Two-sided Bar Constructions for Partial Monoids and Applications to K-Homology Theory

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1 Introduction

The aim of this article is to provide a method for proving a homotopy functor to be linear. The main result of this article is used in [Tam] to prove the Segal's K-homology and its twisted version are linear functors.

Theorem 1. *Let M be a partial topological monoid with a good unit acting on X and Y from the right and the left, respectively. If the inclusions*

$$\begin{aligned} B_n(X,M,Y) &\hookrightarrow X \times M^n \times Y \\ B_n(X,M,*) &\hookrightarrow X \times M^n \\ C_M(Y) &\hookrightarrow M \times Y \end{aligned}$$

are weak equivalences for each n and if the action of $m \in M$ on Y induces a weak equivalence

$$Y_m = \{y \in Y \mid (m,y) \in C_M(Y)\} \xrightarrow{m\cdot} Y$$

for each $m \in M$, then

$$p^Y : |B_*(X,M,Y)| \longrightarrow |B_*(X,M,*)|$$

is a quasifibration.

We remark that Shimakawa studied linear functors arising from partial Abelian monoids in [Shi01, Shi07]. The functors studied by Shimakawa overlap untwisted cases of functors studied in this paper, but he proved linearities by using a result of Woolfson [Woo79] instead of the bar constructions on partial topological monoids. Although Shimakawa's and our results are based on a common idea of Segal's, our approach is more straightforward.

As examples, we verify functors constructed from configuration spaces and Madsen-Tillmann spectrum [MT01] satisfy the conditions of the above theorem, hence are linear.

The paper is organized as follows:

1. In §2, we use May's generalization of the bar construction and prove Theorem 1.
2. In §3, we explore a couple of examples and pose some questions.

We use notations and definitions introduced in a separate paper [Tam] freely.

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2 The Bar Construction on Monad

As we have observed in [Tam], a classical theorem of Dold and Thom [DT58] can be interpreted as follows: the infinite symmetric product functor SP^∞ converts a diagram of based maps

$$X \xleftarrow{f} Y \xrightarrow{g} Z,$$

into a quasifibration

$$\mathrm{SP}^\infty(Z) \longrightarrow \mathrm{SP}^\infty(M_{f,g}) \longrightarrow \mathrm{SP}^\infty(X \cup_f CY).$$

It is also noticed in [Tam] that there is a functorial homeomorphism

$$\mathrm{SP}^\infty(M_{f,g}) \cong |B_*(\mathrm{SP}^\infty(Z), \mathrm{SP}^\infty(X), \mathrm{SP}^\infty(Y))|, \tag{1}$$

where $B_*(-,-,-)$ is the two-sided geometric bar construction and $|-|$ is the geometric realization functor on simplicial spaces. The above quasifibration is obtained from a sequence of simplicial spaces

$$\mathrm{SP}^\infty(Z) \longrightarrow B_*(\mathrm{SP}^\infty(Z), \mathrm{SP}^\infty(X), \mathrm{SP}^\infty(Y)) \longrightarrow B_*(*, \mathrm{SP}^\infty(X), \mathrm{SP}^\infty(Y))$$

by taking the geometric realization functor. Thus the main theorem of Dold and Thom is a consequence of the identification (1) and a general fact on the two-sided bar construction on topological monoid proved, for example, in [Mil67, May75].

Theorem 2. *Let G be a topological monoid acting on X and Y from the right and the left, respectively. If $(G, \{e\})$ is a strong NDR pair and $\pi_0(G)$ is a group, then the following sequence is a quasifibration*

$$Y \longrightarrow |B_*(X, G, Y)| \xrightarrow{p} |B_*(X, G, *)|,$$

where p is given by collapsing Y to a single point.

The functor SP^∞ is exceptionally well-behaved. Although it is well-known [Goo03] that any homology theory $\widetilde{E}_*(-)$ can be expressed as a composition of a space-level functor

$$F_E : \mathbf{Spaces}_* \longrightarrow \mathbf{Spaces}_*$$

and the homotopy groups functor

$$\widetilde{E}_*(X) \cong \pi_*(F_E(X)),$$

it is rarely the case that F_E converts double mapping cylinders to two-sided bar constructions.

For example, consider Segal's K-homology theory [Seg77] $ku(-; H)$ based on a Hilbert space H. It is shown in [Tam] that $ku(-; H)$ converts double mapping cylinders to certain subspaces of the two-sided bar construction

$$ku(M_{f,g}; H) \cong |B_*^\perp(ku(Z; H), ku(X; H), ku(Y; H))|.$$

We can not apply the classical theory of two-sided bar constructions on topological monoids, although we have a sequence of simplicial spaces

$$ku(Z; H) \longrightarrow B_*^\perp(ku(Z; H), ku(X; H), ku(Y; H)) \longrightarrow B_*^\perp(*, ku(X; H), ku(Y; H)).$$

Thus it is natural to ask the following question.

Question 1. When does a map of simplicial spaces

$$f_* : X_* \longrightarrow Y_*$$

induces a quasifibration

$$|f_*| : |X_*| \longrightarrow |Y_*|$$

after taking the geometric realization?

J.P. May gives an answer to this question in [May72] by introducing the notion of simplicial Hurewicz fibration. However, the definition of simplicial Hurewicz fibration is complicated and it is not easy to use May's theorem for our purpose. Another condition was found by Anderson [And78]. His condition involves the fundamental groupoid of the base space.

Let us restrict our attention to the simplicial spaces which arise as "generalized bar constructions" and consider the above question. We would like to be a bit more general and introduce a construction for partial monoids, which is analogous to the two-sided bar construction for topological monoids.

Definition 1. *A partial (topological) monoid is a pointed topological space* $(M, *)$ *together with a subspace* $C(M) \subset M \times M$ *and a map*

$$\mu_M : C(M) \longrightarrow M$$

satisfying the following conditions:

1. $(m_1, m_2) \in C(M)$ *and* $(\mu_M(m_1, m_2), m_3) \in C(M)$ *if and only if* $(m_2, m_3) \in C(M)$ *and* $(m_1, \mu_M(m_2, m_3)) \in C(M)$*. And*

$$\mu_M(\mu_M(m_1, m_2), m_3) = \mu_M(m_1, \mu_M(m_2, m_3))$$

for such a triple $(m_1, m_2, m_3) \in (\mu_M \times 1)^{-1}(C(M)) = 1 \times \mu_M)^{-1}(C(M))$.

2. $M \vee M \subset C(M)$ *and the following diagram is commutative*

$$\begin{array}{ccc} C(M) & \xrightarrow{\mu_M} & M \\ \uparrow & \nearrow_{fold} & \\ M \vee M & & \end{array}$$

The set $C(M)$ *is called the set of composable pairs.*

Definition 2. *Let* M *be a partial monoid. A left action of* M *on a space* X *is map*

$$\mu_X : C_M(X) \longrightarrow X,$$

where $C_M(X) \subset M \times X$*, satisfying the following conditions:*

1. $(m_2, x) \in C_M(X)$ *and* $(m_1, \mu_X(m_2, x)) \in C_M(X)$ *if and only if* $(m_1, m_2) \in C(M)$ *and* $(\mu_M(m_1, m_2), x) \in C_M(X)$*. And*

$$\mu_X(m_1, \mu_X(m_2, x)) = \mu_X(\mu_M(m_1, m_2), x).$$

2. $\{*\} \times X \subset C_M(X)$ *and the following diagram is commutative*

$$\begin{array}{ccc} C_M(X) & \xrightarrow{\mu_X} & X \\ \uparrow & \nearrow & \\ \{*\} \times X & & \end{array}$$

$C_M(X)$ *is also called the set of composable pairs.*

A right action is defined similarly.

Example 1. For a pointed space X, Segal's construction $ku(X;H)$ has a structure of partial monoid with

$$C(ku(X;H)) = ku(X;H) \overset{\perp}{\times} ku(X;H).$$

The multiplication is given by the concatenation.

A based map

$$f : X \longrightarrow Y$$

induces a map of partial monoids

$$f_* : ku(X;H) \longrightarrow ku(Y;H)$$

hence an action of $ku(X;H)$ on $ku(Y;H)$ with the set of composable pairs

$$C_{ku(X;H)}(ku(Y;H)) = ku(X;H) \overset{\perp}{\times} ku(Y;H).$$

Analogously, the twisted version $ku^{AS}(\varphi, s; H)$ constructed in [Tam] is a partial monoid and a morphism $f : (\varphi, s) \longrightarrow (\psi, t)$ induces an action of $ku^{AS}(\varphi, s)$ on $ku^{AS}(\psi, t)$.

A partial monoid naturally gives rise to a monad.

Definition 3. *A monad in a category* $\mathbf{C}$ *is a covariant functor*

$$C : \mathbf{C} \longrightarrow \mathbf{C}$$

together with natural transformations

$$\mu : C^2 \longrightarrow C$$
$$\eta : 1 \longrightarrow C$$

making the following diagrams commutative for each object X

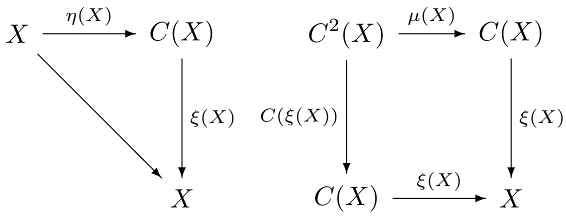
$$\begin{array}{ccccc} C(X) & \xrightarrow{C\eta(X)} & C^2(X) & \xleftarrow{\eta(C(X))} & C(X) \\ & \searrow & \downarrow{\scriptstyle \mu(X)} & \swarrow & \\ & & C(X) & & \end{array}$$

$$\begin{CD} C^3(X) @>{\mu(CX)}>> C^2(X) \\ @V{C(\mu(X))}VV @VV{\mu(X)}V \\ C^2(X) @>{\mu(X)}>> C(X) \end{CD}$$

Definition 4. *Let C be a monad in a category $\mathbf{C}$. An algebra over C is an object X in $\mathbf{C}$ together with a map*

$$\xi : C(X) \longrightarrow X$$

making the following diagrams commutative



Example 2. Let M be a partial monoid and M-**Spaces** be the category of left M-spaces. Then C_M can be regarded as a monad in M-**Spaces** as follows. For an object X in M-**Spaces**, we need to make $C_M(X)$ into an object of M-**Spaces**. The set of composable pairs is

$$C_M(C_M(X)) = \{(m,(n,x)) \in M \times C_M(X) \mid (m,n) \in C(M), (mn,x) \in C_M(X)\}$$

and the action of M is the multiplication of M. Thus C_M becomes an endofunctor

$$C_M : M\text{-}\mathbf{Spaces} \longrightarrow M\text{-}\mathbf{Spaces}.$$

It is obvious that C_M is a monad in M-**Spaces**. In fact any object in M-**Spaces** is an algebra over C_M.

In order to extend the two-sided bar construction from monoids to monads, we also need the concept of a C-functor for a monad C.

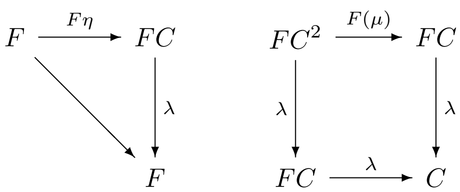
Definition 5. *Let (C, μ, η) be a monad in a category $\mathcal{C}$. A C-functor in a category $\mathcal{D}$ is a functor*

$$F : \mathcal{C} \longrightarrow \mathcal{D}$$

together with a natural transformation

$$\lambda : FC \longrightarrow F$$

making the following diagrams commutative



Example 3. Let M be a partial monoid and X be a right M-space. The functor

$$C_X : M\text{-}\mathbf{Spaces} \longrightarrow \mathbf{Spaces}$$

defined by

$$C_X(Y) = \left\{ (x,y) \in X \times Y \,\middle|\, \begin{array}{l} x = x'm \Longrightarrow (m,y) \in C_M(Y) \\ y = my' \Longrightarrow (x,m) \in C_M(X) \end{array} \right\}$$

is a C_M-functor. The natural transformation

$$\lambda : C_X C_M \longrightarrow C_X$$

is given by the action of M on X.

J.P. May introduced a generalization of the two-sided bar construction to the triple of a monad C, an algebra over C and a C-functor.

Definition 6. *Let (C, μ, η) be a monad in a category $\mathbf{C}$. Given an algebra (X, ξ) over C and a C-functor (F, λ) in $\mathbf{D}$, define a simplicial object $B_*(F, C, X)$ in $\mathbf{D}$ by*

$$B_q(F, C, X) = FC^q(X).$$

The face and degeneracy operators are given by

$$\begin{aligned} d_0 &= \lambda \\ d_i &= FC^{i-1}\mu(C^{q-i}(X)) \text{ for } 1 \leq i \leq q-1 \\ d_q &= FC^{q-1}\xi \\ s_i &= FC^i\eta(C^{q-i}(X)). \end{aligned}$$

Consider a monad C in a category $\mathbf{C}$ and a C-functor

$$F : \mathbf{C} \longrightarrow \mathbf{Spaces}$$

For a map of C-algebras

$$f : X \longrightarrow Y$$

we obtain a map of simplicial spaces

$$f_* : B_*(F, C, X) \longrightarrow B_*(F, C, Y).$$

In order to find a practical condition under which this map induces a quasifibration after taking the geometric realization, let us restrict our attention to the case of partial monoid.

Let M be a partial monoid. Let X and Y be right and left M-spaces, respectively. The bar construction on C_M, Y, and C_X is denoted by

$$B_*(X, M, Y) = B_*(C_X, C_M, Y),$$

for the sake of simplicity. We have the following commutative square of simplicial spaces

$$\begin{array}{ccc} B_*(X,M,Y) & \longrightarrow & B_*(X,M,*) \\ \downarrow & & \downarrow \\ B_*(*,M,Y) & \longrightarrow & B_*(*,M,*). \end{array}$$

We would like to find a condition which makes the geometric realization of the above diagram homotopy Cartesian.

We use the following result of Segal [Seg74]. Recall that $\varDelta$ is the small category of finite totally order sets and order preserving maps so that a simplicial space X_* is nothing but a functor

$$X_* : \varDelta^{\mathrm{op}} \longrightarrow \mathbf{Spaces}.$$

Definition 7. *We say a simplicial space X_* is strongly cofibrant if, for each face operator*

$$d_i : X_{n-1} \hookrightarrow X_n,$$

the pair $(X_n, d_i(X_{n-1}))$ is a strong NDR pair.

Theorem 3. *Let*

$$f_* : X_* \longrightarrow Y_*$$

be a map of strongly cofibrant simplicial spaces satisfying the following condition: for each morphism $\theta : \boldsymbol{m} \longrightarrow \boldsymbol{n}$ in $\varDelta$, the diagram

$$\begin{array}{ccc} X_n & \xrightarrow{\theta^*} & X_m \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f_m} \\ Y_n & \xrightarrow{\theta^*} & Y_m \end{array}$$

is homotopy Cartesian. Then the diagrams

$$\begin{array}{ccc} \varDelta^n \times X_n & \longrightarrow & |X_*| \\ {\scriptstyle 1\times f_n}\downarrow & & \downarrow{\scriptstyle |f_*|} \\ \varDelta_n \times Y_n & \longrightarrow & |Y_*| \end{array}$$

are homotopy Cartesian for all n.

In order to make use of this theorem, we need a condition under which strongly cofibrantness holds. The following is a reasonalbe condition having regard to Theorem 2 and follows immediately from the standard choice of NDR representation for a product of NDR pairs.

Lemma 1. *Let $(M,*)$ be a partial monoid. If $(M,*)$ has a strong NDR representation (h,u) satisfying the condition that, if $(m_1,m_2) \in C(M)$ then $(h(m_1,s),h(m_2,t)) \in C(M)$ for all $s,t \in I$, then $B_*(X,M,Y)$ is a strongly cofibrant simplicial space for any right M-space X and left M-space Y.*

Proof. The standard NDR representation for the product of NDR pairs gives the desired strong NDR representation under the assumption of the Lemma.

We use the following terminology for simplicity.

Definition 8. *We say a partial monoid M has a good unit if $(M,*)$ has a strong NDR representation (h,u) satisfying the condition that, if $(m_1,m_2) \in C(M)$ then $(h(m_1,s),h(m_2,t)) \in C(M)$ for all $s,t \in I$.*

The following is our answer to Question 1 in the case of the two-sided bar construction for partial monoid.

Theorem 4. *Let M be a partial monoid with a good unit acting on X and Y from the right and the left, respectively. If the inclusions*

$$\begin{aligned} B_n(X,M,Y) &\hookrightarrow X \times M^n \times Y \\ B_n(X,M,*) &\hookrightarrow X \times M^n \\ C_M(Y) &\hookrightarrow M \times Y \end{aligned}$$

are weak homotopy equivalences for each n and if the action of $m \in M$ on Y induces a weak homotopy equivalence

$$Y_m = \{y \in Y \mid (m,y) \in C_M(Y)\} \xrightarrow{m\cdot} Y$$

for each $m \in M$, then

$$p^Y : |B_*(X,M,Y)| \longrightarrow |B_*(X,M,*)|$$

is a quasifibration with fiber Y.

Proof. We are going to prove that the homotopy fiber of p_Y over $[\boldsymbol{s},\boldsymbol{x}] \in |B_*(X,M,*)|$ is weakly homotopy equivalent to Y under the canonical map.

Suppose

$$p^Y_* : B_*(X,M,Y) \longrightarrow B_*(X,M,*),$$

satisfies the condition for Theorem 3. (We wiil check this later.) Take an element $[s,x] \in |B_*(X,M,*)|$ and consider the homotopy fiber over this element. This element is represented by an element $(s,x) \in \Delta^n \times B_n(X,M,*)$ for some n. By Theorem 3, we have the following homotopy Cartesian diagram

$$\begin{array}{ccc} \Delta^n \times B_n(X,M,Y) & \longrightarrow & |B_*(X,M,Y)| \\ \downarrow & & \downarrow \\ \Delta^n \times B_n(X,M,*) & \longrightarrow & |B_*(X,M,*)|. \end{array}$$

If we denote F_n and F for the homotopy fibers of

$$\begin{aligned} p_n^Y &: B_n(X,M,Y) \longrightarrow B_n(X,M,*) \\ |p_*^Y| &: |B_*(X,M,Y)| \longrightarrow |B_*(X,M,*)|, \end{aligned}$$

over (s,x) and $[s,x]$, respectively, then it follows that

$$F_n \xrightarrow[w]{\simeq} F.$$

In order to prove that F_n is weakly equivalent to Y, let us extend the commutative diagram

$$\begin{array}{ccc} B_n(X,M,Y) & \longrightarrow & X \times M^n \times Y \\ \downarrow & & \downarrow \\ B_n(X,M,*) & \longrightarrow & X \times M^n \end{array}$$

to the 3×3 diagram of homotopy fiber sequences. Then we have

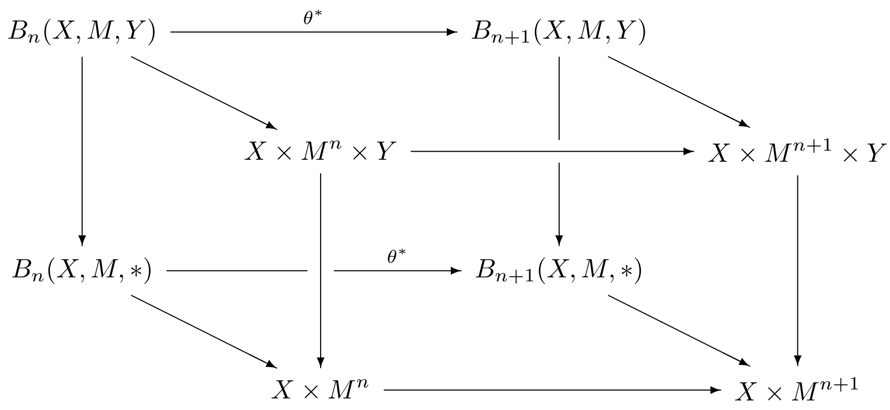
$$\begin{array}{ccccc} T & \longrightarrow & F_n & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ G_1 & \longrightarrow & B_n(X,M,Y) & \xrightarrow[w]{\simeq} & X \times M^n \times Y \\ \downarrow & & \downarrow & & \downarrow \\ G_2 & \longrightarrow & B_n(X,M,*) & \xrightarrow[w]{\simeq} & X \times M^n. \end{array}$$

By assumption G_1 and G_2 are weakly contractible and hence F_n and Y are weakly equivalent.

Now it remains to check that our simplicial map p_*^Y satisfies the condition for Theorem 3. It suffices to prove the following diagram is homotopy Cartesian for $\theta^* = d_i$ and s_i.

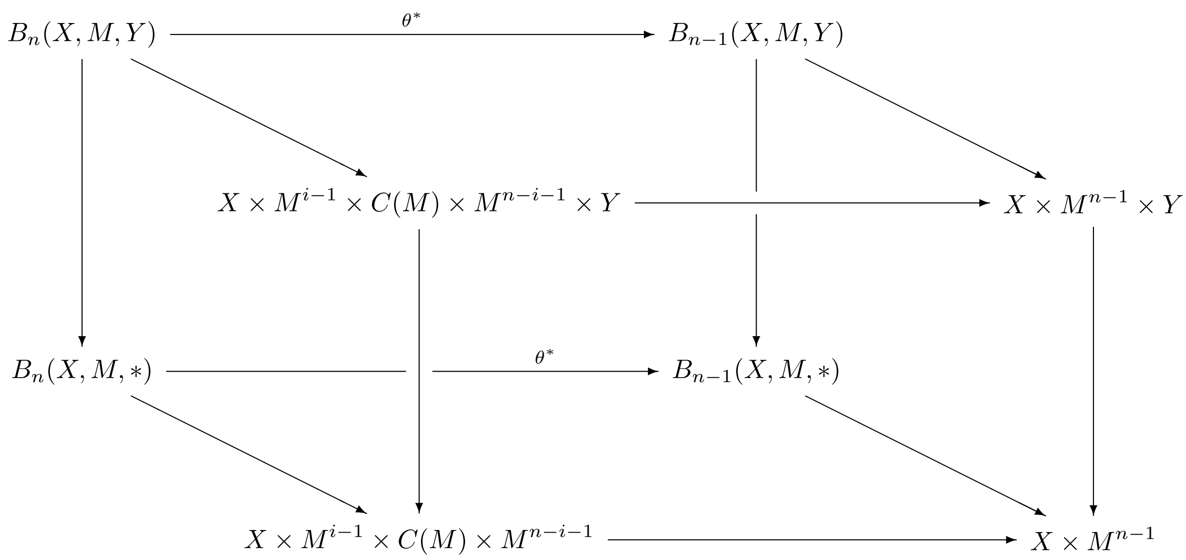
$$\begin{array}{ccc} B_n(X,M,Y) & \xrightarrow{\theta^*} & B_m(X,M,Y) \\ \downarrow & & \downarrow \\ B_n(X,M,*) & \xrightarrow{\theta^*} & B_m(X,M,*). \end{array}$$

When $\theta^* = s_i$ for some i, we have the following commutative diagram



where the horizontal maps in the front diagram are given by inserting $*$ in i-th place of M^{n+1}.

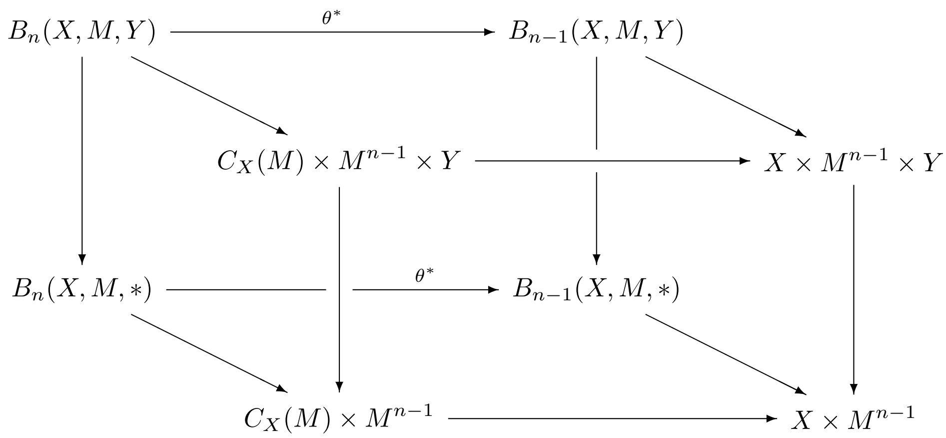
For d_i $(0 < i < n)$, we also have the following diagram



where the front horizontal maps are given by

$$\mu_M : C(M) \longrightarrow M.$$

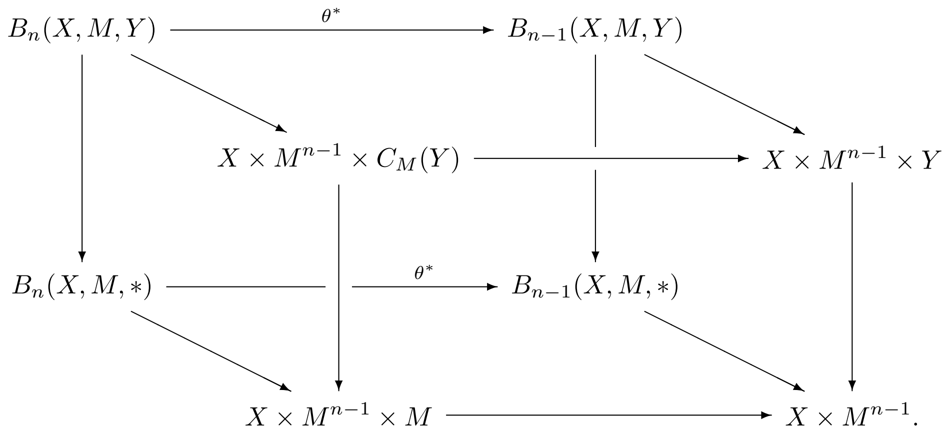
For d_0, we have



where the front horizontal maps are given by the action of M on X.

In all these cases, the vertical maps in the front square of the diagrams are trivial fibrations with fiber Y and the maps of diagrams from the back to the front are all weak equivalences. Thus the homotopy fibers of the vertical maps in the backside diagram are both weakly equivalent to Y hence the backside diagrams are homotopy Cartesian.

The case of d_n is a little bit different. We have the following diagram



Thus it is enough to prove that the following diagram is homotopy Cartesian

$$\begin{array}{ccc} C_M(Y) & \xrightarrow{\mu_Y} & Y \\ {\scriptstyle pr_1}\downarrow & & \downarrow \\ M & \longrightarrow & * \end{array}$$

Note that, by assumption, Y_m is weakly homotopy equivalent to the homotopy fiber of the projection

$$pr_1 : C_M(Y) \longrightarrow Y.$$

By extending this diagram, we obtain a 3×3 diagram of homotopy fiber sequences

$$\begin{array}{ccccc}
T & \longrightarrow & Y_m & \xrightarrow{\mu_Y} & Y \\
\downarrow & & \downarrow & & \downarrow \\
hofiber(\mu_Y) & \longrightarrow & C_M(Y) & \xrightarrow{\mu_Y} & Y \\
\downarrow & & \downarrow{\scriptstyle pr_1} & & \downarrow \\
M & \longrightarrow & M & \longrightarrow & *
\end{array}$$

and the result follows from the assumption that $\mu_Y|_{Y_m}$ is a weak equivalence.

We use Theorem 4 to prove the linearlity of the Segal's functor $ku(-; H)$ and its twisted version $ku^{AS}(-; H)$ in a separate paper [Tam]. Note that we can remove the path-connectedness condition by taking a homotopy theoretic group completion.

Corollary 1. *The following functor is a reduced homology theory*

$$X \longmapsto \pi_*(\Omega ku(\Sigma X; H)).$$

Corollary 2. *The following functor is a reduced homology theory*

$$(\varphi, s) \longmapsto \pi_* \left(\Omega ku^{AS}(\Sigma_{BPU(H)}(\varphi, s; H)) \right)$$

on the category of ex-spaces over $BPU(H)$*.*

3 Examples

We have seen that the facts that the functors

$$\begin{aligned}
X &\longmapsto \pi_*(\Omega \mathrm{SP}^\infty(\Sigma X)) \\
X &\longmapsto \pi_*(\Omega ku(\Sigma X; H)) \\
(\varphi, s) &\longmapsto \pi_* \left(\Omega ku^{AS}(\varphi, s; H) \right)
\end{aligned}$$

define (reduced) homology theories follow immediately from Theorem 4. Note that the dimension function

$$\dim : \{\text{finite dimensional vector subspaces in } H\} \longrightarrow \mathbb{N} \cup \{0\}$$

induces a canonical natural transformation of linear functors

$$\dim : ku(X;H) \longrightarrow \mathrm{SP}^\infty(X),$$

hence a natural transformation of homology theories.

Segal proves that $\pi_*(\Omega ko(\Sigma X))$ is the connective KO-theory by comparing $ko(S^n)$ with the Bott periodicity spaces. In this sense, the connective KO-homology is obtained by a categorification of $\mathbb{N}\cup\{0\}$. It seems that categorifications of $\mathbb{N}\cup\{0\}$, categorifications of categorifications of $\mathbb{N}\cup\{0\}$, and so on, give us interesting examples of homology theories.

Let us begin with the stable homotopy theory, whose construction in terms of configuration spaces have been investigated by many people [Seg73, Shi01].

Example 4. For a vector space H over $\mathbb{R}$, let $\mathcal{C}(j;H)$ be the space of (ordered) distinct j points in H. For a pointed space X, define

$$C(X;H) = \left(\coprod_{j=0}^{\infty} \mathcal{C}(j;H) \times_{\Sigma_j} X^j\right) \Big/ \sim$$

where $\sim$ is the basepoint relation.

$C(X;H)$ is a partial monoid under the concatenation: the set of composable pairs is

$$C(C(X;H)) = \{([\boldsymbol{c},\boldsymbol{x}]),([\boldsymbol{c}',\boldsymbol{x}']) \in C(X;H)\times C(X;H) \mid \boldsymbol{c}\cap\boldsymbol{c}' = \emptyset\}.$$

When $H=\mathbb{R}^\infty$, $C(X;H)$ satisfies the condition for Theorem 4 and it follows that

$$X \longrightarrow \pi_*(\Omega C(\Sigma X;\mathbb{R}^\infty))$$

is a homology theory. This is the stable homotopy group of X, $\pi_*^S(X)$.

Note that an element of $C(X;\mathbb{R}^\infty)$ can be written as

$$([c_1,\cdots,c_k;x_1,\cdots,x_k])$$

where $c_i \in \mathcal{C}(j_i;\mathbb{R}^\infty)$ and $x_i \neq x_{i'}$ if $i\neq i'$. The cardinality function

$$\coprod \mathcal{C}(j;\mathbb{R}^\infty) \longrightarrow \mathbb{N}$$

induces a natural transformation of homology theories

$$\pi_*^S(X) \longrightarrow \widetilde{H}_*(X;\mathbb{Z})$$

which is nothing but the stable Hurewicz homomorphism.

We can also consider configuration spaces with multiplicity at most i and obtain a homology theory $\pi_*(\Omega^\infty\mathrm{SP}^i(\Sigma^\infty X))$ lying in between these two

$$\pi_*^S(X) \longrightarrow \pi_*(\Omega^\infty\mathrm{SP}^i(\Sigma^\infty X)) \longrightarrow \widetilde{H}_*(X;\mathbb{Z}).$$

Example 5. Consider the infinite loop space structure on

$$\Omega B\left(\coprod_g B\Gamma_{g,1}\right) \simeq \mathbb{Z} \times B\Gamma_\infty^+$$

studied by Tillmann [Til97, MT01], where $\Gamma_{g,1}$ is the mapping class group of oriented surface of genus g with one boundary component,

$$\Gamma_\infty = \operatorname*{colim}_g \Gamma_{g,1}$$

by sewing a torus with two boundary components, and + in $B\Gamma_\infty^+$ denotes the plus-construction of Quillen.

Following the construction of Madsen and Tillmann, we can form a topological category $\mathcal{Y}$ as follows: objects are given by

$$\mathrm{Ob}(\mathcal{Y}) = \coprod_{j=0}^{\infty} \mathrm{Emb}(S_j, \mathbb{R}^\infty)/\mathrm{Diff}(S_j)$$

where $S_j = \underbrace{S^1 \amalg \cdots \amalg S^1}_{j}$, Emb is the space of smooth embeddings, and Diff is the group of diffeomorphisms. The space of morphisms is given by cobordisms in $\mathbb{R}^\infty$. To be more precise, let F be an oriented cobordism from S_n to S_m, namely

$$\partial_- F = S_n,$$
$$\partial_+ F = S_m.$$

For $t > 0$, let

$$\mathrm{Emb}^\Omega(F; [0,t] \times \mathbb{R}^\infty)$$

be the space of embeddings $h\ F \hookrightarrow [0,t] \times \mathbb{R}^\infty$ with

$$h(\partial_- F) = h(F) \cap \{0\} \times \mathbb{R}^\infty$$
$$h(\partial_+ F) = h(F) \cap \{t\} \times \mathbb{R}^\infty.$$

We assume that the boundary of F has good collar neighborhood and elements in $\mathrm{Emb}^\Omega(F; [0,t] \times \mathbb{R}^\infty)$ map these collars into the good collars of $h(F)$. Let $\mathrm{Diff}^\Omega(F)$ be the group of diffeomorphisms of F which restricts to give diffeomorphisms of the form

$$\partial_- \phi \times 1_{[0,\varepsilon]},$$
$$\partial_+ \phi \times 1_{[0,\varepsilon]}$$

on the collar neighborhood of the incoming and outgoing boundaries. The space of morphisms in $\mathcal{Y}$ is given by

$$\mathrm{Mor}(\mathcal{Y}) = \mathrm{Ob}(\mathcal{Y}) \amalg \coprod_{F,t>0} \mathrm{Emb}^{\Omega}(F,[0,t]\times\mathbb{R}^{\infty})/\mathrm{Diff}^{\Omega}(F)$$

with a suitable topology.

Note that the space of morphism of this category has two operations: disjoint union and sewing. Of course, the sewing operation is the composition of morphisms. The disjoint union can be taken only when the images are disjoint in $\mathbb{R}^{\infty}$. Thus the disjoint union operation gives a partial monoid structure on the nerve of $\mathcal{Y}$. In order to form a "balanced product" of $N_*\mathcal{Y}$ and a pointed space X as we have done to define $ku(X;H)$ or $C(X;\mathbb{R}^{\infty})$, we need to label the connected components of surfaces in the nerves of $\mathcal{Y}_b$. Thus we define a simplicial space $Y(j)_*$ by

$$Y(j)_q = \{([\boldsymbol{h}],\lambda) \mid [\boldsymbol{h}] \in N_q\mathcal{Y}, \lambda : \pi_0\mathrm{Im}(h_q \circ \cdots \circ h_1) \to \{1,\cdots,j\}\}.$$

For a pointed space X, define

$$M_q(X) = \left(\coprod_{j=0}^{\infty} Y(j)_q \times_{\Sigma_j} X^j\right)\Big/\sim$$

where $\sim$ is the basepoint relation. Thus we have a simplicial space $M_*(X)$ with a partial monoid structure on each $M_q(X)$ defined by disjoint union. As in the case of Segal's K-homology, given pointed maps

$$X \xleftarrow{f} Y \xrightarrow{g} Z$$

we have a homeomorphism

$$M_q(Y \cup_f (Y\times I) \cup_g Z) \cong |B_*(M_q(X), M_q(Y), M_q(Z))|.$$

Note that

$$B_*(M_*(X), M_*(Y), M_*(Z)) = \{B_p(M_q(X), M_q(Y), M_q(Z))\}_{p,q}$$

is a bisimplicial space and by the standard property of bisimplicial space

$$\begin{aligned} |M_*(Y \cup_f Y\times I \cup_g Z)| &\cong \left(\coprod_{q\geq 0} \Delta^q \times |B_*(M_q(X), M_q(Y), M_q(Z))|\right)\Big/\sim \\ &\cong |B_*(|M_*(X)|, |M_*(Y)|, |M_*(Z)|)|. \end{aligned}$$

It is essentially proved by Madsen and Tillmann that the inclusion

$$B_q(|M_*(X)|, |M_*(Y)|, |M_*(Z)|) \hookrightarrow |M_*(X)| \times |M_*(Y)|^q \times |M_*(Z)|$$

is a homotopy equivalence (Theorem 2.3 of [MT01]). Thus we can apply Theorem 4 and

$$X \longrightarrow \pi_*(\Omega|M_*(\Sigma X)|)$$

is a homology theory.

Note that this homology theory is larger than the stable homotopy theoy (away from 2) $\pi_*^S(X)_{[\frac{1}{2}]}$, due to the following fact proved in [Til99]

$$\mathbb{Z} \times B\Gamma_\infty^+ \simeq \Omega^\infty \Sigma^\infty (S^0)_{\frac{1}{2}} \times W$$

for some infinite loop space W.

Let us conclude this paper by a couple of questions.

Question 2. Find a more general and practically applicable condition on a map of simplicial spaces

$$p_* : X_* \longrightarrow Y_*$$

under which p_* induces a quasifibration after taking the geometric realization functor. It is also interesting if we could find a condition under which p_* induces a Serre fibration.

This question was discussed in a mailing list of algebraic topologists maintained by Don Davis. The question is

`http://www.lehigh.edu/~dmd1/ah24.txt`

And a response is

`http://www.lehigh.edu/~dmd1/pm24.txt`

Question 3. Find a nice categorification of Segal's K-homology.

As we have seen, Segal's K-homology $ku(X; H)$ is derived from the infinite symmetric product $\mathrm{SP}^\infty(X)$ by replacing the "coefficients" $n_i \in \mathbb{N}$ of elements

$$n_1 x_1 + \cdots + n_j x_j \in \mathrm{SP}^\infty(X)$$

by objects of a categorification of $\mathbb{N}$, i.e. the category $\mathbf{Vect}(H)$ of finite dimensional vector spaces in H.

From the view point of stable homotopy theory, the (co)homology theory next to K-theory is the elliptic (co)homology theory. There is an attempt by Baas, Dundas and Rognes [BDR04] to define a form of elliptic cohomology theory by using a categorification of $\mathbf{Vect}(H)$.

They used the category of 2-vector spaces introduced by Kapranov and Voevodsky [KV94] and introduces a notion of 2-vector bundle. They succeeded in defining a cohomology theory, although their cohomology theory fails to be complex oriented.

It is natural to try to categorify Segal's K-homology to obtain a form of elliptic homology. Note that the definition of Segal's K-homology involves with metric. We would need a categorification of the category of vector spaces with inner product.

Question 4. Explore the properties of the homology theory defined by Madsen-Tillmann.

One of the reasons why their homology theory contains the stable homotopy is that the construction of $M_q(X)$ is similar to that of $C(X;H)$ in the sense that the disjointness condition in $Y(j)_q$ is independent of the labelling points in X. On the other hand, the orthogonality conditions in Segal's K-homology depends on the labelling points in X.

There might be a modification of Madsen-Tillmann construction which is directly related to K-theory or elliptic (co)homology. Note that a conformal field theory is a functor from a certain cobordism category to the category of vector spaces.

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