

# On the Homology of Configuration Spaces Associated to Centers of Mass

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## Abstract.

The aim of this paper is to make sample computations with the Salvetti complex of the “center of mass” arrangement introduced in [CK07] by Cohen and Kamiyama. We compute the homology of the Salvetti complex of these arrangements with coefficients in the sign representation of the symmetric group on  $\mathbb{F}_p$  in the case of four particles. We show, when  $p$  is an odd prime, the homology is isomorphic to the homology of the configuration space  $F(\mathbb{C}, 4)$  of distinct four points in  $\mathbb{C}$  with the same coefficients. When  $p = 2$ , we show the homology is different from the equivariant homology of  $F(\mathbb{C}, 4)$ , hence we obtain an alternative and more direct proof of a theorem of Cohen and Kamiyama in [CK07].

## §1. Introduction

The configuration spaces of distinct points in  $\mathbb{C}$

$$F(\mathbb{C}, n) = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ if } i \neq j\}$$

have been playing important roles in homotopy theory. For example, F. Cohen [Coh83] used the weak equivalence

$$\Omega^2 \Sigma^2 X \underset{w}{\simeq} \left( \prod_n F(\mathbb{C}, n) \times_{\Sigma_n} X^n \right) / \sim$$

to construct an unstable splitting map

$$(1) \quad \Sigma^{2n} \Omega^2 \Sigma^2 X \longrightarrow \Sigma^{2n} F(\mathbb{C}, n)_+ \wedge_{\Sigma_n} X^{\wedge n}.$$

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This is a desuspension of the well-known stable splitting

$$\Sigma^\infty \Omega^2 \Sigma^2 X \underset{w}{\simeq} \Sigma^\infty \left( \bigvee_n F(\mathbb{C}, n)_+ \wedge_{\Sigma_n} X^{\wedge n} \right)$$

due to Snaith [Sna74].

These stable and unstable splitting maps can be used to construct important maps in unstable homotopy theory. See [Mah77, Coh83], for example. It is also known that the map (1) cannot be desuspended further [CM82]. There is a chance of desuspending this map, however, if we localize at an appropriate prime. B. Gray observed in [Gra93a, Gra93b] that if we could construct a map

$$(2) \quad \Sigma^2 \Omega^2 S^3 \longrightarrow \Sigma^2 F(\mathbb{C}, p)_+ \wedge_{\Sigma_p} S^p$$

after localizing at an odd prime  $p$ , we would be able to refine results of Cohen, Moore, and Neisendorfer [CMN79b, CMN79a] and construct higher order EHP sequences.

The difficulty is to construct a localized model of  $\Omega^2 S^3$  in terms of configuration spaces. We do not know very much about localizations of configuration spaces. As an attempt to construct such a localized model, F. Cohen and Kamiyama introduced a subspace  $M_\ell(\mathbb{C}, n)$  of  $F(\mathbb{C}, n)$  in [CK07], for natural numbers  $n$  and  $\ell$  with  $\ell < n$ . It can be defined as the complement in  $\mathbb{C}^n$  of the complexification of the real central hyperplane arrangement defined by

$$\mathcal{C}_{n-1}^\ell = \{L_{I,J} \mid I, J \subset \{1, \dots, n\}, |I| = |J| = \ell, I \neq J\},$$

where

$$L_{I,J} = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid |J| \sum_{i \in I} x_i = |I| \sum_{j \in J} x_j \right\}$$

for  $I, J \subset \{1, \dots, n\}$ . For  $\ell \geq n$ , we define  $\mathcal{C}_{n-1}^\ell$  to be the braid arrangement  $\mathcal{A}_{n-1}$ .

Notice that

$$\mathcal{C}_{n-1}^\ell = \begin{cases} \{L_{I,J} \mid I, J \subset \{1, \dots, n\}, |I| = |J| \leq \ell, I \cap J = \emptyset\}, & \ell \leq \frac{n}{2}, \\ \{L_{I,J} \mid I, J \subset \{1, \dots, n\}, |I| = |J| \leq n - \ell, I \cap J = \emptyset\}, & \ell > \frac{n}{2} \end{cases}$$

and we have the duality

$$\mathcal{C}_{n-1}^\ell = \mathcal{C}_{n-1}^{n-\ell}$$

for  $\ell < n$ . Thus we have the following inclusions of arrangements:

$$\mathcal{A}_{n-1} = \mathcal{C}_{n-1}^1 \subset \mathcal{C}_{n-1}^2 \subset \cdots \subset \mathcal{C}_{n-1}^{\lfloor \frac{n}{2} \rfloor} \supset \cdots \supset \mathcal{C}_{n-1}^{n-2} \supset \mathcal{C}_{n-1}^{n-1} = \mathcal{A}_{n-1}.$$

By taking the complements, we obtain

$$(3) \quad M_\ell(\mathbb{C}, n) \subset M_{\ell-1}(\mathbb{C}, n) \subset \cdots \subset M_2(\mathbb{C}, n) \subset M_1(\mathbb{C}, n) = F(\mathbb{C}, n)$$

if  $\ell \leq \lfloor \frac{n}{2} \rfloor$ . When  $n > \ell > \lfloor \frac{n}{2} \rfloor$ , we have

$$(4) \quad M_\ell(\mathbb{C}, n) = M_{n-\ell}(\mathbb{C}, n) \subset M_{n-\ell-1}(\mathbb{C}, n) \subset \cdots \subset M_2(\mathbb{C}, n) \subset M_1(\mathbb{C}, n) = F(\mathbb{C}, n).$$

We denote these inclusions by  $i_{\ell,n} : M_\ell(\mathbb{C}, n) \hookrightarrow F(\mathbb{C}, n)$ .

**Conjecture 1.1** (Cohen-Kamiyama). *For an odd prime  $p$ , the natural inclusion*

$$i_{p,n} : M_p(\mathbb{C}, n) \hookrightarrow F(\mathbb{C}, n)$$

*induces an isomorphism*

$$(i_{p,n})_* : H_*(S_*(M_p(\mathbb{C}, n)) \otimes_{\Sigma_n} \mathbb{F}_p(\pm 1)) \longrightarrow H_*(S_*(F(\mathbb{C}, n)) \otimes_{\Sigma_n} \mathbb{F}_p(\pm 1))$$

*for all  $n$ , where  $S_*(-)$  is the singular chain complex functor,  $\Sigma_n$  is the symmetric group on  $n$  letters, and  $\mathbb{F}_p(\pm 1)$  is  $\mathbb{F}_p$  regarded as a  $\Sigma_n$ -module via sign representation.*

They proved that the above conjecture implies the existence of the desired map (2). They also initiated the analysis of the homology of  $M_p(\mathbb{C}, n)$  and proved the statement of the conjecture does not hold when  $p = 2$ . Notice that  $M_2(\mathbb{C}, 2) = F(\mathbb{C}, 2)$  by definition and  $M_2(\mathbb{C}, 3) = M_1(\mathbb{C}, 3) = F(\mathbb{C}, 3)$  by the duality. Thus  $M_2(\mathbb{C}, 4)$  is the first nontrivial case.

**Theorem 1.2.** *The class in  $H_3(S_*(F(\mathbb{C}, 4)) \otimes_{\Sigma_4} \mathbb{F}_2)$  corresponding to  $Q_1^2(x)$  in  $H_*(\Omega^2 S^{n+2}; \mathbb{F}_2)$  is not in the image of the map*

$$H_3(S_*(M_2(\mathbb{C}, 4)) \otimes_{\Sigma_4} \mathbb{F}_2) \longrightarrow H_3(S_*(F(\mathbb{C}, 4)) \otimes_{\Sigma_4} \mathbb{F}_2)$$

*induced by the natural inclusion. Hence this map is not surjective.*

Their method is indirect in the sense that they proved it by contradiction by calculating homology and cohomology operations. In order to find a way to attack the conjecture, a more direct method is desirable.

For a real central hyperplane arrangement  $\mathcal{A}$  in general, Salvetti [Sal87] constructed a finite cell complex  $\text{Sal}(\mathcal{A})$  embedded in the complement of the complexification of  $\mathcal{A}$  as a deformation retract. The aim

of this paper is to determine the maps induced on homology groups in the last step of the inclusions (3) and (4) by using the Salvetti complex.

We reprove Theorem 1.2 by analyzing the cellular structure of the Salvetti complex. In fact, we show that  $H_3(S_*(M_2(\mathbb{C}, 4)) \otimes_{\Sigma_4} \mathbb{F}_2) \cong \mathbb{F}_2$  and the map takes the generator to 0.

For odd primes, we obtain the following result.

**Theorem 1.3.** *For an odd prime  $p$ , the inclusion  $M_2(\mathbb{C}, 4) \hookrightarrow F(\mathbb{C}, 4)$  induces an isomorphism*

$$H_*(S_*(M_2(\mathbb{C}, 4)) \otimes_{\Sigma_4} \mathbb{F}_p(\pm 1)) \cong H_*(S_*(F(\mathbb{C}, 4)) \otimes_{\Sigma_4} \mathbb{F}_p(\pm 1)).$$

The paper is organized as follows:

- Basic properties of the Salvetti complex used in this paper are recalled in §2.
- We describe the cell decomposition of the Salvetti complex for the braid arrangement in §3 by using the notations in [Tam].
- The Salvetti complex for the center of mass arrangement is studied in §4, including the computation of the homology of  $\mathcal{C}_3^2$ . We also include a computation with trivial  $\mathbb{F}_p$  coefficients for  $p$  an odd prime, following the suggestion by the referee.

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## §2. Salvetti Complex and Oriented Matroid

Let us recall the definition and basic properties of the Salvetti complex used in this paper.

### 2.1. Salvetti Complex for Real Central Arrangements

A hyperplane arrangement  $\mathcal{A}$  in a real vector space  $V$  defines a stratification of  $V$  by

$$\begin{aligned} S_0 &= V - \bigcup_{L \in \mathcal{A}} L \\ S_1 &= \bigcup_{L \in \mathcal{A}} L - \bigcup_{L, L' \in \mathcal{A}} L \cap L' \\ &\vdots \\ S_{|\mathcal{L}|} &= \bigcap_{L \in \mathcal{A}} L. \end{aligned}$$

Each stratum is a disjoint union of convex regions. These connected components are called faces and the faces in the top stratum are called chambers or topes. The set of faces is denoted by  $\mathcal{L}(\mathcal{A})$  and the subset of chambers is denoted by  $\mathcal{L}^{(0)}(\mathcal{A})$ .  $\mathcal{L}(\mathcal{A})$  has a structure of poset by

$$F \leq G \iff F \subset \overline{G}$$

and is called the face poset of  $\mathcal{A}$ .

When  $\mathcal{A}$  is central, i.e. hyperplanes in  $\mathcal{A}$  are vector subspaces, Salvetti [Sal87] constructed a finite regular cell complex  $\text{Sal}(\mathcal{A})$  embedded in  $V \otimes \mathbb{C} - \bigcup_{L \in \mathcal{A}} L \otimes \mathbb{C}$  as a deformation retract. Its cellular structure is determined by the face poset  $\mathcal{L}(\mathcal{A})$  of the real arrangement  $\mathcal{A}$ .

A rough idea of the original construction of the Salvetti complex  $\text{Sal}(\mathcal{A})$  is as follows: For each face  $F$  in  $\mathcal{A}$ , choose a point  $w(F)$  in  $F$ . For each pair of a face  $F$  and a chamber  $C$  with  $F \leq C$ , define a point in  $V \otimes \mathbb{C}$  by

$$v(F, C) = w(F) \otimes 1 + (w(C) - w(F)) \otimes i.$$

The set of all such points is denoted by

$$\text{sk}_0 \text{Sal}(\mathcal{A}) = \{v(F, C) \mid F \in \mathcal{L}(\mathcal{A}), C \in \mathcal{L}^{(0)}(\mathcal{A}), F \leq C\}.$$

The Salvetti complex  $\text{Sal}(\mathcal{A})$  is constructed as a Euclidean simplicial complex embedded in  $V \otimes \mathbb{C} - \bigcup_{L \in \mathcal{A}} L \otimes \mathbb{C}$  by forming simplices by choosing vertices from these points in a certain manner. Salvetti defines a structure of a finite cell complex on  $\text{Sal}(\mathcal{A})$  by combining several simplices together. Since all we need is this cell decomposition, we recall this cellular structure, instead of describing the rule for simplices. The cell decomposition can be described in terms of the face-chamber pairing, or the matroid product. The definition of the matroid product

can be found in [Sal87, Arv91]. An alternative description will be given later in Lemma 2.4.

**Definition 2.1.** For  $F \in \mathcal{L}(\mathcal{A})$  and  $C \in \mathcal{L}^{(0)}(\mathcal{A})$  with  $F \leq C$ , define a subset of  $\text{sk}_0(\text{Sal}(\mathcal{A}))$  by

$$\mathcal{D}(F, C) = \{v(G, G \circ C) \mid G \geq F\},$$

where  $\circ$  is the matroid product. This set is regarded as a poset by

$$v(G, G \circ C) \leq v(H, H \circ C) \iff G \leq H.$$

The (geometric realization of the) order complex of  $\mathcal{D}(F, C)$  is denoted by  $D(F, C)$ .

**Lemma 2.2.** *The complex  $D(F, C)$  has the following properties:*

- (1) *The inclusion of vertices induces a simplicial embedding*

$$D(F, C) \hookrightarrow \text{Sal}(\mathcal{A}).$$

- (2)  *$D(F, C)$  is homeomorphic to a disk of dimension  $\text{codim } F$ .*

- (3) *The boundary of  $D(F, C)$  is given by*

$$\partial D(F, C) = \bigcup_{G > F} D(G, G \circ C).$$

- (4) *The decomposition*

$$\text{Sal}(\mathcal{A}) = \bigcup_{v(F, C) \in \text{sk}_0(\text{Sal}(\mathcal{A} \otimes \mathbb{C}))} (D(F, C) - \partial D(F, C))$$

*defines a structure of a finite regular cell complex on  $\text{Sal}(\mathcal{A})$ .*

In order to compute the boundary, therefore, we need to understand the matroid product. The following elementary fact is very useful.

**Lemma 2.3.** *Given a real central hyperplane arrangement  $\mathcal{A} = \{L_1, \dots, L_n\}$  in a real inner product space  $(V, \langle -, - \rangle)$ , choose a normal vector  $a_i$  for each hyperplane  $L_i$*

$$L_i = \{x \in V \mid \langle a_i, x \rangle = 0\}$$

*and define  $\mathcal{V}(\mathcal{A}) = \{a_1, \dots, a_n\}$ .*

*Let  $S_1 = \{0, +1, -1\}$  be the poset with  $0 < +1, -1$ . For  $F \in \mathcal{L}(\mathcal{A})$ , define*

$$\tau_F : \mathcal{V}(\mathcal{A}) \longrightarrow S_1$$

by

$$\tau_F(a) = \text{sign}\langle a, F \rangle,$$

where

$$\text{sign} : \mathbb{R} \longrightarrow S_1$$

is the sign function.

Then we obtain an embedding

$$\tau : \mathcal{L}(\mathcal{A}) \hookrightarrow \text{Map}(\mathcal{V}(\mathcal{A}), S_1).$$

It is possible to give an explicit description of the subset  $\tau(\mathcal{L}(\mathcal{A}))$  by using the language of oriented matroid. See the paper [GR89] by Gel'fand and Rybnikov or the book on oriented matroids [BLVS<sup>+</sup>99] by five authors for more details.

With this identification, the face-chamber pairing in the face lattice  $\mathcal{L}(\mathcal{A})$  can be translated into the following product of functions.

**Lemma 2.4.** *Let  $E$  be a set. For  $\varphi, \psi \in \text{Map}(E, S_1)$  define  $\varphi \circ \psi \in \text{Map}(E, S_1)$  by*

$$(\varphi \circ \psi)(x) = \begin{cases} \varphi(x), & \varphi(x) \neq 0 \\ \psi(x), & \varphi(x) = 0. \end{cases}$$

Then for a real central arrangement  $\mathcal{A}$ , we have

$$\tau_F \circ \tau_G = \tau_{F \circ G}$$

for  $F, G \in \mathcal{L}(\mathcal{A})$ .

We may formally complexify the poset  $\mathcal{L}(\mathcal{A})$  by using the poset  $S_2 = \{0, +1, -1, +i, -i\}$  with ordering  $0 < \pm 1 < \pm i$ .

**Definition 2.5.** Define  $\Sigma_2$ -equivariant inclusions

$$i_1, i_2 : S_1 \hookrightarrow S_2$$

by

$$\begin{aligned} i_1(0) &= i_2(0) = 0 \\ i_1(\pm 1) &= \pm 1 \\ i_2(\pm 1) &= \pm i. \end{aligned}$$

**Definition 2.6.** Let  $E$  be a set and  $L$  be a subset of  $\text{Map}(E, S_1)$ . Define a subposet  $L \otimes \mathbb{C}$  of  $\text{Map}(E, S_2)$  by

$$L \otimes \mathbb{C} = \{(i_1)_*(X) \circ (i_2)_*(Y) \mid X, Y \in L\},$$

where

$$(i_1)_*, (i_2)_* : \text{Map}(E, S_1) \longrightarrow \text{Map}(E, S_2)$$

are the maps induced by  $i_1$  and  $i_2$  and the matroid product  $\circ$  in  $\text{Map}(E, S_2)$  is defined by

$$(\varphi \circ \psi)(x) = \begin{cases} \varphi(x), & \varphi(x) \not\leq \psi(x) \\ \psi(x), & \varphi(x) \leq \psi(x). \end{cases}$$

Another useful way of describing the Salvetti complex is as follows. See [BZ92], for example.

**Proposition 2.7.** *Let  $\mathcal{A}$  be a real central arrangement in an inner product space  $V$  and  $\mathcal{V}(\mathcal{A})$  be a set of unit normal vectors of hyperplanes in  $\mathcal{A}$ . Define a subposet  $\mathcal{L}^{(1)}(\mathcal{A})$  of  $\mathcal{L}(\mathcal{A}) \otimes \mathbb{C}$  by*

$$\mathcal{L}^{(1)}(\mathcal{A}) = \{X \in \mathcal{L}(\mathcal{A}) \otimes \mathbb{C} \mid X(v) \neq 0 \text{ for all } v \in \mathcal{V}(\mathcal{A})\}.$$

*Then  $\mathcal{L}^{(1)}(\mathcal{A})$  is isomorphic to the face poset  $F(\text{Sal}(\mathcal{A}))$  of the Salvetti complex of  $\mathcal{A}$  as posets. Thus we have an isomorphism of simplicial complexes*

$$B\mathcal{L}^{(1)}(\mathcal{A}) \cong \text{Sd}(\text{Sal}(\mathcal{A})),$$

*where  $B(-)$  is the classifying space (order complex) functor and  $\text{Sd}(-)$  is the barycentric subdivision.*

## 2.2. Salvetti Complex for Reflection Arrangements

When  $\mathcal{A}$  is a reflection arrangement, Salvetti analyzed the cellular structure of  $\text{Sal}(\mathcal{A})/G(\mathcal{A})$  in [Sal94], where  $G(\mathcal{A})$  is the reflection group associated with  $\mathcal{A}$ . In particular, he described the boundary in the cellular cochain complex. Salvetti's work includes the case of affine reflection groups. Here we only consider central arrangements.

Let  $\mathcal{A}$  be a real central reflection arrangement in  $V$  and  $\mathcal{L}(\mathcal{A})$  be the face poset. We have a cellular decomposition (stratification) of  $V$

$$V = \coprod_{F \in \mathcal{L}(\mathcal{A})} F.$$

The cell complex dual to this cellular decomposition is denoted by  $C(\mathcal{A})$ , whose face poset is denoted by  $F(C(\mathcal{A}))$ . One of the ways to construct  $C(\mathcal{A})$  is to choose a chamber  $C_0 \in \mathcal{L}^{(0)}(\mathcal{A})$  and a point  $v_0$  inside of  $C_0$  and to take the convex hull of the  $G$ -orbit of  $v_0$

$$C(\mathcal{A}) = \text{Conv}(gv_0 \mid g \in G(\mathcal{A})).$$

The cellular structure of  $C(\mathcal{A})$  is given by that of this convex polytope.

When  $e \in F(C(\mathcal{A}))$  is the face dual to  $F \in \mathcal{L}(\mathcal{A})$  we denote  $e = F^*$  and  $F = e^*$ .



**Definition 2.8.** For each face  $e \in F(C(\mathcal{A}))$ , let  $\gamma(e) \in G(\mathcal{A})$  be the unique element of minimal length such that

$$\gamma(e)^{-1}(e^*) \subset \overline{C_0}.$$

Salvetti found the following description of  $\text{Sal}(\mathcal{A})/G(\mathcal{A})$ .

**Theorem 2.9.** *The cell complex  $\text{Sal}(\mathcal{A})/G(\mathcal{A})$  can be identified with the cell complex given by identifying those two cells  $e, e'$  in  $C(\mathcal{A})$  which are in the same  $G(\mathcal{A})$ -orbit by using the homeomorphism induced by the element  $\gamma(e')\gamma(e)^{-1}$ .*

In order to describe the boundary homomorphism in the cellular chain complex, Salvetti defined an orientation on each cell in  $C(\mathcal{A})$ .

**Definition 2.10.** Fix an ordering of hyperplanes  $H_1, \dots, H_n$  bounding the chosen chamber  $C_0$ . Let  $v_i$  be the projection of  $v_0$  onto  $H_i$ . Define

$$\begin{aligned} F(C_0) &= \{F \in \mathcal{L}(\mathcal{A}) \mid F \subset \overline{C_0}\}, \\ F^*(C_0) &= \{F^* \in F(C(\mathcal{A})) \mid F \in F(C_0)\}. \end{aligned}$$

For a cell  $e \in F^*(C_0)$ , define an orientation on  $e$  as follows. Let  $H_{i_1}, \dots, H_{i_k}$  be hyperplanes with

$$e^* \subset H_{i_1} \cap \dots \cap H_{i_k}$$

and  $i_1 < \dots < i_k$ . The orientation of  $e$  is induced from the ordering  $v_0, v_{i_1}, \dots, v_{i_k}$  under the inclusion

$$\text{Conv}(v_0, v_{i_1}, \dots, v_{i_k}) \subset \bar{e}.$$

In general, define an orientation on  $e \in C(\mathcal{A})$  in such a way that  $\gamma(e)^{-1}$  is orientation preserving.

Under the above orientations, the incidence numbers among cells in  $C(\mathcal{A})$  are described as follows.

**Proposition 2.11.** *Let  $F \in F(C_0)$  and  $G \in \mathcal{L}(\mathcal{A})$  with*

$$\overline{gG} \supset F$$

*for an element  $g \in G(\mathcal{A})$  of the shortest length and  $\dim G = \dim F + 1$ . Then*

$$[F^* : G^*] = (-1)^{\ell(g)} [F^*, (gG)^*].$$

### 2.3. Maps between Salvetti complexes

The “center of mass” arrangement  $\mathcal{C}_{n-1}^\ell$  is obtained by adding hyperplanes to the braid arrangement  $\mathcal{A}_{n-1}$ . And contravariantly we have an inclusion

$$M_\ell(\mathbb{C}, n) \hookrightarrow F(\mathbb{C}, n)$$

of the complements. We also have corresponding maps on the Salvetti complexes.

**Lemma 2.12.** *Let  $\mathcal{A}$  be a real central arrangement in a vector space  $V$  and  $\mathcal{B} \subset \mathcal{A}$  be a subarrangement. Then the inclusion*

$$i : \mathcal{B} \hookrightarrow \mathcal{A}$$

*induces a cellular map*

$$i^* : \text{Sal}(\mathcal{A}) \longrightarrow \text{Sal}(\mathcal{B})$$

*which makes the following diagram commutative up to homotopy*

$$\begin{array}{ccc} \text{Sal}(\mathcal{A}) & \xrightarrow{i^*} & \text{Sal}(\mathcal{B}) \\ \downarrow & & \downarrow \\ V \otimes \mathbb{C} - \bigcup_{H \in \mathcal{A}} H \otimes \mathbb{C} & \longrightarrow & V \otimes \mathbb{C} - \bigcup_{H \in \mathcal{B}} H \otimes \mathbb{C} \end{array}$$

*Proof.* The inclusion  $i : \mathcal{B} \hookrightarrow \mathcal{A}$  induces a map of face posets

$$i^* : \mathcal{L}(\mathcal{A}) \longrightarrow \mathcal{L}(\mathcal{B})$$

(i.e. it induces a strong map between oriented matroids) which induces a map of posets

$$i^* : \mathcal{L}(\mathcal{A}) \otimes \mathbb{C} \longrightarrow \mathcal{L}(\mathcal{B}) \otimes \mathbb{C}.$$

Since  $i^*$  is given by restriction, we obtain

$$i^* : \mathcal{L}^{(1)}(\mathcal{A}) \longrightarrow \mathcal{L}^{(1)}(\mathcal{B})$$

and hence a map

$$i^* : \text{Sal}(\mathcal{A}) \longrightarrow \text{Sal}(\mathcal{B})$$

by Proposition 2.7.

The embeddings of the Salvetti complexes depend on choices of simplicial vertices corresponding to faces in the face posets. We obtain embeddings by choosing  $w(F)$  for  $F \in \mathcal{L}(\mathcal{A})$  first and then by choosing vertices for  $\mathcal{B}$  among  $\{w(F)\}$  which make the required diagram commutative. Q.E.D.

We would like to know the behavior of the chain map

$$i_* : C_*(\text{Sal}(\mathcal{A})) \longrightarrow C_*(\text{Sal}(\mathcal{B})).$$

**Lemma 2.13.** *Let  $\mathcal{A}$  be a real central arrangement in  $V$  and  $\mathcal{B} \subset \mathcal{A}$  be a subarrangement. Then the inclusion*

$$i : \mathcal{B} \hookrightarrow \mathcal{A}$$

*induces a surjective chain map*

$$i^* : C_*(\text{Sal}(\mathcal{A})) \longrightarrow C_*(\text{Sal}(\mathcal{B})).$$

*Proof.* Generators of  $C_*(\text{Sal}(\mathcal{B}))$  are in one-to-one correspondence with pairs  $(F, C)$  of a face  $F$  and a chamber  $C$  in  $\mathcal{L}(\mathcal{B})$ . Since

$$i^* : \mathcal{L}(\mathcal{A}) \longrightarrow \mathcal{L}(\mathcal{B})$$

is surjective, it induces a surjective map on the cellular chain complexes of Salvetti complexes. Q.E.D.

### §3. The Salvetti Complex for the Braid Arrangement

We need to understand the cellular structure of the Salvetti complex of the braid arrangement in order to compare it with that of the center of mass arrangement.

#### 3.1. The Structure of Cell Complex

The braid arrangement is a typical example of reflection arrangements and the results of §2.2 apply. In particular, the cell structure of the Salvetti complex for the braid arrangement can be described in terms of partitions. The following symbols are introduced in [Tam].

**Definition 3.1.** A partition of  $\{1, \dots, n\}$  is a surjective map

$$\lambda : \{1, \dots, n\} \longrightarrow \{1, \dots, n - r\}$$

for some  $0 \leq r < n$ . The number  $r$  is called the rank of this partition.

The set of partitions of  $\{1, \dots, n\}$  is denoted by  $\Pi_n$ . The subset of rank  $r$  partitions is denoted by  $\Pi_{n,r}$ .  $\Pi_n$  becomes a poset under refinement. Note that rank 0 partitions are nothing but elements of  $\Sigma_n$ .

**Definition 3.2.** For a partition  $\lambda \in \Pi_n$  of rank  $r$  and  $\sigma \in \Sigma_n$  with  $\sigma \geq \lambda$ , define a symbol  $S(\lambda, \sigma)$  as follows:

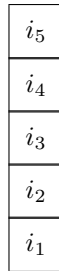
- (1) For each  $1 \leq i \leq n - r$ , draw vertically stacked squares  $S_i$  of length  $|\lambda^{-1}(i)|$ .



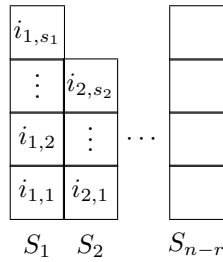
- (2) Order  $\lambda^{-1}(i)$  according to  $\sigma$  and label each square in  $S_i$  from bottom to top by elements in  $\lambda^{-1}(i)$ . For example, when  $\lambda^{-1}(i) = \{i_1, i_2, i_3, i_4, i_5\}$  and if these numbers appear in  $(\sigma(1), \dots, \sigma(n))$  in the order

$$i_1, i_2, i_3, i_4, i_5$$

then  $S_i$  is labeled as



- (3) Place  $S_1, \dots, S_{n-r}$  side by side from left to right.  $S(\lambda, \sigma)$  is the resulting picture.



The following observation played an essential role in [Tam].

**Lemma 3.3.** *There is a bijection between the set of vertices (of the simplicial structure)  $\text{sk}_0 \text{Sal}(\mathcal{A}_{n-1})$  and the set of symbols*

$$\{S(\lambda, \sigma) \mid \lambda \in \Pi_n, \sigma \in \Sigma_n, \lambda \leq \sigma\}.$$

Thus we obtain a bijection between the cells of  $\text{Sal}(\mathcal{A}_{n-1})$  and the symbols  $S(\lambda, \sigma)$ .

In order to compute the boundary maps of the cellular chain complex of  $\text{Sal}(\mathcal{A}_{n-1})$ , we need to fix orientations of cells. We follow the orientations defined in §2.2.

We choose the chamber

$$C_0 = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 < x_2 < \dots < x_n\}$$

and define  $v_0 = (1, 2, \dots, n) \in C_0$ . Then

$$C(\mathcal{A}_{n-1}) = \text{Conv}(v_0\sigma \mid \sigma \in \Sigma_n).$$

We have the following refinement of Theorem 2.9.

**Proposition 3.4.** *We have the following isomorphism of  $\mathbb{F}_p$ -modules*

$$\begin{aligned} C_*(\text{Sal}(\mathcal{A}_{n-1})) \otimes_{\Sigma_n} \mathbb{F}_p(\pm 1) &\cong \mathbb{F}_p \langle [e] \mid [e] \in F(C(\mathcal{A}_{n-1}))/\Sigma_n \rangle \\ &\cong \mathbb{F}_p \langle e \mid e \in F^*(C_0) \rangle. \end{aligned}$$

*Proof.* Let  $S$  be a set with an action of  $\Sigma_n$ . Then we have an isomorphism of  $\mathbb{F}_p$ -modules

$$\mathbb{Z}\langle S \rangle \otimes_{\Sigma_n} \mathbb{F}_p(\pm 1) \cong \mathbb{Z}\langle S/\Sigma_n \rangle \otimes \mathbb{F}_p.$$

And the result follows by the identifications of cells in the proof of Theorem 2.9 in [Sal94]. Q.E.D.

The bounding hyperplanes of  $C_0$  are

$$L_{1,2}, L_{2,3}, \dots, L_{n,n-1}.$$

This ordering of hyperplanes determines orientations of cells in  $C(\mathcal{A}_{n-1})$ . Under the correspondence in Lemma 3.3, cells in  $C(\mathcal{A}_{n-1})$  correspond to  $S(\lambda, 1)$  with  $\lambda \in \Pi_n$  and  $\lambda \leq 1$ . Those cells in  $F^*(C_0)$  corresponds to ordered partitions.

**Definition 3.5.** An order preserving surjective map

$$\lambda : \{1, \dots, k\} \longrightarrow \{1, \dots, k-r\}$$

is called an ordered partition of rank  $r$ . The set of ordered partitions of  $\{1, \dots, k\}$  of rank  $r$  is denoted by  $O_{k,r}$ .

**Corollary 3.6.** *Under the identification in Lemma 3.3, we have the following isomorphism of  $\mathbb{F}_p$ -modules*

$$C_s(\text{Sal}(\mathcal{A}_{n-1})) \otimes_{\Sigma_n} \mathbb{F}_p(\pm 1) \cong \mathbb{F}_p \langle D(\lambda, (1|\dots|n)) \mid \lambda \in O_{n,n-s} \rangle.$$

The following formula for the boundary map follows from Proposition 2.11.

**Lemma 3.7.** *For  $\lambda \in O_{n,n-s}$ , we have*

$$\begin{aligned} \partial(D(\lambda, (1|\cdots|n))) &= \\ & \sum_{\tau \in O_{n,n-s-1}, \lambda < \tau} \sum_{\sigma \in \Sigma(\lambda)} \text{sgn}(\sigma) [D(\lambda, (1|\cdots|n)) : D(\tau, (1|\cdots|n))] D(\tau\sigma, \sigma) \end{aligned}$$

in  $C_*(\text{Sal}(\mathcal{A}_{n-1}))$ , where

$$\Sigma(\lambda) = \{\sigma \in \Sigma_n \mid \lambda\sigma = \lambda\}$$

is the set of permutations preserving the partition  $\lambda$ .

*Proof.* The set of faces in  $\text{Sal}(\mathcal{A}_{n-1})$  contained in  $D(\lambda, (1|\cdots|n))$  as a face is given by  $D(\tau\sigma, \sigma)$  for  $\lambda < \tau$  and  $\sigma \in \Sigma(\lambda)$ . Thus

$$\begin{aligned} & \partial D(\lambda, (1|\cdots|n)) \\ &= \sum_{\tau \in O_{n,n-s-1}, \lambda < \tau} \sum_{\sigma \in \Sigma(\tau)} [D(\lambda, (1|\cdots|n)) : D(\tau\sigma, \sigma)] D(\tau\sigma, \sigma) \\ &= \sum_{\tau \in O_{n,n-s-1}, \lambda < \tau} \sum_{\sigma \in \Sigma(\tau)} \text{sgn}(\sigma) [D(\lambda, (1|\cdots|n)) : D(\tau, (1|\cdots|n))] D(\tau\sigma, \sigma) \end{aligned}$$

by Proposition 2.11.

Q.E.D.

Note that the incidence number  $[D(\lambda, (1|\cdots|n)) : D(\tau, (1|\cdots|n))]$  can be determined by comparing the “positions of =” in  $\lambda$  and  $\tau$ .

### 3.2. The Homology of $F(\mathbb{C}, 4)$

The homology  $H_*(S_*(F(\mathbb{C}, 4)) \otimes_{\Sigma_4} \mathbb{F}_p(\pm 1))$  is well-known. We need, however, an explicit description in order to compare it with  $H_*(C_*(\text{Sal}(\mathcal{C}_3^2)) \otimes_{\Sigma_4} \mathbb{F}_p(\pm 1))$  in the next section.

Let us compute  $H_*(C_*(\text{Sal}(\mathcal{A}_3)) \otimes_{\Sigma_4} \mathbb{F}_p(\pm 1))$  by using the symbols introduced in the previous section.  $\text{Sal}(\mathcal{A}_3)$  has the following cells:

- 0-cells are in one-to-one correspondence with the symbols

$$\boxed{\sigma(1) \mid \sigma(2) \mid \sigma(3) \mid \sigma(4)}$$

- 1-cells are in one-to-one correspondence with the symbols

$$\begin{array}{|c|} \hline \sigma(2) \\ \hline \sigma(1) \mid \sigma(3) \mid \sigma(4) \\ \hline \end{array}, \begin{array}{|c|} \hline \sigma(3) \\ \hline \sigma(1) \mid \sigma(2) \mid \sigma(4) \\ \hline \end{array}, \begin{array}{|c|} \hline \sigma(4) \\ \hline \sigma(1) \mid \sigma(2) \mid \sigma(3) \\ \hline \end{array}$$

- 2-cells are in one-to-one correspondence with the symbols

$$\begin{array}{ccccc}
 & \sigma(4) & & & \sigma(3) \\
 & \sigma(3) & \sigma(2)\sigma(4) & & \sigma(2) \\
 \sigma(1)\sigma(2) & & \sigma(1)\sigma(3) & & \sigma(1)\sigma(4)
 \end{array}$$

- 3-cells are in one-to-one correspondence with the symbols

$$\begin{array}{c}
 \sigma(4) \\
 \sigma(3) \\
 \sigma(2) \\
 \sigma(1)
 \end{array}$$

Thus the chain complex  $C_*(\text{Sal}(\mathcal{A}_3)) \otimes_{\Sigma_4} \mathbb{F}_p(\pm 1)$  has the following basis.

$$\begin{aligned}
 C_0(\text{Sal}(\mathcal{A}_3)) \otimes_{\Sigma_4} \mathbb{F}_p(\pm 1) &= \left\langle \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array} \right\rangle \\
 C_1(\text{Sal}(\mathcal{A}_3)) \otimes_{\Sigma_4} \mathbb{F}_p(\pm 1) &= \left\langle \begin{array}{|c|} \hline 2 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline & 3 & \\ \hline 1 & 2 & 4 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline & & 4 \\ \hline 1 & 2 & 3 \\ \hline \end{array} \right\rangle \\
 C_2(\text{Sal}(\mathcal{A}_3)) \otimes_{\Sigma_4} \mathbb{F}_p(\pm 1) &= \left\langle \begin{array}{|c|} \hline 4 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}, \begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline \end{array} \right\rangle \\
 C_3(\text{Sal}(\mathcal{A}_3)) \otimes_{\Sigma_4} \mathbb{F}_p(\pm 1) &= \left\langle \begin{array}{|c|} \hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} \right\rangle.
 \end{aligned}$$

The boundaries can be computed by using Lemma 3.7 as follows. For 0-cells, we obviously have

$$\partial_0 \left( \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array} \right) = 0.$$

For 1-cells, we have

$$\begin{aligned}
 \partial_1 \left( \begin{array}{|c|} \hline 2 \\ \hline \end{array} \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline \end{array} \right) &= \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline 2 & 1 & 3 & 4 \\ \hline \end{array} \\
 &= 2 \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array}.
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}\partial_1 \left( \begin{array}{|c|c|c|} \hline & 3 & \\ \hline 1 & 2 & 4 \\ \hline \end{array} \right) &= 2 \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array} \\ \partial_1 \left( \begin{array}{|c|c|c|} \hline & & 4 \\ \hline 1 & 2 & 3 \\ \hline \end{array} \right) &= 2 \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array}.\end{aligned}$$

For 2-cells, we have

$$\begin{aligned}\partial_2 \left( \begin{array}{|c|c|} \hline & 4 \\ \hline & 3 \\ \hline 1 & 2 \\ \hline \end{array} \right) &= 3 \begin{array}{|c|c|c|} \hline & 4 & \\ \hline 1 & 2 & 3 \\ \hline \end{array} - 3 \begin{array}{|c|c|c|} \hline & 3 & \\ \hline 1 & 2 & 4 \\ \hline \end{array} \\ \partial_2 \left( \begin{array}{|c|c|} \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array} \right) &= 2 \begin{array}{|c|c|c|} \hline & 4 & \\ \hline 1 & 2 & 3 \\ \hline \end{array} - 2 \begin{array}{|c|c|c|} \hline 2 & & \\ \hline 1 & 3 & 4 \\ \hline \end{array} \\ \partial_2 \left( \begin{array}{|c|c|} \hline 3 & \\ \hline 2 & \\ \hline 1 & 4 \\ \hline \end{array} \right) &= 3 \begin{array}{|c|c|c|} \hline & 3 & \\ \hline 1 & 2 & 4 \\ \hline \end{array} - 3 \begin{array}{|c|c|c|} \hline 2 & & \\ \hline 1 & 3 & 4 \\ \hline \end{array}.\end{aligned}$$

Finally, for 3-cells, we have

$$\partial_3 \left( \begin{array}{|c|} \hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} \right) = 4 \begin{array}{|c|c|} \hline & 4 \\ \hline 1 & 2 \\ \hline \end{array} - 6 \begin{array}{|c|c|} \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array} + 4 \begin{array}{|c|c|} \hline 3 & \\ \hline 1 & 4 \\ \hline \end{array}.$$

Thus we have the following well-known result.

**Proposition 3.8.** *When  $p > 3$ , we have*

$$H_i(C_*(\text{Sal}(\mathcal{A}_3)) \otimes_{\Sigma_4} \mathbb{F}_p(\pm 1)) = 0$$

for all  $i$ .

*Proof.* Since  $p \neq 2$ ,

$$\begin{aligned}\text{Im } \partial_1 &= \left\langle 2 \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array} \right\rangle = \left\langle \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array} \right\rangle \\ \text{Ker } \partial_1 &= \left\langle \begin{array}{|c|c|c|} \hline & 3 & \\ \hline 1 & 2 & 4 \\ \hline \end{array} - \begin{array}{|c|c|c|} \hline & 4 & \\ \hline 1 & 2 & 3 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline & 3 & \\ \hline 1 & 2 & 4 \\ \hline \end{array} - \begin{array}{|c|c|c|} \hline 2 & & \\ \hline 1 & 3 & 4 \\ \hline \end{array} \right\rangle.\end{aligned}$$



Thus we have

$$H_0(C_*(\text{Sal}(\mathcal{C}_3^2)) \otimes_{\Sigma_4} \mathbb{F}_p(\pm 1)) = 0.$$

Since  $p \neq 3$ , we have

$$\text{Im } \partial_2 = \left\langle \begin{array}{|c|c|c|} \hline & 3 & \\ \hline 1 & 2 & 4 \\ \hline \end{array} - \begin{array}{|c|c|c|} \hline & & 4 \\ \hline 1 & 2 & 3 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline & 3 & \\ \hline 1 & 2 & 4 \\ \hline \end{array} - \begin{array}{|c|c|c|} \hline & 2 & \\ \hline 1 & 3 & 4 \\ \hline \end{array} \right\rangle = \text{Ker } \partial_1$$

and

$$H_1(C_*(\text{Sal}(\mathcal{C}_3^2)) \otimes_{\Sigma_4} \mathbb{F}_p(\pm 1)) = 0.$$

We also have

$$\text{Ker } \partial_2 = \left\langle \frac{1}{3} \begin{array}{|c|c|} \hline 4 \\ \hline 3 \\ \hline 1 & 2 \\ \hline \end{array} - \frac{1}{2} \begin{array}{|c|c|} \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array} + \frac{1}{3} \begin{array}{|c|c|} \hline 3 \\ \hline 2 \\ \hline 1 & 4 \\ \hline \end{array} \right\rangle = \text{Im } \partial_3$$

and

$$H_2(C_*(\text{Sal}(\mathcal{C}_3^2)) \otimes_{\Sigma_4} \mathbb{F}_p(\pm 1)) = 0.$$

Since  $\text{Ker } \partial_3 = 0$ ,

$$H_3(C_*(\text{Sal}(\mathcal{C}_3^2)) \otimes_{\Sigma_4} \mathbb{F}_p(\pm 1)) = 0.$$

Q.E.D.

**Proposition 3.9.** *When  $p = 3$ , we have*

$$H_0(C_*(\text{Sal}(\mathcal{C}_3^2)) \otimes_{\Sigma_4} \mathbb{F}_3(\pm 1)) = 0$$

$$H_1(C_*(\text{Sal}(\mathcal{C}_3^2)) \otimes_{\Sigma_4} \mathbb{F}_3(\pm 1)) = \left\langle \left[ \begin{array}{|c|c|c|} \hline & 3 & \\ \hline 1 & 2 & 4 \\ \hline \end{array} - \begin{array}{|c|c|c|} \hline & & 4 \\ \hline 1 & 2 & 3 \\ \hline \end{array} \right] \right\rangle \cong \mathbb{F}_3$$

$$H_2(C_*(\text{Sal}(\mathcal{C}_3^2)) \otimes_{\Sigma_4} \mathbb{F}_3(\pm 1)) = \left\langle \left[ \begin{array}{|c|c|} \hline 4 & 3 \\ \hline 3 & 2 \\ \hline 1 & 2 \\ \hline \end{array} - \begin{array}{|c|c|} \hline 3 & 2 \\ \hline 1 & 4 \\ \hline \end{array} \right] \right\rangle \cong \mathbb{F}_3$$

$$H_3(C_*(\text{Sal}(\mathcal{C}_3^2)) \otimes_{\Sigma_4} \mathbb{F}_3(\pm 1)) = 0.$$

*Proof.* The differences are the computation of  $H_2$  and  $H_3$ . The result follows from

$$\text{Ker } \partial_2 = \left\langle \begin{array}{|c|c|} \hline 4 & 3 \\ \hline 3 & 2 \\ \hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 3 & 2 \\ \hline 1 & 4 \\ \hline \end{array} \right\rangle.$$

The details are omitted.

Q.E.D.

**Remark 3.10.** It is well-known that

$$H_*(\Omega^2 S^{2n+1}; \mathbb{F}_3) \cong \Lambda(Q_1^a(x_{2n-1}) \mid a \geq 0) \otimes \mathbb{F}_3[\beta Q_1^{a+1}(x) \mid a \geq 0].$$

Under the Snaith splitting (and dimension shifts), the generators in  $H_1(C_*(\text{Sal}(\mathcal{C}_3^2)) \otimes_{\Sigma_4} \mathbb{F}_3(\pm 1))$  and  $H_2(C_*(\text{Sal}(\mathcal{C}_3^2)) \otimes_{\Sigma_4} \mathbb{F}_3(\pm 1))$  correspond to  $x_{2n-1}\beta Q_1(x_{2n-1})$  and  $x_{2n-1}Q_1(x_{2n-1})$ , respectively, since

$$\begin{aligned} \deg x_{2n-1}\beta Q_1(x_{2n-1}) &= (2n-1) + 3(2n-1) + (p-2) \\ &= 4(2n-1) + 1 \\ \deg x_{2n-1}Q_1(x_{2n-1}) &= (2n-1) + 3(2n-1) + (p-1) \\ &= 4(2n-1) + 2. \end{aligned}$$

The 2-primary case is simpler, since we don't have to worry about the signs. We have

$$\begin{aligned} H_*(C_*(\text{Sal}(\mathcal{A}_3)) \otimes_{\Sigma_4} \mathbb{F}_2(\pm 1)) &= H_*(C_*(\text{Sal}(\mathcal{A}_3)) \otimes_{\Sigma_4} \mathbb{F}_2) \\ &= H_*(C_*(\text{Sal}(\mathcal{A}_3)/\Sigma_4) \otimes \mathbb{F}_2) \\ &= H_*(F(\mathbb{C}, 4)/\Sigma_4; \mathbb{F}_2). \end{aligned}$$

We obtain the following well-known result by elementary calculations. Details are omitted.

**Proposition 3.11.** *The homology  $H_*(F(\mathbb{C}, 4)/\Sigma_4; \mathbb{F}_2)$  has the following description:*

$$\begin{aligned} H_0(F(\mathbb{C}, 4)/\Sigma_4; \mathbb{F}_2) &= \left\langle \boxed{1 \mid 2 \mid 3 \mid 4} \right\rangle \\ &\cong \mathbb{F}_2 \\ H_1(F(\mathbb{C}, 4)/\Sigma_4; \mathbb{F}_2) &= \left\langle \begin{array}{|c|c|c|} \hline 2 & & \\ \hline 1 & 3 & 4 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline & 3 & \\ \hline 1 & 2 & 4 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline & & 4 \\ \hline 1 & 2 & 3 \\ \hline \end{array} \right\rangle \\ &\cong \mathbb{F}_2 \\ H_2(F(\mathbb{C}, 4)/\Sigma_4; \mathbb{F}_2) &= \left\langle \begin{array}{|c|c|} \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array} \right\rangle \\ &\cong \mathbb{F}_2 \\ H_3(F(\mathbb{C}, 4)/\Sigma_4; \mathbb{F}_2) &= \left\langle \begin{array}{|c|} \hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} \right\rangle \\ &\cong \mathbb{F}_2 \end{aligned}$$

**Remark 3.12.** Under the stable splitting

$$\Omega^2 S^{n+2} \underset{S}{\simeq} \bigvee_j F(\mathbb{C}, j)_+ \wedge_{\Sigma_j} (S^n)^{\wedge j},$$

the elements in the mod 2 homology of  $F(\mathbb{C}, 4)/\Sigma_4$ , up to a shift of degree, correspond to elements in  $H_*(\Omega^2 S^{n+2}; \mathbb{F}_2)$  as follows:

$$\begin{aligned} Q_0^2(x) &\longleftrightarrow \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array} \\ Q_0(x)Q_1(x) &\longleftrightarrow \begin{array}{|c|} \hline 2 \\ \hline \end{array} \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline & 3 & \\ \hline 1 & 2 & 4 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline & & 4 \\ \hline 1 & 2 & 3 \\ \hline \end{array} \\ Q_0Q_1(x) &\longleftrightarrow \begin{array}{|c|c|} \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array} \\ Q_1^2(x) &\longleftrightarrow \begin{array}{|c|} \hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} \end{aligned}$$

#### §4. The Center of Mass Configuration

Let us recall the definition of the center of mass configuration space introduced by F. Cohen and Kamiyama in [CK07].

**Definition 4.1.** For  $I, J \subset \{1, \dots, n\}$ , define

$$L_{I,J} = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid |J| \sum_{i \in I} x_i = |I| \sum_{j \in J} x_j \right\}$$

For  $\ell < n$ , define a real central hyperplane arrangement  $\mathcal{C}_{n-1}^\ell$  by

$$\mathcal{C}_{n-1}^\ell = \{L_{I,J} \mid I, J \subset \{1, \dots, n\}, |I| = |J| = p, I \neq J\}.$$

The configuration space of  $n$  points with distinct center of mass of  $\ell$  points is defined as the complement of the complexification of  $\mathcal{C}_{n-1}^\ell$

$$M_\ell(\mathbb{C}, n) = \mathbb{C}^n - \bigcup_{L_{I,J} \in \mathcal{C}_{n-1}^\ell} L_{I,J} \otimes \mathbb{C}$$

As we have seen in §1, we have the following inclusions of arrangements

$$\mathcal{A}_{n-1} = \mathcal{C}_{n-1}^1 \subset \mathcal{C}_{n-1}^2 \subset \dots \subset \mathcal{C}_{n-1}^{\lfloor \frac{n}{2} \rfloor} \supset \dots \supset \mathcal{C}_{n-1}^{n-1} = \mathcal{A}_{n-1}.$$

By Proposition 2.7, we obtain a sequence of maps between the Salvetti complexes

$$\mathrm{Sal}(\mathcal{C}_{n-1}^p) \rightarrow \cdots \rightarrow \mathrm{Sal}(\mathcal{C}_{n-1}^2) \rightarrow \mathrm{Sal}(\mathcal{C}_{n-1}^1) = \mathrm{Sal}(\mathcal{A}_{n-1})$$

when  $p \leq \lfloor \frac{n}{2} \rfloor$ . We also have

$$\mathrm{Sal}(\mathcal{C}_{n-1}^p) = \mathrm{Sal}(\mathcal{C}_{n-1}^{n-p}) \rightarrow \cdots \rightarrow \mathrm{Sal}(\mathcal{C}_{n-1}^2) \rightarrow \mathrm{Sal}(\mathcal{C}_{n-1}^1) = \mathrm{Sal}(\mathcal{A}_{n-1})$$

when  $p > \lfloor \frac{n}{2} \rfloor$ .

We would like to know if these inclusions induce isomorphisms of homology groups with coefficients in  $\mathbb{F}_p(\pm 1)$ . Our strategy is to compute the homology of the kernel of the map

$$i_n^k : C_*(\mathrm{Sal}(\mathcal{C}_{n-1}^k)) \otimes_{\Sigma_n} \mathbb{F}_p(\pm 1) \longrightarrow C_*(\mathrm{Sal}(\mathcal{C}_{n-1}^{k-1})) \otimes_{\Sigma_n} \mathbb{F}_p(\pm 1),$$

for  $k \leq \lfloor \frac{n}{2} \rfloor$ . By Lemma 2.13, these chain maps are surjective.

**Corollary 4.2.** *For  $k \leq \lfloor \frac{n}{2} \rfloor$ , the map*

$$i_n^k : C_*(\mathrm{Sal}(\mathcal{C}_{n-1}^k)) \otimes_{\Sigma_n} \mathbb{F}_p(\pm 1) \longrightarrow C_*(\mathrm{Sal}(\mathcal{C}_{n-1}^{k-1})) \otimes_{\Sigma_n} \mathbb{F}_p(\pm 1),$$

*is surjective.*

In the rest of this article, we consider the first stage, i.e.

$$\begin{aligned} i_n^2 : C_*(\mathrm{Sal}(\mathcal{C}_{n-1}^2)) \otimes_{\Sigma_n} \mathbb{F}_p(\pm 1) &\longrightarrow \\ C_*(\mathrm{Sal}(\mathcal{C}_{n-1}^1)) \otimes_{\Sigma_n} \mathbb{F}_p(\pm 1) &= C_*(\mathrm{Sal}(\mathcal{A}_{n-1})) \otimes_{\Sigma_n} \mathbb{F}_p(\pm 1). \end{aligned}$$

When  $n = 3$ ,

$$\mathcal{C}_2^2 = \mathcal{C}_2^{3-2} = \mathcal{C}_2^1 = \mathcal{A}_2$$

and there is nothing to compute. The first nontrivial case is

$$\begin{aligned} i_4^2 : C_*(\mathrm{Sal}(\mathcal{C}_3^2)) \otimes_{\Sigma_4} \mathbb{F}_p(\pm 1) &\longrightarrow \\ C_*(\mathrm{Sal}(\mathcal{C}_3^1)) \otimes_{\Sigma_4} \mathbb{F}_p(\pm 1) &= C_*(\mathrm{Sal}(\mathcal{A}_3)) \otimes_{\Sigma_4} \mathbb{F}_p(\pm 1). \end{aligned}$$

By Corollary 4.2, it suffices to calculate the kernel of  $i_4^2$  in order to compare  $H_*(C_*(\mathrm{Sal}(\mathcal{C}_3^2)) \otimes_{\Sigma_4} \mathbb{F}_p(\pm 1))$  and  $H_*(C_*(\mathrm{Sal}(\mathcal{A}_3)) \otimes_{\Sigma_4} \mathbb{F}_p(\pm 1))$ .

**Definition 4.3.** We denote

$$K_*^{n,k} = \mathrm{Ker}(i_n^k : C_*(\mathrm{Sal}(\mathcal{C}_{n-1}^k)) \otimes_{\Sigma_n} \mathbb{F}_p(\pm 1) \rightarrow C_*(\mathrm{Sal}(\mathcal{C}_{n-1}^{k-1})) \otimes_{\Sigma_n} \mathbb{F}_p(\pm 1)).$$

For simplicity, we also abbreviate

$$C_*^{n,k} = C_*(\mathrm{Sal}(\mathcal{C}_{n-1}^k)) \otimes_{\Sigma_n} \mathbb{F}_p(\pm 1).$$

Thus we have a short exact sequence of chain complexes

$$0 \longrightarrow K_*^{n,k} \longrightarrow C_*^{n,k} \xrightarrow{i_n^k} C_*^{n,k-1} \longrightarrow 0.$$

#### 4.1. The Face Poset of $\mathcal{C}_3^2$

In order to compute  $H_*(\text{Sal}(\mathcal{C}_3^2) \otimes_{\Sigma_4} \mathbb{F}_p(\pm 1))$ , the first step is to determine the face poset of  $\mathcal{C}_3^2$ .

Since  $\mathcal{C}_3^2 = \mathcal{A}_3 \cup \{L_{\{1,2\},\{3,4\}}, L_{\{1,3\},\{2,4\}}, L_{\{1,4\},\{2,3\}}\}$ , the faces of  $\mathcal{C}_3^2$  are given by splitting the faces of  $\mathcal{A}_3$  by the hyperplanes

$$L_{\{1,2\},\{3,4\}}, L_{\{1,3\},\{2,4\}}, L_{\{1,4\},\{2,3\}}.$$

In order to understand these cuttings, let us see how the chamber

$$\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 < x_2 < x_3 < x_4\}$$

is cut. Notice that under the action of  $\Sigma_4$ , the cells in  $M_2(\mathbb{C}, 4)$  can be represented by cells related to this chamber of  $\mathcal{A}_3$ .

The only hyperplane among  $L_{\{1,2\},\{3,4\}}, L_{\{1,3\},\{2,4\}}, L_{\{1,4\},\{2,3\}}$  that intersects with this chamber is  $L_{\{1,4\},\{2,3\}}$  and the chamber is cut into two pieces:

$$\begin{aligned} & \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 < x_2 < x_3 < x_4\} \\ &= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 < x_2 < x_3 < x_4, x_1 + x_4 \leq x_2 + x_3\} \\ & \cup \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 < x_2 < x_3 < x_4, x_1 + x_4 \geq x_2 + x_3\}. \end{aligned}$$

We denote these chambers by the following symbols:

$$\begin{array}{|c|c|c|c|} \hline 1 & & 2 & 3 & 4 \\ \hline \end{array} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 < x_2 < x_3 < x_4, x_1 + x_4 < x_2 + x_3\}$$

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & & 4 \\ \hline \end{array} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 < x_2 < x_3 < x_4, x_1 + x_4 > x_2 + x_3\}.$$

The faces of these chambers of  $\mathcal{C}_3^2$  are also denoted by analogous symbols. The chamber  $\begin{array}{|c|c|c|c|} \hline 1 & & 2 & 3 & 4 \\ \hline \end{array}$  has five 3-dimensional faces, but under the action of  $\Sigma_4$ , we only need the following three faces:

$$\begin{array}{|c|c|c|} \hline & & 4 \\ \hline 1 & & 2 & 3 \\ \hline \end{array} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 < x_2 < x_3 = x_4\},$$

$$\begin{array}{|c|c|} \hline & 3 \\ \hline 1 & & 2 & 4 \\ \hline \end{array} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 < x_2 = x_3 < x_4, x_1 + x_4 < x_2 + x_3\},$$

1	2		3	4
---	---	--	---	---

$$= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 < x_2 < x_3 < x_4, x_1 + x_4 = x_2 + x_3\}.$$

Similarly, we need the following three faces for the chamber 

1	2	3		4
---	---	---	--	---

.

2				
1	3		4	

$$= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 = x_2 < x_3 < x_4\},$$

	3			
1	2		4	

$$= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 < x_2 = x_3 < x_4, x_1 + x_4 > x_2 + x_3\},$$

1	2		3	4
---	---	--	---	---

$$= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 < x_2 < x_3 < x_4, x_1 + x_4 = x_2 + x_3\}.$$

The 2-dimensional faces we need are the following:

		4		
		3		
1		2		

$$= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 < x_2 = x_3 = x_4\},$$

2		4		
1		3		

$$= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 = x_2 < x_3 = x_4, x_1 + x_4 = x_2 + x_3\},$$

		3		
1		2		4

$$= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 < x_2 = x_3 < x_4, x_1 + x_4 = x_2 + x_3\},$$

3				
2				
1		4		

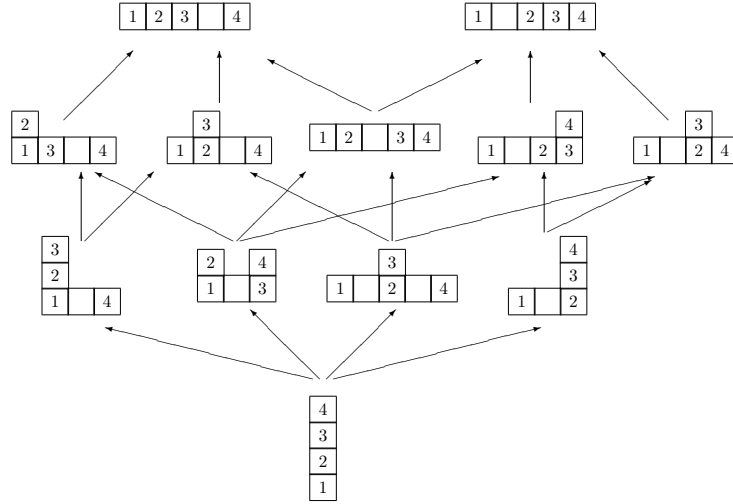
$$= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 = x_2 = x_3 < x_4\}.$$

All these faces have the following 1-dimensional face in common.

$$\begin{array}{|c|} \hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 = x_2 = x_3 = x_4\}.$$

Notice that  $\Sigma_4$  acts on  $\mathcal{L}(\mathcal{C}_3^2)$  and the action is compatible with the ordering. We have the following description of the poset  $\mathcal{L}(\mathcal{C}_3^2)/\Sigma_4$ .

**Lemma 4.4.** *The poset  $\mathcal{L}(\mathcal{C}_3^2)/\Sigma_4$  has the following structure:*



#### 4.2. The Cellular Structure on $\text{Sal}(\mathcal{C}_3^2)/\Sigma_4$

Let us determine the cellular structure of  $\text{Sal}(\mathcal{C}_3^2)/\Sigma_4$ . The cell decomposition of the Salvetti complex for  $\mathcal{C}_3^2$  is compatible with the action of  $\Sigma_4$  and the quotient  $\text{Sal}(\mathcal{C}_3^2)/\Sigma_4$  has the induced cell decomposition.

The cells of the Salvetti complex are labeled by pairs of a face  $F$  and a chamber  $C$  with  $C \geq F$ . The cell for the pair  $(F, C)$  is denoted by  $D(F, C)$  in §2.1. Thus the cells of  $\text{Sal}(\mathcal{C}_3^2)/\Sigma_4$  are in one-to-one correspondence with elements in

$$\{([F], [C]) \mid [F] \in \mathcal{L}(\mathcal{C}_3^2)/\Sigma_4, [C] \in \mathcal{L}^{(0)}(\mathcal{C}_3^2)/\Sigma_4, F \leq C\}.$$

In the case of  $\mathcal{C}_3^2$ , there are only two chambers in  $\mathcal{L}^{(0)}(\mathcal{C}_3^2)/\Sigma_4$ , and we denote the cells corresponding to the pair  $\left([F], \left[\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array}\right]\right)$

and  $([F], \left[ \left[ \begin{array}{|c|c|c|c|c|} \hline 1 & & 2 & 3 & 4 \\ \hline \end{array} \right] \right])$  by  $F^+$  and  $F^-$ , respectively. To be more efficient, we simply denote them by  $F$  when  $F$  is contained in only one chamber.

More explicitly,

**Lemma 4.5.**  $\text{Sal}(\mathcal{C}_3^2)/\Sigma_4$  has

- two 0-cells

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & & 4 \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline 1 & & 2 & 3 & 4 \\ \hline \end{array}.$$

- six 1-cells

$$\begin{array}{|c|c|c|c|} \hline & & 3 & \\ \hline 1 & & 2 & 4 \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline & & & 4 \\ \hline 1 & & 2 & 3 \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & & 3 & 4 \\ \hline \end{array}^+,$$

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 2 & & 3 & 4 \\ \hline \end{array}^-, \begin{array}{|c|c|c|c|} \hline & & 3 & \\ \hline 1 & 2 & & 4 \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline & & 2 & \\ \hline 1 & 3 & & 4 \\ \hline \end{array}.$$

- six 2-cells

$$\begin{array}{|c|c|c|} \hline & & 4 \\ \hline & & 3 \\ \hline 1 & & 2 \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline & & 3 & \\ \hline 1 & & 2 & 4 \\ \hline \end{array}^+, \begin{array}{|c|c|c|c|} \hline & & 3 & \\ \hline 1 & & 2 & 4 \\ \hline \end{array}^-,$$

$$\begin{array}{|c|c|c|} \hline 2 & & 4 \\ \hline 1 & & 3 \\ \hline \end{array}^+, \begin{array}{|c|c|c|} \hline 2 & & 4 \\ \hline 1 & & 3 \\ \hline \end{array}^-, \begin{array}{|c|c|c|} \hline 3 & & \\ \hline 2 & & \\ \hline 1 & & 4 \\ \hline \end{array}.$$

- two 3-cells

$$\begin{array}{|c|} \hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}^+, \begin{array}{|c|} \hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}^-.$$

We use above symbols as representatives of cells in  $\text{Sal}(\mathcal{C}_3^2)$ . We define orientations on these cells and then transfer orientations to other cells in  $\text{Sal}(\mathcal{C}_3^2)$  via the action of  $\Sigma_4$ . Those cells which are mapped to cells of the same dimensions in  $\text{Sal}(\mathcal{A}_3)$  by the map  $i_4^2$  are oriented in such a way  $i_4^2$  is orientation preserving. Then remaining four types of cells

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 2 & & 3 & 4 \\ \hline \end{array}^+, \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & & 3 & 4 \\ \hline \end{array}^-, \begin{array}{|c|c|c|c|} \hline & & 3 & \\ \hline 1 & & 2 & 4 \\ \hline \end{array}^+,$$

$$\begin{array}{|c|c|c|c|} \hline & & 3 & \\ \hline 1 & & 2 & 4 \\ \hline \end{array}^-.$$



are oriented as follows. As we will see below, the first two 1-cells have  $\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array}$  and  $\begin{array}{|c|c|c|c|} \hline 1 & & 2 & 3 & 4 \\ \hline \end{array}$  as vertices. We orient these 1-cells from  $\begin{array}{|c|c|c|c|} \hline 1 & & 2 & 3 & 4 \\ \hline \end{array}$  to  $\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & & 4 \\ \hline \end{array}$ . The 2-cells

$\begin{array}{|c|c|c|c|} \hline & 3 & & \\ \hline 1 & & 2 & & 4 \\ \hline \end{array}$  and  $\begin{array}{|c|c|c|c|} \hline & & 3 & & \\ \hline 1 & & 2 & & 4 \\ \hline \end{array}$  contains  $\begin{array}{|c|c|c|c|} \hline & 3 & & \\ \hline 1 & 2 & & & 4 \\ \hline \end{array}$  in the boundary, as we will see later. We orient these 2-cells in such a way the incidence number to this 1-cell is positive.

Now we are ready to consider the boundaries. This can be done by using the formula for the boundary in Lemma 2.2 and a formula analogous to the case of the braid arrangement (Lemma 3.7).

The first nontrivial case is the boundaries of 1-cells.

**Lemma 4.6.** *We have the following formula in  $C_*^{4,2}$ .*

$$\begin{aligned} \partial_1 \left( \begin{array}{|c|c|c|c|} \hline & 3 & & \\ \hline 1 & & 2 & 4 \\ \hline \end{array} \right) &= 2 \begin{array}{|c|c|c|c|} \hline 1 & & 2 & 3 & 4 \\ \hline \end{array} \\ \partial_1 \left( \begin{array}{|c|c|c|c|} \hline & & & 4 \\ \hline 1 & & 2 & 3 \\ \hline \end{array} \right) &= 2 \begin{array}{|c|c|c|c|} \hline 1 & & 2 & 3 & 4 \\ \hline \end{array} \\ \partial_1 \left( \begin{array}{|c|c|c|c|} \hline 1 & 2 & & 3 & 4 \\ \hline \end{array} \right)^+ &= \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & & 4 \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline 1 & & 2 & 3 & 4 \\ \hline \end{array} \\ \partial_1 \left( \begin{array}{|c|c|c|c|} \hline 1 & 2 & & 3 & 4 \\ \hline \end{array} \right)^- &= \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & & 4 \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline 1 & & 2 & 3 & 4 \\ \hline \end{array} \\ \partial_1 \left( \begin{array}{|c|c|c|c|} \hline & 3 & & \\ \hline 1 & 2 & & 4 \\ \hline \end{array} \right) &= 2 \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & & 4 \\ \hline \end{array} \\ \partial_1 \left( \begin{array}{|c|c|c|c|} \hline 2 & & & \\ \hline 1 & 3 & & 4 \\ \hline \end{array} \right) &= 2 \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & & 4 \\ \hline \end{array}. \end{aligned}$$

*Proof.* By Lemma 2.2

$$\partial_1 \left( \begin{array}{|c|c|c|c|} \hline & 3 & & \\ \hline 1 & & 2 & 4 \\ \hline \end{array} \right) = \sum_F \varepsilon_F D \left( F, F \circ \begin{array}{|c|c|c|c|} \hline 1 & & 2 & 3 & 4 \\ \hline \end{array} \right)$$

where  $F$  runs over all faces containing  $\begin{array}{|c|c|c|c|} \hline & 3 & & \\ \hline 1 & & 2 & 4 \\ \hline \end{array}$  in  $\mathcal{L}(C_3^2)$  and

$$\varepsilon_F = \left[ \begin{array}{|c|c|c|c|} \hline & 3 & & \\ \hline 1 & & 2 & 4 \\ \hline \end{array} : D \left( F, F \circ \begin{array}{|c|c|c|c|} \hline 1 & & 2 & 3 & 4 \\ \hline \end{array} \right) \right]$$

In this case, we have

$$\begin{aligned} & \partial_1 \left( \begin{array}{|c|c|c|} \hline & & 3 \\ \hline 1 & & 2 & 4 \\ \hline \end{array} \right) \\ &= \varepsilon_1 D \left( \begin{array}{|c|c|c|c|c|} \hline 1 & & 2 & 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline 1 & & 2 & 3 & 4 \\ \hline \end{array} \circ \begin{array}{|c|c|c|c|c|} \hline 1 & & 2 & 3 & 4 \\ \hline \end{array} \right) \\ & \quad + \varepsilon_2 D \left( \begin{array}{|c|c|c|c|c|} \hline 1 & & 3 & 2 & 4 \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline 1 & & 3 & 2 & 4 \\ \hline \end{array} \circ \begin{array}{|c|c|c|c|c|} \hline 1 & & 2 & 3 & 4 \\ \hline \end{array} \right), \end{aligned}$$

where  $\varepsilon_1, \varepsilon_2$  are appropriate incidence numbers.

In order to compute the boundaries, therefore, we need to understand the matroid product in the face lattice  $\mathcal{L}(\mathcal{C}_3^2)$ .

As we have recalled in §2, the cellular structure of the Salvetti complex was originally described by face-chamber pairings. For computations, however, it is much more convenient to regard faces as functions from the set of normal vectors to the poset of three elements  $S_1 = \{0, +1, -1\}$  and use the matroid product of these functions, as we have seen in Lemma 2.3 and 2.4.

We choose the following set of normal vectors for the arrangement  $\mathcal{C}_3^2$ :

$$\begin{aligned} \mathbf{a}_1 &= (1, -1, 0, 0) \\ \mathbf{a}_2 &= (0, 1, -1, 0) \\ \mathbf{a}_3 &= (0, 0, 1, -1) \\ \mathbf{a}_4 &= (1, 0, -1, 0) \\ \mathbf{a}_5 &= (1, 0, 0, -1) \\ \mathbf{a}_6 &= (0, 1, 0, -1) \\ \mathbf{a}_7 &= (1, 1, -1, -1) \\ \mathbf{a}_8 &= (1, -1, 1, -1) \\ \mathbf{a}_9 &= (1, -1, -1, 1). \end{aligned}$$

Then a face  $F \in \mathcal{L}(\mathcal{C}_3^2)$  can be regarded as a function

$$\tau_F : \{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_9\} \longrightarrow S_1.$$

For simplicity, we denote elements  $+1, -1$  in  $S_1$  by  $+, -$ , respectively, and denote the above function by the symbol

$$(\tau_F(\mathbf{a}_1), \tau_F(\mathbf{a}_2), \dots, \tau_F(\mathbf{a}_6) \mid \tau_F(\mathbf{a}_7), \tau_F(\mathbf{a}_8), \tau_F(\mathbf{a}_9)).$$

For example, the face  $\begin{array}{|c|c|c|c|} \hline 1 & & 2 & 3 & 4 \\ \hline \end{array}$  corresponds to the symbol

$$(-, -, -, -, -, - \mid -, -, -).$$

With these notations, the matroid product of a face and a chamber is given by replacing 0's in the face by the values at the same position in the chamber. For example,

$$\begin{aligned}
& \begin{array}{|c|c|c|c|c|} \hline 1 & & 3 & 2 & 4 \\ \hline \end{array} \circ \begin{array}{|c|c|c|c|c|} \hline 1 & & 2 & 3 & 4 \\ \hline \end{array} \\
&= (-, +, -, -, -, - \mid -, -, -) \circ (-, -, -, -, -, - \mid -, -, -) \\
&= (-, +, -, -, -, - \mid -, -, -) \\
&= \begin{array}{|c|c|c|c|c|} \hline 1 & & 3 & 2 & 4 \\ \hline \end{array}.
\end{aligned}$$

Thus, by taking the orientations into an account, we have

$$\begin{aligned}
\partial_1 \left( \begin{array}{|c|c|c|c|} \hline & & 3 & \\ \hline 1 & & 2 & 4 \\ \hline \end{array} \right) &= \begin{array}{|c|c|c|c|c|} \hline 1 & & 2 & 3 & 4 \\ \hline \end{array} - \begin{array}{|c|c|c|c|c|} \hline 1 & & 3 & 2 & 4 \\ \hline \end{array} \\
&= 2 \begin{array}{|c|c|c|c|c|} \hline 1 & & 2 & 3 & 4 \\ \hline \end{array}.
\end{aligned}$$

By analogous calculations, we obtain

$$\begin{aligned}
\partial_1 \left( \begin{array}{|c|c|c|c|} \hline & & & 4 \\ \hline 1 & & 2 & 3 \\ \hline \end{array} \right) &= 2 \begin{array}{|c|c|c|c|c|} \hline 1 & & 2 & 3 & 4 \\ \hline \end{array} \\
\partial_1 \left( \begin{array}{|c|c|c|c|} \hline & 3 & & \\ \hline 1 & 2 & & 4 \\ \hline \end{array} \right) &= 2 \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & & 4 \\ \hline \end{array} \\
\partial_1 \left( \begin{array}{|c|c|c|c|} \hline 2 & & & \\ \hline 1 & 3 & & 4 \\ \hline \end{array} \right) &= 2 \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & & 4 \\ \hline \end{array}.
\end{aligned}$$

On the other hand,  $\begin{array}{|c|c|c|c|} \hline 1 & 2 & & 3 & 4 \\ \hline \end{array}^+$  has

$$D \left( \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & & 4 \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & & 4 \\ \hline \end{array} \circ \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & & 4 \\ \hline \end{array} \right)$$

and

$$D \left( \begin{array}{|c|c|c|c|c|} \hline 1 & & 2 & 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline 1 & & 2 & 3 & 4 \\ \hline \end{array} \circ \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & & 4 \\ \hline \end{array} \right)$$

as its boundary. By computing the matroid products, we see these 0-cells are  $\begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & & 4 \\ \hline \end{array}$  and  $\begin{array}{|c|c|c|c|c|} \hline 1 & & 2 & 3 & 4 \\ \hline \end{array}$ , respectively. By the definition of the orientation, we obtain

$$\begin{aligned}
\partial_1 \left( \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & & 3 & 4 \\ \hline \end{array}^+ \right) &= \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & & 4 \\ \hline \end{array} - \begin{array}{|c|c|c|c|c|} \hline 1 & & 2 & 3 & 4 \\ \hline \end{array} \\
\partial_1 \left( \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & & 3 & 4 \\ \hline \end{array}^- \right) &= \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & & 4 \\ \hline \end{array} - \begin{array}{|c|c|c|c|c|} \hline 1 & & 2 & 3 & 4 \\ \hline \end{array}.
\end{aligned}$$

Q.E.D.

**Remark 4.7.** Note that in the above proof, the computation of  $\partial_1 \left( \begin{array}{|c|c|c|} \hline & 3 & \\ \hline 1 & & 2 & 4 \\ \hline \end{array} \right)$  is exactly the same as that of  $\partial_1 \left( \begin{array}{|c|c|c|} \hline & 3 & \\ \hline 1 & 2 & 4 \\ \hline \end{array} \right)$  in  $C_*(\text{Sal}(\mathcal{A}_3)) \otimes_{\Sigma_4} \mathbb{F}_p(\pm 1)$ .

In general, boundaries of those cells which do not have + or - sign on the shoulder can be computed by the same formulas for the corresponding cells in the braid arrangement.

Let us consider 2-cells next.

**Lemma 4.8.** *We have the following formula in  $C_*^{4,2}$ .*

$$\begin{aligned}
\partial_2 \left( \begin{array}{|c|c|c|} \hline & & 4 \\ \hline & 3 & \\ \hline 1 & & 2 \\ \hline \end{array} \right) &= 3 \begin{array}{|c|c|c|} \hline & & 4 \\ \hline 1 & & 2 & 3 \\ \hline \end{array} - 3 \begin{array}{|c|c|c|} \hline & 3 & \\ \hline 1 & & 2 & 4 \\ \hline \end{array} \\
\partial_2 \left( \begin{array}{|c|c|c|} \hline & 3 & \\ \hline 1 & & 2 & 4 \\ \hline \end{array} \right)^+ &= - \begin{array}{|c|c|c|} \hline & 3 & \\ \hline 1 & & 2 & 4 \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & 3 & \\ \hline 1 & 2 & & 4 \\ \hline \end{array} \\
&\quad - 2 \begin{array}{|c|c|c|} \hline 1 & 2 & & 3 & 4 \\ \hline \end{array}^+ \\
\partial_2 \left( \begin{array}{|c|c|c|} \hline & 3 & \\ \hline 1 & & 2 & 4 \\ \hline \end{array} \right)^- &= - \begin{array}{|c|c|c|} \hline & 3 & \\ \hline 1 & & 2 & 4 \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & 3 & \\ \hline 1 & 2 & & 4 \\ \hline \end{array} \\
&\quad - 2 \begin{array}{|c|c|c|} \hline 1 & 2 & & 3 & 4 \\ \hline \end{array}^- \\
\partial_2 \left( \begin{array}{|c|c|c|} \hline 2 & & 4 \\ \hline 1 & & 3 \\ \hline \end{array} \right)^+ &= 2 \begin{array}{|c|c|c|} \hline & & 4 \\ \hline 1 & & 2 & 3 \\ \hline \end{array} - 2 \begin{array}{|c|c|c|} \hline 2 & & \\ \hline 1 & 3 & & 4 \\ \hline \end{array} \\
&\quad + 3 \begin{array}{|c|c|c|} \hline 1 & 2 & & 3 & 4 \\ \hline \end{array}^+ + \begin{array}{|c|c|c|} \hline 1 & 2 & & 3 & 4 \\ \hline \end{array}^- \\
\partial_2 \left( \begin{array}{|c|c|c|} \hline 2 & & 4 \\ \hline 1 & & 3 \\ \hline \end{array} \right)^- &= 2 \begin{array}{|c|c|c|} \hline & & 4 \\ \hline 1 & & 2 & 3 \\ \hline \end{array} - 2 \begin{array}{|c|c|c|} \hline 2 & & \\ \hline 1 & 3 & & 4 \\ \hline \end{array} \\
&\quad + \begin{array}{|c|c|c|} \hline 1 & 2 & & 3 & 4 \\ \hline \end{array}^+ + 3 \begin{array}{|c|c|c|} \hline 1 & 2 & & 3 & 4 \\ \hline \end{array}^- \\
\partial_2 \left( \begin{array}{|c|c|c|} \hline 3 & & \\ \hline 2 & & \\ \hline 1 & & 4 \\ \hline \end{array} \right) &= 3 \begin{array}{|c|c|c|} \hline & 3 & \\ \hline 1 & 2 & & 4 \\ \hline \end{array} - 3 \begin{array}{|c|c|c|} \hline 2 & & \\ \hline 1 & 3 & & 4 \\ \hline \end{array}.
\end{aligned}$$

*Proof.* The computations of  $\partial_2 \left( \begin{array}{c} 4 \\ 3 \\ 1 \quad 2 \end{array} \right)$  and  $\partial_2 \left( \begin{array}{c} 3 \\ 2 \\ 1 \quad 4 \end{array} \right)$  are essentially the same as those of  $\partial_2 \left( \begin{array}{c} 4 \\ 3 \\ 1 \quad 2 \end{array} \right)$  and  $\partial_2 \left( \begin{array}{c} 3 \\ 2 \\ 1 \quad 4 \end{array} \right)$  in §3.2 as is noted in Remark 4.7 and are omitted.

There are four faces in  $\mathcal{L}(\mathcal{C}_3^2)$  that contain  $\begin{array}{c} 3 \\ 1 \quad 2 \quad 4 \end{array}$  as faces  
Thus

$$\begin{aligned}
& \partial_2 \left( \begin{array}{c} 3 \\ 1 \quad 2 \quad 4 \end{array} \right)^+ \\
&= \varepsilon_1 D \left( \begin{array}{c} 3 \\ 1 \quad 2 \quad 4 \end{array}, \begin{array}{c} 3 \\ 1 \quad 2 \quad 4 \end{array} \circ \begin{array}{c} 1 \quad 2 \quad 3 \quad 4 \end{array} \right) \\
&+ \varepsilon_2 D \left( \begin{array}{c} 3 \\ 1 \quad 2 \quad 4 \end{array}, \begin{array}{c} 3 \\ 1 \quad 2 \quad 4 \end{array} \circ \begin{array}{c} 1 \quad 2 \quad 3 \quad 4 \end{array} \right) \\
&+ \varepsilon_3 D \left( \begin{array}{c} 1 \quad 2 \quad 3 \quad 4 \end{array}, \begin{array}{c} 1 \quad 2 \quad 3 \quad 4 \end{array} \circ \begin{array}{c} 1 \quad 2 \quad 3 \quad 4 \end{array} \right) \\
&+ \varepsilon_4 D \left( \begin{array}{c} 1 \quad 3 \quad 2 \quad 4 \end{array}, \begin{array}{c} 1 \quad 3 \quad 2 \quad 4 \end{array} \circ \begin{array}{c} 1 \quad 2 \quad 3 \quad 4 \end{array} \right) \\
&= \varepsilon_1 D \left( \begin{array}{c} 3 \\ 1 \quad 2 \quad 4 \end{array}, \begin{array}{c} 1 \quad 2 \quad 3 \quad 4 \end{array} \right) \\
&+ \varepsilon_2 D \left( \begin{array}{c} 3 \\ 1 \quad 2 \quad 4 \end{array}, \begin{array}{c} 1 \quad 2 \quad 3 \quad 4 \end{array} \right) \\
&+ (\varepsilon_3 + \varepsilon_4) D \left( \begin{array}{c} 1 \quad 2 \quad 3 \quad 4 \end{array}, \begin{array}{c} 1 \quad 2 \quad 3 \quad 4 \end{array} \right) \\
&= \varepsilon_1 \begin{array}{c} 3 \\ 1 \quad 2 \quad 4 \end{array} + \varepsilon_2 \begin{array}{c} 3 \\ 1 \quad 2 \quad 4 \end{array} + (\varepsilon_3 + \varepsilon_4) \begin{array}{c} 1 \quad 2 \quad 3 \quad 4 \end{array}^+,
\end{aligned}$$

where  $\varepsilon_i \in \{\pm 1\}$  ( $i = 1, 2, 3, 4$ ) are certain signs. By the definition of the orientation of  $\begin{array}{c} 3 \\ 1 \quad 2 \quad 4 \end{array}^+$ , we have  $\varepsilon_2 = 1$ . Since  $\partial_1 \circ \partial_2 = 0$ , we see that  $\varepsilon_1 = -1$  and  $\varepsilon_3 + \varepsilon_4 = 2$ .

By the same calculation, we also have

$$\partial_2 \left( \begin{array}{|c|c|c|c|} \hline & 3 & & \\ \hline 1 & & 2 & 4 \\ \hline \end{array} \right)^{-} =$$

$$- \begin{array}{|c|c|c|c|} \hline & 3 & & \\ \hline 1 & & 2 & 4 \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline & 3 & & \\ \hline 1 & 2 & & 4 \\ \hline \end{array} + 2 \begin{array}{|c|c|c|c|} \hline 1 & 2 & & 3 & 4 \\ \hline \end{array}^{-}.$$

Finally, we have

$$\partial_2 \left( \begin{array}{|c|c|c|} \hline 2 & & 4 \\ \hline 1 & & 3 \\ \hline \end{array} \right)^{+}$$

$$= -2D \left( \begin{array}{|c|c|c|c|} \hline 2 & & & \\ \hline 1 & 3 & & 4 \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline 2 & & & \\ \hline 1 & 3 & & 4 \\ \hline \end{array} \circ \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array} \right)$$

$$+ D \left( \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & & 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & & 3 & 4 \\ \hline \end{array} \circ \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array} \right)$$

$$- D \left( \begin{array}{|c|c|c|c|c|} \hline 2 & 1 & & 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline 2 & 1 & & 3 & 4 \\ \hline \end{array} \circ \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array} \right)$$

$$- D \left( \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & & 4 & 3 \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & & 4 & 3 \\ \hline \end{array} \circ \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array} \right)$$

$$+ D \left( \begin{array}{|c|c|c|c|c|} \hline 2 & 1 & & 4 & 3 \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline 2 & 1 & & 4 & 3 \\ \hline \end{array} \circ \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array} \right)$$

$$+ 2D \left( \begin{array}{|c|c|c|c|} \hline & & & 4 \\ \hline 1 & 2 & & 3 \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline & & & 4 \\ \hline 1 & 2 & & 3 \\ \hline \end{array} \circ \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array} \right)$$

$$= -2 \begin{array}{|c|c|c|c|} \hline 2 & & & \\ \hline 1 & 3 & & 4 \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline 1 & 2 & & 3 & 4 \\ \hline \end{array}^{+} - \begin{array}{|c|c|c|c|} \hline 2 & 1 & & 3 & 4 \\ \hline \end{array}^{+}$$

$$- \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & & 4 & 3 \\ \hline \end{array}^{-} + \begin{array}{|c|c|c|c|c|} \hline 2 & 1 & & 4 & 3 \\ \hline \end{array}^{+} + 2 \begin{array}{|c|c|c|c|} \hline & & & 4 \\ \hline 1 & 2 & & 3 \\ \hline \end{array}$$

$$= -2 \begin{array}{|c|c|c|c|} \hline 2 & & & \\ \hline 1 & 3 & & 4 \\ \hline \end{array} - + 2 \begin{array}{|c|c|c|c|} \hline & & & 4 \\ \hline 1 & 2 & & 3 \\ \hline \end{array}$$

$$+ 3 \begin{array}{|c|c|c|c|} \hline 1 & 2 & & 3 & 4 \\ \hline \end{array}^{+} + \begin{array}{|c|c|c|c|} \hline 1 & 2 & & 3 & 4 \\ \hline \end{array}^{-},$$

where coefficients (signs) of the first two terms are determined by comparing with the braid arrangement. The coefficients of the last two terms are determined in the same way as in the previous calculation.

The calculation of  $\partial_2 \left( \begin{array}{|c|c|c|} \hline 2 & & 4 \\ \hline 1 & & 3 \\ \hline \end{array} \right)^{-}$  can be done in the same manner. Q.E.D.

It remains to compute boundaries on 3-cells.

**Lemma 4.9.** *We have*

$$\begin{aligned} \partial_3 \left( \begin{array}{|c|} \hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} \right)^+ &= 4 \begin{array}{|c|c|c|} \hline & 4 & \\ \hline & 3 & \\ \hline 1 & & 2 \\ \hline \end{array} + 4 \begin{array}{|c|c|c|} \hline 3 & & \\ \hline 2 & & \\ \hline 1 & & 4 \\ \hline \end{array} \\ &- 5 \begin{array}{|c|c|c|} \hline 2 & & 4 \\ \hline 1 & & 3 \\ \hline \end{array}^+ - 5 \begin{array}{|c|c|c|} \hline 2 & & 4 \\ \hline 1 & & 3 \\ \hline \end{array}^- \\ &+ 8 \begin{array}{|c|c|c|c|} \hline & 3 & & \\ \hline 1 & & 2 & & 4 \\ \hline \end{array} + 4 \begin{array}{|c|c|c|c|} \hline & & 3 & \\ \hline 1 & & 2 & & 4 \\ \hline \end{array}^-, \\ \partial_3 \left( \begin{array}{|c|} \hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} \right)^- &= 4 \begin{array}{|c|c|c|} \hline & 4 & \\ \hline & 3 & \\ \hline 1 & & 2 \\ \hline \end{array} + 4 \begin{array}{|c|c|c|} \hline 3 & & \\ \hline 2 & & \\ \hline 1 & & 4 \\ \hline \end{array} \\ &- \begin{array}{|c|c|c|} \hline 2 & & 4 \\ \hline 1 & & 3 \\ \hline \end{array}^+ - 5 \begin{array}{|c|c|c|} \hline 2 & & 4 \\ \hline 1 & & 3 \\ \hline \end{array}^- \\ &+ 4 \begin{array}{|c|c|c|c|} \hline & 3 & & \\ \hline 1 & & 2 & & 4 \\ \hline \end{array} + 8 \begin{array}{|c|c|c|c|} \hline & & 3 & \\ \hline 1 & & 2 & & 4 \\ \hline \end{array}^-. \end{aligned}$$

*Proof.* Let us consider  $\partial_3 \left( \begin{array}{|c|} \hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} \right)^+$ . There are many faces contain-

ing  $\begin{array}{|c|} \hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}$ . For example, there are  $\binom{4}{2} = 6$  faces of the shape  $\begin{array}{|c|c|c|} \hline 2 & & 4 \\ \hline 1 & & 3 \\ \hline \end{array}$ .

Matroid products can be computed, for example,

$$\begin{aligned}
& \begin{array}{|c|c|} \hline 3 & 4 \\ \hline 2 & 1 \\ \hline \end{array} \circ \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array} \\
&= (+, 0, -, +, 0, - \mid 0, 0, +) \circ (-, -, -, -, -, - \mid -, -, +) \\
&= (+, -, -, +, -, - \mid -, -, +) \\
&= \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid \begin{array}{l} x_1 > x_2, x_2 < x_3, x_3 < x_4, \\ x_1 > x_3, x_1 < x_4, x_2 < x_4, \\ x_1 + x_2 < x_3 + x_4, \\ x_1 + x_3 < x_2 + x_4, \\ x_1 + x_4 > x_2 + x_3 \end{array} \right\} \\
&= \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid \begin{array}{l} x_2 < x_3 < x_1 < x_4, \\ x_1 + x_3 < x_2 + x_4 \end{array} \right\} \\
&= \begin{array}{|c|c|c|c|} \hline 2 & 3 & 1 & 4 \\ \hline \end{array}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& \begin{array}{|c|c|} \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array} \circ \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array} \\
& \begin{array}{|c|c|} \hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array} \circ \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 2 & 4 \\ \hline \end{array} \\
& \begin{array}{|c|c|} \hline 4 & 3 \\ \hline 1 & 2 \\ \hline \end{array} \circ \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline 1 & 4 & 2 & 3 \\ \hline \end{array} \\
& \begin{array}{|c|c|} \hline 4 & 3 \\ \hline 2 & 1 \\ \hline \end{array} \circ \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline 2 & 4 & 1 & 3 \\ \hline \end{array} \\
& \begin{array}{|c|c|} \hline 4 & 2 \\ \hline 3 & 1 \\ \hline \end{array} \circ \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline 3 & 4 & 1 & 2 \\ \hline \end{array}
\end{aligned}$$



This implies that, in  $C_3(\text{Sal}(\mathcal{C}_3^2)) \otimes_{\Sigma_4} \mathbb{F}_p$ , if we write

$$\begin{aligned} \partial_3 \left( \begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \end{array} \right)^+ &= A_1 \begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \end{array} + A_2 \begin{array}{c} 3 \\ 2 \\ 1 \end{array} + \\ &+ A_3 \begin{array}{c} 3 \\ 2 \\ 1 \end{array} - A_4 \begin{array}{c} 2 \\ 1 \end{array} + A_5 \begin{array}{c} 2 \\ 1 \end{array} + A_6 \begin{array}{c} 3 \\ 2 \\ 1 \end{array}, \end{aligned}$$

we have

$$A_4 = -5, \quad A_5 = -1.$$

By similar calculations of matroid products and by that fact  $\partial_2 \partial_3 = 0$ , we can determine other coefficients and we have

$$\begin{aligned} \partial_3 \left( \begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \end{array} \right)^+ &= 4 \begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \end{array} + 4 \begin{array}{c} 3 \\ 2 \\ 1 \end{array} - 5 \begin{array}{c} 2 \\ 1 \end{array} + 8 \begin{array}{c} 3 \\ 2 \\ 1 \end{array} + 4 \begin{array}{c} 3 \\ 2 \\ 1 \end{array}. \end{aligned}$$

Analogously we have

$$\begin{aligned}
\partial_3 \left( \begin{array}{|c|} \hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} \right) &= 4 \begin{array}{|c|c|c|} \hline & & 4 \\ \hline & & 3 \\ \hline 1 & & 2 \\ \hline \end{array} + 4 \begin{array}{|c|c|c|} \hline & & 3 \\ \hline & & 2 \\ \hline 1 & & 4 \\ \hline \end{array} \\
&- \begin{array}{|c|c|c|} \hline 2 & & 4 \\ \hline 1 & & 3 \\ \hline \end{array}^+ - 5 \begin{array}{|c|c|c|} \hline 2 & & 4 \\ \hline 1 & & 3 \\ \hline \end{array}^- \\
&+ 4 \begin{array}{|c|c|c|c|} \hline & & 3 & \\ \hline 1 & & 2 & 4 \\ \hline \end{array} + 8 \begin{array}{|c|c|c|c|} \hline & & 3 & \\ \hline 1 & & 2 & 4 \\ \hline \end{array}^-.
\end{aligned}$$

Q.E.D.

### 4.3. The homology of $\mathcal{C}_3^2$

In this section, we compare  $H_*(\text{Sal}(\mathcal{C}_3^2) \otimes_{\Sigma_4} \mathbb{F}_p(\pm 1))$  and  $H_*(\text{Sal}(\mathcal{A}_3) \otimes_{\Sigma_4} \mathbb{F}_p(\pm 1))$  for  $p$  a prime, following the strategy described in the beginning of §4.

We have analyzed the cell structure of  $\text{Sal}(\mathcal{C}_3^2)$  in the previous section based on the structure of  $\mathcal{L}(\mathcal{C}_3^2)/\Sigma_4$  investigated in §4.2.

In this section, we compute the homology of

$$K_*^{4,2} = \text{Ker}(i_4^2 : C_*(\text{Sal}(\mathcal{C}_3^2)) \otimes_{\Sigma_4} \mathbb{F}_p(\pm 1) \rightarrow C_*(\text{Sal}(\mathcal{A}_3)) \otimes_{\Sigma_4} \mathbb{F}_p(\pm 1))$$

for an odd prime  $p$ . We first need to know generators for  $K_*^{4,2}$ .

The generators of the cellular chain complex  $C_*(\text{Sal}(\mathcal{C}_3^2)) \otimes_{\Sigma_4} \mathbb{F}_p(\pm 1)$  are in one-to-one correspondence with cells in  $\text{Sal}(\mathcal{C}_3^2)/\Sigma_4$  even when  $p$  is odd. Of course, we have to take the sign representation into account, when we compute the boundary homomorphisms.

**Lemma 4.10.** *Define*

$$\begin{aligned}
 x_0 &= \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline 1 & & 2 & 3 & 4 \\ \hline \end{array} \\
 x_{11} &= \begin{array}{|c|c|c|c|} \hline & 3 & & \\ \hline 1 & 2 & & 4 \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline & & & 3 \\ \hline 1 & & 2 & 4 \\ \hline \end{array} \\
 x_{12}^+ &= \begin{array}{|c|c|c|c|} \hline 1 & 2 & & 3 & 4 \\ \hline \end{array}^+ \\
 x_{12}^- &= \begin{array}{|c|c|c|c|} \hline 1 & 2 & & 3 & 4 \\ \hline \end{array}^- \\
 x_{21} &= \begin{array}{|c|c|c|} \hline 2 & & 4 \\ \hline 1 & & 3 \\ \hline \end{array}^+ - \begin{array}{|c|c|c|} \hline 2 & & 4 \\ \hline 1 & & 3 \\ \hline \end{array}^- \\
 x_{22}^+ &= \begin{array}{|c|c|c|c|} \hline & & 3 & \\ \hline 1 & & 2 & 4 \\ \hline \end{array}^+ \\
 x_{22}^- &= \begin{array}{|c|c|c|c|} \hline & & 3 & \\ \hline 1 & & 2 & 4 \\ \hline \end{array}^- \\
 x_3 &= \begin{array}{|c|} \hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}^+ - \begin{array}{|c|} \hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}^-.
 \end{aligned}$$

Then these are generators for  $K_*^{4,2}$

$$\begin{aligned}
 K_0^{4,2} &= \langle x_0 \rangle, \\
 K_1^{4,2} &= \langle x_{11}, x_{12}^+, x_{12}^- \rangle, \\
 K_2^{4,2} &= \langle x_{21}, x_{22}^+, x_{22}^- \rangle, \\
 K_3^{4,2} &= \langle x_3 \rangle.
 \end{aligned}$$

Thanks to the calculations in the previous section, we can easily compute the boundaries on these generators.

**Lemma 4.11.** *Boundaries are given by*

$$\begin{aligned}
\partial_1(x_{11}) &= 2x_0 \\
\partial_1(x_{12}^+) &= x_0 \\
\partial_1(x_{12}^-) &= x_0 \\
\partial_2(x_{21}) &= 2(x_{12}^+ - x_{12}^-) \\
\partial_2(x_{22}^+) &= x_{11} - 2x_{12}^+ \\
\partial_2(x_{22}^-) &= x_{11} - 2x_{12}^- \\
\partial_3(x_3) &= -4x_{21} + 4(x_{22}^+ - x_{22}^-).
\end{aligned}$$

By an elementary calculation, we obtain the homology of  $K_*^{4,2}$ .

**Proposition 4.12.** *When  $p$  is odd,*

$$H_i(K_*^{4,2}) \cong 0$$

for all  $i$ .

As a corollary, we obtain Theorem 1.3.

**Corollary 4.13** (Theorem 1.3). *For an odd prime  $p$ , the inclusion  $M_2(\mathbb{C}, 4) \hookrightarrow F(\mathbb{C}, 4)$  induces an isomorphism*

$$H_*(S_*(M_2(\mathbb{C}, 4)) \otimes_{\Sigma_4} \mathbb{F}_p(\pm 1)) \cong H_*(S_*(F(\mathbb{C}, 4)) \otimes_{\Sigma_4} \mathbb{F}_p(\pm 1)).$$

When  $p = 2$ , Lemma 4.11 implies that the boundaries in  $K_*^{4,2}$  are given by.

$$\begin{aligned}
\partial_1(x_{11}) &= 0 \\
\partial_1(x_{12}^+) &= x_0 \\
\partial_1(x_{12}^-) &= x_0 \\
\partial_2(x_{21}) &= 0 \\
\partial_2(x_{22}^+) &= x_{11} \\
\partial_2(x_{22}^-) &= x_{11} \\
\partial_3(x_3) &= 0.
\end{aligned}$$

In particular,  $x_3$  represents a nontrivial cycle in  $C_3^{4,2}$  that is mapped to 0 under the map  $i_4^2$ . Thus we obtain a proof of Theorem 1.2.

Following the suggestion by the referee, we conclude this paper by briefly describing the case of the trivial  $\mathbb{F}_p$  coefficients. We have

$$C_*(\text{Sal}(C_{n-1}^k)) \otimes_{\Sigma_n} \mathbb{F}_p \cong C_*(\text{Sal}(C_{n-1}^k)/\Sigma_n) \otimes \mathbb{F}_p.$$

By ignoring the changes of signs when we permute labels in the calculations in §4.2, we obtain the following. The details are omitted.

**Lemma 4.14.** *We have the following formula in  $C_*(\text{Sal}(\mathcal{C}_3^2)/\Sigma_4) \otimes \mathbb{F}_p$ .*

$$\begin{aligned}
\partial_1 \left( \begin{array}{|c|c|c|c|} \hline & & 3 & \\ \hline 1 & & 2 & 4 \\ \hline \end{array} \right) &= 0 \\
\partial_1 \left( \begin{array}{|c|c|c|c|} \hline & & & 4 \\ \hline 1 & & 2 & 3 \\ \hline \end{array} \right) &= 0 \\
\partial_1 \left( \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & & 3 & 4 \\ \hline \end{array} \right)^+ &= \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & & 4 \\ \hline \end{array} - \begin{array}{|c|c|c|c|c|} \hline 1 & & 2 & 3 & 4 \\ \hline \end{array} \\
\partial_1 \left( \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & & 3 & 4 \\ \hline \end{array} \right)^- &= \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & & 4 \\ \hline \end{array} - \begin{array}{|c|c|c|c|c|} \hline 1 & & 2 & 3 & 4 \\ \hline \end{array} \\
\partial_1 \left( \begin{array}{|c|c|c|c|} \hline & & 3 & \\ \hline 1 & 2 & & 4 \\ \hline \end{array} \right) &= 0 \\
\partial_1 \left( \begin{array}{|c|c|c|c|} \hline 2 & & & \\ \hline 1 & 3 & & 4 \\ \hline \end{array} \right) &= 0, \\
\\
\partial_2 \left( \begin{array}{|c|c|c|c|} \hline & & & 4 \\ \hline & & 3 & \\ \hline 1 & & 2 & \\ \hline \end{array} \right) &= \begin{array}{|c|c|c|c|} \hline & & & 4 \\ \hline 1 & & 2 & 3 \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline & & 3 & \\ \hline 1 & & 2 & 4 \\ \hline \end{array} \\
\partial_2 \left( \begin{array}{|c|c|c|c|} \hline 3 & & & \\ \hline 2 & & & \\ \hline 1 & & & 4 \\ \hline \end{array} \right) &= \begin{array}{|c|c|c|c|} \hline & 3 & & \\ \hline 1 & 2 & & 4 \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline 2 & & & \\ \hline 1 & 3 & & 4 \\ \hline \end{array} \\
\partial_2 \left( \begin{array}{|c|c|c|c|c|} \hline & & 3 & & \\ \hline 1 & & 2 & & 4 \\ \hline \end{array} \right)^+ &= - \begin{array}{|c|c|c|c|c|} \hline & & 3 & & \\ \hline 1 & & 2 & & 4 \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|} \hline & & 3 & & \\ \hline 1 & 2 & & & 4 \\ \hline \end{array} \\
\partial_2 \left( \begin{array}{|c|c|c|c|c|} \hline & & 3 & & \\ \hline 1 & & 2 & & 4 \\ \hline \end{array} \right)^- &= - \begin{array}{|c|c|c|c|c|} \hline & & 3 & & \\ \hline 1 & & 2 & & 4 \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|} \hline & & 3 & & \\ \hline 1 & 2 & & & 4 \\ \hline \end{array} \\
\partial_2 \left( \begin{array}{|c|c|c|c|} \hline 2 & & 4 & \\ \hline 1 & & 3 & \\ \hline \end{array} \right)^+ &= \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & & 3 & 4 \\ \hline \end{array}^+ - \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & & 3 & 4 \\ \hline \end{array}^- \\
\partial_2 \left( \begin{array}{|c|c|c|c|} \hline 2 & & 4 & \\ \hline 1 & & 3 & \\ \hline \end{array} \right)^- &= - \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & & 3 & 4 \\ \hline \end{array}^+ + \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & & 3 & 4 \\ \hline \end{array}^-,
\end{aligned}$$

$$\partial_3 \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}^+ = - \begin{array}{|c|c|} \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}^+ - \begin{array}{|c|c|} \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}^-$$

$$\partial_3 \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}^- = - \begin{array}{|c|c|} \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}^+ - \begin{array}{|c|c|} \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}^-.$$

Let  $(\bar{K}_*^{n,k}, \bar{\partial})$  denote the chain complex given by the kernel of

$$C_*(\text{Sal}(\mathcal{C}_{n-1}^k)/\Sigma_n) \otimes \mathbb{F}_p \longrightarrow C_*(\text{Sal}(\mathcal{C}_{n-1}^{k-1})/\Sigma_n) \otimes \mathbb{F}_p.$$

Then we obtain the homology of  $\bar{K}_*^{4,2}$  as follows.

**Proposition 4.15.** *We have*

$$\begin{aligned} \bar{\partial}(x_0) &= 0 \\ \bar{\partial}(x_{11}) &= 0 \\ \bar{\partial}(x_{12}^+) &= x_0 \\ \bar{\partial}(x_{12}^-) &= x_0 \\ \bar{\partial}(x_{21}) &= 2(x_{12}^+ - x_{12}^-) \\ \bar{\partial}(x_{22}^+) &= x_{11} \\ \bar{\partial}(x_{22}^-) &= x_{11} \\ \bar{\partial}(x_3) &= 0. \end{aligned}$$

Hence we obtain

$$\begin{aligned} H_0(\bar{K}_*^{4,2}) &= 0 \\ H_1(\bar{K}_*^{4,2}) &= 0 \\ H_2(\bar{K}_*^{4,2}) &= \langle [x_{22}^+ - x_{22}^-] \rangle \\ H_3(\bar{K}_*^{4,2}) &= \langle [x_3] \rangle. \end{aligned}$$

when  $p$  is odd, and

$$\begin{aligned} H_0(\bar{K}_*^{4,2}) &= 0 \\ H_1(\bar{K}_*^{4,2}) &= \langle [x_{12}^+ - x_{12}^-] \rangle \\ H_2(\bar{K}_*^{4,2}) &= \langle [x_{21}], [x_{22}^+ - x_{22}^-] \rangle \\ H_3(\bar{K}_*^{4,2}) &= \langle [x_3] \rangle \end{aligned}$$

when  $p = 2$ .

**Corollary 4.16.** *The map*

$$H_*(M_2(\mathbb{C}, 4)/\Sigma_4; \mathbb{F}_p) \longrightarrow H_*(F(\mathbb{C}, 4)/\Sigma_4; \mathbb{F}_p)$$

*induced by the inclusion is not an isomorphism for any prime  $p$ .*

### References

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