Proposition 4.2. To prove the first inequality, take an open covering $X \times X = A_0 \cup \cdots \cup A_k$ where each A_i is symmetric and has a continuous equivariant section of e_2 .

The $\mathbb{Z}/2$ -action τ on $X \times X$ yields the orbit map $\rho_2 : X \times X \rightarrow (X \times X)/\tau$. Then, for each $i = 0, \dots, k$, $\rho_2(A_i - d_2(X))$ is open and has a section of ε_2 , and thus $\mathbf{genus}(\varepsilon_2) \leq \mathrm{TC}^\Sigma(X)$. For the second inequality, take B_0, \dots, B_l , with $B_0 \cup \dots \cup B_l = \rho_2(X \times X - d_2(X))$ where each B_i is open and has a section of ε_2 . Then each $\rho_2^{-1}(B_i)$ is symmetric, open in $X \times X$, and admits an equivariant section of e_2 , cf. [FG07, Lemma 8]. Further, since X is an ENR, there is a symmetric open neighborhood of $d_2(X)$ supporting an equivariant section of e_2 (see the proof of [FG07, Corollary 9]). Consequently $\mathrm{TC}^\Sigma(X) \leq 1 + \mathbf{genus}(\varepsilon_2)$. \square

The two examples below show that both bounds in Proposition 4.2 are optimal in general.

Example 4.4. For X contractible we have $\mathrm{TC}_2(X) = \mathrm{TC}^\Sigma(X) = 0$ while $\mathrm{TC}_2^S(X) = 1$. Indeed, take a point $x_0 \in X$ and a contraction $H : X \times I \rightarrow X$, with $H(x, 0) = x$ and $H(x, 1) = x_0$ for all $x \in X$. Given $(a, b) \in X \times X$, take the path $\sigma = s(a, b) : I \rightarrow X$ such that $\sigma(t) = H(a, 2t)$ for $0 \leq t \leq 1/2$ and $\sigma(t) = H(b, 2 - 2t)$ for $1/2 \leq t \leq 1$. Then s is an equivariant section for e_2^X and, in view of the general inequality

$$\mathrm{TC}_2(X) \leq \mathrm{TC}^\Sigma(X),$$

this gives $\mathrm{TC}_2(X) = \mathrm{TC}^\Sigma(X) = 0$. The same argument, but now using (6), gives $\mathrm{TC}_2^S(X) = 1$ (see [FG08, Example 7]).

Example 4.5. The numbers $\mathrm{TC}_2^S(S^k)$ and $\mathrm{TC}_2(S^k)$ have been computed in [FG07, Corollary 18] and [Fa03], respectively. Here we use the inequalities $\mathrm{TC}_2 \leq \mathrm{TC}^\Sigma \leq \mathrm{TC}^S$ together with the fact that $\mathrm{TC}_2^S(S^k) = 2 = \mathrm{TC}_2(S^{2k})$ to deduce $\mathrm{TC}^\Sigma(S^{2k}) = \mathrm{TC}_2^S(S^{2k}) = 2$ for all k . On the other hand, since $\mathrm{TC}_2(S^{2k+1}) = 1$, the above argument only gives $1 \leq \mathrm{TC}^\Sigma(S^{2k+1}) \leq \mathrm{TC}_2^S(S^{2k+1}) = 2$. Incidentally, note that the construction in [Fa08, Example 4.8] gives an open covering $S^{2k+1} \times S^{2k+1} = A_0 \cup A_1$ by symmetric sets A_i , and continuous sections of e_2 over each A_i , $i = 0, 1$. However, one of these sections is not equivariant, which prevents us from deducing $\mathrm{TC}^\Sigma(S^k) = 1$.

We next define higher analogues of TC^Σ . Recall that for a given n , the symmetric group Σ_n acts on X^n and on X^{J_n} by permuting coordinates and paths, respectively. Further, the fibration e_n in (2) is Σ_n -equivariant. We now work with symmetric subsets $A \subseteq X^n$ (i.e. those for which $\sigma A = A$ for all $\sigma \in \Sigma_n$), and equivariant maps $s : A \rightarrow X^{J_n}$ (i.e. those satisfying $\sigma(s(a)) = s(\sigma(a))$ for all $a \in A$ and $\sigma \in \Sigma_n$). Definition 4.1 can now be extended to:

Definition 4.6. $\mathrm{TC}_n^\Sigma(X)$ is the least integer k such that $X^n = A_0 \cup A_1 \cup \dots \cup A_k$ where each A_i is open, symmetric and admits a continuous equivariant section $s_i : A_i \rightarrow X^{J_n}$ for the map e_n in (2).

Proposition 4.7. $\mathrm{TC}_n^\Sigma(X)$ is a homotopy invariant of X .

Proof. It suffices to prove that, given $f : Y \rightarrow X$ and $g : X \rightarrow Y$ with $gf \simeq 1_Y$, we have $\mathrm{TC}_n^\Sigma(X) \geq \mathrm{TC}_n^\Sigma(Y)$ for all n . Let $H : 1_Y \simeq gf$ be a homotopy $H : Y \times [0, 1] \rightarrow Y$ such that $H(y, 0) = y$ and $H(y, 1) = gf(y)$.

Let A be an open symmetric subset of X^n , and let $s : A \rightarrow X^{J_n}$ be an equivariant section of e_n^X over A . Given $a = (a_1, \dots, a_n) \in A$, let $s_i(a)$ denote the restriction of $s(a) \in X^{J_n}$ to the i^{th} wedge summand of J_n (this is a path in X joining x_0 and a_i for some $x_0 \in X$ that depends continuously on a). Note that the equivariance of s gives

$$(7) \quad s_i(a_{\sigma(1)}, \dots, a_{\sigma(n)}) = s_{\sigma(i)}(a_1, \dots, a_n) \quad \text{for } \sigma \in \Sigma_n.$$

Take $B := (f^n)^{-1}(A)$, where f^n stands for the n -th cartesian power of f , and consider the map $s' : B \rightarrow Y^{J_n}$ which, at a given $b \in B$ with $f^n(b) = a$, has $s'_i(b) := (g \circ s_i(a)) \cdot \gamma_i$ as its restriction to the i^{th} wedge summand of J_n , where γ_i is the path in Y given by

$$\gamma_i(t) = H(b_i, 1 - t).$$

Then, s' is an equivariant continuous section of e_n^Y over B in view of (7)).

In this setting, if $X = A_0 \cup \dots \cup A_k$ where each A_j ($j = 0, \dots, k$) is open, symmetric, and admits a continuous equivariant section of e_n^X , then $Y = B_0 \cup \dots \cup B_k$ where each B_j —defined as above using A_j —is open, symmetric, and admits a continuous equivariant section of e_n^Y . Hence, $\text{TC}_n^\Sigma(X) \geq \text{TC}_n^\Sigma(Y)$. \square

The following assertion is our higher analogue of Proposition 4.2.

Theorem 4.8. *If X is an ENR, and ε_n is the map on the right hand side of (3), then*

$$(8) \quad \mathbf{genus}(\varepsilon_n) \leq \text{TC}_n^\Sigma(X) \leq \mathbf{genus}(\varepsilon_n) + \dots + \mathbf{genus}(\varepsilon_2) + n - 1.$$

The first inequality in (8) follows just as in the proof of Proposition 4.2: If e_n admits an equivariant section over $A \subset X^n$, then ε_n admits a section over $\rho_n(A \cap C_n(X))$ where $\rho_n : X^n \rightarrow X^n/\Sigma_n$ stands for the canonical projection. Our efforts will therefore focus on the second inequality in (8), whose proof requires some preparation.

Definition 4.9. A topological space X with an action of a compact Lie group G is called a *euclidean neighborhood G -retract* (G -ENR) if X can be G -equivariantly embedded, as a G -equivariant retract of a G -symmetric neighborhood of X , into an orthogonal representation of G .

In what follows we will make implicit use of the following fact: if a G -ENR X is G -equivariantly embedded in a given orthogonal representation \mathbb{R}^N of G , then there exists a G -symmetric neighborhood U of X in \mathbb{R}^N and a G -equivariant retraction $U \rightarrow X$. As noticed at the end of the introduction in [J76], such a property follows by applying the equivariant version of the Tietze Theorem (Tietze-Gleason Theorem, [Br72, Gl50]) to the non-equivariant argument in [Do95, Proposition and Definition IV.8.5].

We shall use the following weaker version of [J76, Theorem 2.1]².

Theorem 4.10 (Jaworowski). *Let L be a finite group acting on an ENR Z . Then Z is an L -ENR if for every subgroup G of L , the fixed point set Z^G is an ENR.*

²Although Jaworowski's theorem was originally set in terms of a combination of the concepts of ANR's and ENR's, for our formulation the reader should keep in mind the fact that any ENR is an ANR (which is elementary in view of the Tietze Theorem).

Next, consider the Σ_n -equivariant filtration

$$(9) \quad d_n(X) = D^1(X) \subset \dots \subset D^{n-1}(X) \subset D^n(X) = X^n$$

where, for $i \in \{1, \dots, n\}$, $D^i(X)$ is the closed set consisting of the n -tuples (x_1, x_2, \dots, x_n) such that the set $\{x_1, x_2, \dots, x_n\}$ has cardinality at most i . For instance, $D^{n-1}(X)$ is the so called fat diagonal in X^n , otherwise denoted by $\Delta_n(X)$. Compare the filtration in (9) with the one considered at the end of Section 1 in [Ka08].

Set $D^0(X) = \emptyset$, and for $1 \leq i \leq n$ let C^i stand for the difference $D^i(X) - D^{i-1}(X)$, the subspace of n -tuples (x_1, x_2, \dots, x_n) such that the set $\{x_1, x_2, \dots, x_n\}$ has cardinality i . Note that $C^n = \text{Conf}_n(X)$ and that for $i < n$, each partition $\mathcal{P} = \{P_1, \dots, P_i\}$ of $\{1, 2, \dots, n\}$ into i nonempty sets determines a closed subspace $C_{\mathcal{P}}^i \subset C^i$ formed by those tuples (x_1, \dots, x_n) in C^i satisfying $x_r = x_s$ whenever both r and s lie in the same part P_j for some j .

Note that C^i is the disjoint union of the $C_{\mathcal{P}}^i$'s, each of which maps homeomorphically onto $\text{Conf}_i(X)$ under a suitable coordinate projection. [For instance, for $n = 3$ the three closed subspaces partitioning C^2 are determined by the three requirements $x_1 = x_2$, $x_1 = x_3$, and $x_2 = x_3$; in the latter case, the required projection can be chosen to be $(x_1, x_2, x_3) \mapsto (x_1, x_2)$.] Therefore, we have a continuous (surjective) map $\pi_i: C^i \rightarrow C_i(X)$.

Let P^i denote the subspace of $e_n^{-1}(C^i)$ consisting of those multipaths $\alpha = \{\alpha_i\}_{i=1}^n$ satisfying $\alpha_k = \alpha_\ell$ whenever $\alpha_k(1_k) = \alpha_\ell(1_\ell)$. Proceeding as above, we get a continuous surjection $\Pi_i: P^i \rightarrow e_i^{-1}(\text{Conf}_i(X))$ in such a way that in the following commutative diagram

$$(10) \quad \begin{array}{ccccccc} X^{J_n} & \longleftarrow & P^i & \xrightarrow{\Pi_i} & e_i^{-1}(\text{Conf}_i(X)) & \longrightarrow & Y_i(X) \\ e_n \downarrow & & e_n \downarrow & & e_i \downarrow & & \downarrow \varepsilon_i \\ X^n & \longleftarrow & C^i & \xrightarrow{\pi_i} & \text{Conf}_i(X) & \longrightarrow & \text{Braid}_i(X) \end{array}$$

the second and third squares are pullbacks, and the two left-most horizontal maps are inclusions but do not determine a pullback square.

Our last ingredient in preparation for the proof of (8) is given by taking an arbitrary open subset W of $\text{Braid}_i(X)$. We then let $A = \pi_i^{-1}(W)$ where W' stands for the inverse image of W under the projection $\text{Conf}_i(X) \rightarrow \text{Braid}_i(X)$. Clearly W' is Σ_i -symmetric and A is Σ_n -symmetric. This setup will be in force in the following two auxiliary results, which are the basis of our proof of the second inequality in (8).

Lemma 4.11. *The space A is a Σ_n -ENR.*

Proof. Note first that every $C_{\mathcal{P}}^i$ is an ENR, because it is homeomorphic to $C_i(X)$ which, in turn, is an open subset of the ENR X^i . Now, every $g \in \Sigma_n$ yields a homeomorphism from any given $C_{\mathcal{P}}^i$ onto some $C_{\mathcal{P}'}^i$. In particular for $\mathcal{P} = \mathcal{P}'$, if there is some $x \in C_{\mathcal{P}}^i$ fixed by g , then $g \cdot y = y$ for all $y \in C_{\mathcal{P}}^i$, i.e. $(C_{\mathcal{P}}^i)^g = C_{\mathcal{P}}^i$. Hence, for any subgroup G of Σ_n , the set $(C_{\mathcal{P}}^i)^G$ is either empty or the whole of $C_{\mathcal{P}}^i$, and therefore an ENR. Consequently, $(C^i)^G$

is an ENR since C^i is the disjoint union of the various C_p^i 's, and A^G is an ENR since A is open in C^i . Thus, by Theorem 4.10, A is a Σ_n -ENR, as asserted. \square

Lemma 4.12. *Assume $s : A \rightarrow P^i$ is a Σ_n -equivariant section of the second vertical map in (10). Then there is a Σ_n -symmetric neighborhood U of A in X^n that admits a Σ_n -equivariant section $\sigma : U \rightarrow X^{J_n}$ of the first vertical map in (10).*

Proof. We begin by noticing that, as a consequence of Theorem 4.10, X^n is a Σ_n -ENR. Indeed, for any subgroup G of Σ_n , the fixed point set of G on X^n is an intersection of hyperplanes $x_i = x_j$ in X^n . Hence, $(X^n)^G$ is an ENR since it is homeomorphic to X^m for $m \leq n$. Thus, we can take Σ_n -equivariant embeddings $A \rightarrow X^n \rightarrow \mathbb{R}^N$, and a Σ_n -equivariant retraction $r' : O \rightarrow A$ of a Σ_n -symmetric neighborhood O of A in \mathbb{R}^N , where \mathbb{R}^N is an orthogonal representation of Σ_n .

Set $V = O \cap X^n$. Then V is a Σ_n -symmetric neighborhood of A in X^n , and $r = r'|_V : V \rightarrow A$ is a Σ_n -equivariant retraction. Note that V is an open Σ_n -symmetric subset of the Σ_n -ENR X^n , and so V is a Σ_n -ENR too. We can then choose an open Σ_n -symmetric neighborhood Y of V in \mathbb{R}^N , and a Σ_n -equivariant retraction $\rho : Y \rightarrow V$. Let $U \subset V$ consist of all points $v \in V$ such that the segment from v to $i \circ r(v)$ lies in Y where i stands for the inclusion $A \hookrightarrow V$ (cf. [Do95, Corollary IV.8.7]). Clearly U is a neighborhood of A in V , and hence in X^n . Furthermore, the composition $i \circ r|_U$ and the inclusion $U \hookrightarrow V$ are homotopic via the homotopy

$$\Phi : U \times I \rightarrow V, \quad \Phi(u, t) = \rho(t \cdot u + (1 - t) \cdot i \circ r(u)).$$

Note that U is Σ_n -symmetric and Φ is Σ_n -equivariant, since the Σ_n -action on \mathbb{R}^N is orthogonal and so it maps lines to lines.

We use the homotopy Φ in order to construct a Σ_n -equivariant section $\sigma : U \rightarrow X^{J_n}$ of the first vertical map in (10). For $x \in U$, consider the path $\beta : I \rightarrow V$, $\beta(t) = \Phi(x, t)$, starting at $y = \beta(0) = r(x) \in A$ and ending at x . Since V is a subset of X^n , we can set $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, and $\beta = (\beta_1, \dots, \beta_n)$, so each β_i is a path in X from y_i to x_i . Further, $s(y)$ gives a multipath $\{\alpha_i\}_{i=1}^n$ with $\alpha_i(1) = y_i$ and $\alpha_i(0) = \alpha_j(0)$ for all $1 \leq i, j \leq n$. Then the multipath $\{\alpha_i \cdot \beta_i\}_{i=1}^n$ determines an element $\sigma(x) \in X^{J_n}$ with $e_n(\sigma(x)) = x$. This defines the required Σ_n -equivariant section over U . \square

Note that the two pull-back squares in (10) imply that the hypothesis in Lemma 4.12 holds whenever W (the arbitrary open subset of $\text{Braid}_i(X)$ taken in the paragraph previous to Lemma 4.11) is chosen to admit a section of the fourth vertical map in (10). Thus we obtain the following:

Proof of Theorem 4.8 (conclusion). In view of Lemmas 4.11 and 4.12 we can choose $1 + \mathbf{genus}(\varepsilon_i)$ Σ_n -equivariant local sections for e_n whose domains cover C^i , and thus a total of

$$(11) \quad \sum_{i=2}^n (1 + \mathbf{genus}(\varepsilon_i)) + 1 = \mathbf{genus}(\varepsilon_n) + \dots + \mathbf{genus}(\varepsilon_2) + n$$

Σ_n -equivariant local sections for e_n whose domains cover X^n . Here the “+1” on the left-hand side of (11) accounts for the obvious equivariant section on the diagonal $D^1(X)$. The theorem follows. \square

A comparison of Proposition 4.2 and Theorem 4.8 suggests the following generalization of (6):

Definition 4.13. For $n \geq 2$ set

$$\mathrm{TC}_n^S(X) = \mathbf{genus}(\varepsilon_n) + \cdots + \mathbf{genus}(\varepsilon_2) + n - 1.$$

This is a minor variation of the one proposed in the short final section in [Ru10], and will be explored next for X a sphere.

5. SCHWARZ GENUS OF ε_n AND CONFIGURATION SPACES OF SPHERES

The following result, which is a specialization of [Sv66, Theorem 5, page 75] (recalling that $(\Omega X)^{n-1}$ is the homotopy fiber of the map $\varepsilon_n = \varepsilon_n^X: Y_n(X) \rightarrow \mathrm{Braid}_n(X)$ in (3)), gives a general upper bound for $\mathbf{genus}(\varepsilon_n)$ analogous to that in Theorem 3.9.

Proposition 5.1. *If X is an $(s - 1)$ -connected space and $\mathrm{Braid}_n(X)$ has the homotopy type of a d -dimensional CW space, then $\mathbf{genus}(\varepsilon_n) \leq d/s$.*

For instance, $\mathbf{genus}(\varepsilon_n^X) = 0$ for any contractible space X . This generalizes the phenomenon noted in Example 4.4. Part of the goal of this section is to show that the bound in Proposition 5.1 becomes an equality in some concrete situations—other than those noted for a contractible space X . Yet, the following considerations are written in conjectural terms; non-conjectural statements start from equation (14) on.

The conjectural inequality in (1) is based on Proposition 5.1. To illustrate the idea, start by recalling from Example 4.5 the equality $\mathrm{TC}_2^S(S^k) = 2$ valid for any k . Farber and Grant prove that $\mathrm{TC}_2^S(S^k)$ is no greater than 2 by producing a symmetric motion planner with two local rules. Their construction makes use of a well-known explicit Σ_2 -equivariant deformation retraction $\mathrm{Conf}_2(S^k) \rightarrow S^k$ that implies a corresponding homotopy equivalence

$$(12) \quad \mathrm{Braid}_2(S^k) \simeq \mathbb{RP}^k.$$

Here we note that Proposition 5.1 gives an alternative direct way to deduce the inequality $\mathrm{TC}_2^S(S^k) \leq 2$: all that is needed is the fact that $\mathrm{hdim}(\mathrm{Braid}_2(S^k)) = k$. In order to extend this simple argument for higher TC_n^S we would need to have a good hold on the homotopy dimension of $\mathrm{Braid}_n(S^k)$. Remark 5.3 below provides evidence toward the following:

Conjecture 5.2. *For $n \geq 2$ and $k \geq 1$, $\mathrm{hdim}(\mathrm{Braid}_n(S^k)) = (k - 1)(n - 1) + 1$.*

Remark 5.3. Note that the validness of Conjecture 5.2 for $n = 2$ follows from (12). Likewise, the case $k = 1$ of Conjecture 5.2 is well known: $\mathrm{Braid}_n(S^1)$ has the homotopy type of S^1 (cf. [Ka08, Proposition 2.5]). On the other hand, from the calculations of homology groups in [FZ00], it can be proved that Conjecture 5.2 is true if $\mathrm{Braid}_n(S^k)$ is

replaced by $\text{Conf}_n(S^k)$ when $k \geq 3$. At any rate, since the homotopy dimension of a space is not less than the homotopy dimension of any of its covering spaces, we have

$$\text{hdim}(\text{Braid}_n(S^k)) \geq \text{hdim}(\text{Conf}_n(S^k)) = (k-1)(n-1) + 1.$$

Therefore the crux of the matter in settling Conjecture 5.2 (and, as a consequence, the equality $\text{hdim}(\text{Conf}_n(S^k)) = (n-1)(k-1) + 1$) rests in producing a CW complex of dimension $(k-1)(n-1) + 1$ which has the Σ_n -equivariant homotopy type of $\text{Conf}_n(S^k)$. The second and fourth authors of this paper have an ongoing project aiming at such a goal; the basic ideas have been presented in the second half of [BGRT11]. However, it turns out that those ideas require an important tuning and have actually become a completely independent paper (which will appear elsewhere). The present paper then focuses on the first half of [BGRT11], i.e. the development of the properties of the sequential topological complexity.

We have mentioned that the validness of (1) would follow from Conjecture 5.2. In fact, in view of Proposition 5.1, we see that Conjecture 5.2 would actually imply the validness of the more detailed but still conjectural estimate:

$$(13) \quad \text{genus}(\varepsilon_i) \leq i - 1 - \frac{i-2}{k}, \text{ for } X = S^k \text{ and } i \geq 2.$$

The remainder of the section is devoted to presenting evidence for the validness and general optimality of (13).

We have observed that (13) holds true for $i = 2$. As for its optimality, it is worth observing that Farber and Grant prove in [FG07, Section 3] the inequality

$$(14) \quad \text{TC}_2^S(S^k) \geq 2$$

by means of an involved extension of Haefliger's calculation of the mod 2 cohomology ring $H^*(\text{Braid}_2(M); \mathbb{Z}/2)$ for M a closed smooth manifold. But a simpler argument is available. Start by observing that if (14) were to fail, then there would exist a continuous section σ for $\varepsilon_2^{S^k}$. In such a situation we could consider the composite

$$S^k \xrightarrow{\alpha} \text{Conf}_2(S^k) \xrightarrow{\tilde{\sigma}} e_2^{-1}(\text{Conf}_2(S^k)) \hookrightarrow PS^k$$

where $\alpha(x) = (x, -x)$ and $\tilde{\sigma}$ would be the ($\mathbb{Z}/2$ -equivariant) pull-back of σ under (3). The adjoint of this composite would then yield a homotopy $H: S^k \times [0, 1] \rightarrow S^k$ between the identity $H(-, 0)$ and the antipodal map $H(-, 1)$, and which would in addition satisfy the relation

$$(15) \quad H(x, t) = H(-x, 1 - t).$$

But this is impossible since the identity on S^k (which has degree 1) cannot be homotopic to the presumed map $H(-, 1/2)$ which, in view of (15), would factor as

$$S^k \xrightarrow{\text{proj}} \mathbb{RP}^k \rightarrow S^k,$$

and would therefore have even degree.

The above argument, as well as the closely related proof of Proposition 5.4 below, were pointed out to the authors by Peter Landweber.

Proposition 5.4. *Let k be a positive odd integer. For $X = S^k$ and $i \geq 2$, $\mathbf{genus}(\varepsilon_i) \geq 1$. Further, $\mathbf{genus}(\varepsilon_i) = 1$ provided $k = 1$.*

Proof of Proposition 5.4. The second assertion follows from the first one in view of Proposition 5.1 and the first part of Remark 5.3. To prove the first assertion, we derive a contradiction from the assumption that ε_i admits a global continuous section σ . Consider the map $c: S^k \rightarrow (S^k)^{J_i}$ given as the composite

$$S^k \xrightarrow{\alpha} \mathrm{Conf}_i(S^k) \xrightarrow{\tilde{\sigma}} e_i^{-1}(\mathrm{Conf}_i(S^k)) \hookrightarrow (S^k)^{J_i}.$$

Here $\alpha(x) = (x, zx, z^2x, \dots, z^{i-1}x)$, where $z \in S^1$ is a primitive i^{th} root of unity acting on S^k in the standard way (recall k is odd), and $\tilde{\sigma}$ is the Σ_n -equivariant section of the map $e_i: e_i^{-1}(\mathrm{Conf}_i(S^k)) \rightarrow \mathrm{Conf}_i(S^k)$ obtained as the pull-back in (3) of the assumed σ . Thus, for each $x \in S^k$, $c(x)$ is a multipath $\{c_j(x)\}_{j=0}^{i-1} \in (S^k)^{J_i}$, where each $c_j(x)$ is a path in S^k starting at a point $s(x) \in S^k$ and ending at z^jx , for a continuous map $s: S^k \rightarrow S^k$. Note that the equivariance of $\tilde{\sigma}$ gives

$$(16) \quad c_j(zx) = c_{j+1}(x)$$

for all $x \in S^k$ —here the value of j is to be interpreted modulo i . Then the map $H: S^k \times [0, 1] \rightarrow S^k$ defined by $H(x, t) = c_0(x)(t)$ is a homotopy starting at s and ending at the identity. In particular, $s: S^k \rightarrow S^k$ has degree 1. The contradiction comes by observing that the degree of s would be divisible by i . Indeed, (16) gives

$$s(zx) = c_0(zx)(0) = c_1(x)(0) = s(x),$$

so that s factors as $S^k \xrightarrow{\mathrm{proj}} L^k(i) \rightarrow S^k$ where $L^k(i)$ is the standard lens space $S^k/(\mathbb{Z}/i)$. \square

Corollary 5.5. *The known equality $\mathrm{TC}_2^S(S^k) = 2$ (valid for any integer $k > 0$) extends to $\mathrm{TC}_n^S(S^k) = 2(n-1)$ for $k = 1$.*

Remark 5.6. The first conclusion in Proposition 5.4 is partially extended by Karasev-Landweber's result in [KL12] asserting that $\mathbf{genus}(\varepsilon_3^{S^k}) \geq 1$ for k not of the form $4 \cdot 3^e$ with $e \geq 0$. Note that the conjectural (13) would in fact sharpen the above estimate to an equality.

REFERENCES

- [BGRT11] Basabe, I.; González, J.; Rudyak, Yu.; Tamaki, D.: Higher topological complexity and homotopy dimension of configuration spaces on spheres. arXiv:1009.1851v5 [math.AT].
- [Br72] Bredon, G. E.: *Introduction to compact transformation groups*. Pure and Applied Mathematics **46**, Academic Press, New York-London, 1972.
- [CLOT03] Cornea, O.; Lupton, G.; Oprea, J.; Tanré, D.: *Lusternik-Schnirelmann category*. Mathematical Surveys and Monographs **103**, American Mathematical Society, Providence, RI, 2003.
- [Do95] Dold, A.: *Lectures on algebraic topology*, reprint of the 1972 edition. Classics in Mathematics, Springer-Verlag, Berlin, 1995.

- [Fa03] Farber, M.: Topological complexity of motion planning. *Discrete Comput. Geom.* **29** (2003) 211–221.
- [Fa04] Farber, M.: Instabilities of robot motion. *Topology Appl.* **140** (2004) 245–266.
- [Fa06] Farber, M.: *Topology of robot motion planning*. In: Morse theoretic methods in nonlinear analysis and in symplectic topology, NATO Science Series II: Mathematics, Physics and Chemistry **217**, 185–230, Springer, Dordrecht, 2006.
- [Fa08] Farber, M.: *Invitation to topological robotics*. Zurich Lectures in Advanced Mathematics, EMS, Zürich, 2008.
- [FG07] Farber, M.; Grant, M.: Symmetric motion planning. In: Topology and Robotics, *Contemp. Math.* **438**, 85–104, Amer. Math. Soc., Providence, RI, 2007.
- [FG08] Farber, M.; Grant, M.: Robot motion planning, weights of cohomology classes, and cohomology operations. *Proc. Amer. Math. Soc.* **136** (2008) 3339–3349.
- [FTY03] Farber, M.; Tabachnikov, S.; Yuzvinsky, S. Topological robotics: motion planning in projective spaces. *Int. Math. Res. Not.* **34** (2003) 1853–1870.
- [FY04] Farber, M.; Yuzvinsky, S.: Topological robotics: subspace arrangements and collision free motion planning. In: Geometry, Topology, and Mathematical Physics, *Amer. Math. Soc. Transl.* (2) **212**, 145–156, Amer. Math. Soc., Providence, RI, 2004.
- [FZ00] Feichtner, E. M.; Ziegler, G. M.: The integral cohomology algebras of ordered configuration spaces of spheres. *Doc. Math.* **5** (2000) 115–139.
- [Gl50] Gleason, A.: Spaces with a compact Lie group of transformations. *Proc. Amer. Math. Soc.* **1** (1950) 35–43.
- [GL09] González, J.; Landweber, P.: Symmetric topological complexity of projective and lens spaces. *Algebr. Geom. Topol.* **9** (2009) 473–494.
- [IS10] Iwase, N.; Sakai, M.: Topological complexity is a fibrewise L-S category. *Topology Appl.* **157** (2010) 10–21.
- [J76] Jaworowski, J.: Extensions of G -maps and Euclidean G -retracts. *Math. Z.* **146** (1976) 143–148.
- [Ka08] Kallel, S.: Symmetric products, duality and homological dimension of configuration spaces. *Geom. Topol. Monogr.* **13** (2008) 499–527.
- [KL12] Karasev, R.; Landweber, P.: Estimating the higher symmetric topological complexity of spheres. *Algebraic & Geometric Topology* **12** (2012) 75–94.
- [LV06] LaValle, S.: *Planning algorithms*. Cambridge University Press, Cambridge, 2006.
- [La91] Latombe, J.-C.: *Robot motion planning*. The Kluwer international series in engineering and computer science **124**, Kluwer Academic Publishers, Boston, 1991.
- [LS13] Lupton, G.; Scherer, Jérôme.: *Topological complexity of H -spaces*. Proceedings of the American Mathematical Society **141** No. 5 (2013) 1827–1838.
- [Ru10] Rudyak, Yu. B.: On higher analogs of topological complexity. *Topology and its Applications* **157** (2010) 916–920 (erratum in *Topology and its Applications* **157** (2010) 1118).
- [Sv66] Švarc (Schwarz), A. S.: The genus of a fiber space. *Amer. Math. Soc. Transl. Series 2* **55** (1966) 49–140.

IBAI BASABE

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA
 358 LITTLE HALL, GAINESVILLE, FL 32611-8105, USA
E-mail address: iebasabe1@ufl.edu

JESÚS GONZÁLEZ

DEPARTAMENTO DE MATEMÁTICAS, CINVESTAV-IPN

A.P. 14-740, MÉXICO CITY 07000, MÉXICO

E-mail address: jesus@math.cinvestav.mx

YULI B. RUDYAK

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA

358 LITTLE HALL, GAINESVILLE, FL 32611-8105, USA

E-mail address: rudyak@ufl.edu

DAI TAMAKI

DEPARTMENT OF MATHEMATICAL SCIENCES, SHINSHU UNIVERSITY

MATSUMOTO, 390-8621, JAPAN

E-mail address: rivulus@shinshu-u.ac.jp