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Spectral Analysis of the Dirac Polaron

Dedicated to Professor Asao Arai, on the occasion of his 60th birthday.

by

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Abstract

A system of a Dirac particle interacting with the radiation field is considered. The Hamiltonian of the system is defined by $H = \boldsymbol{\alpha} \cdot (\hat{\mathbf{p}} - q\mathbf{A}(\hat{\mathbf{x}})) + m\beta + H_f$, where $q \in \mathbb{R}$ is a coupling constant, $\mathbf{A}(\hat{\mathbf{x}})$ the quantized vector potential and H_f the free photon Hamiltonian. Since the total momentum is conserved, H is decomposed with respect to the total momentum with fiber Hamiltonian $H(\mathbf{p}), (\mathbf{p} \in \mathbb{R}^3)$. Since the self-adjoint operator $H(\mathbf{p})$ is bounded from below, one can define the lowest energy $E(\mathbf{p}, m) := \inf \sigma(H(\mathbf{p}))$. We prove that $E(\mathbf{p}, m)$ is an eigenvalue of $H(\mathbf{p})$ under the following conditions: (i) infrared regularization and (ii) $E(\mathbf{p}, m) < E(\mathbf{p}, 0)$. We also discuss the polarization vectors and the angular momentums.

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§1. Introduction

We consider a quantum system of a Dirac particle interacting with the radiation field. An example of a Dirac particle is the free electron. The Hilbert space for the Dirac particle is

$$\mathcal{H}_{\mathrm{p}} := L^2(\mathbb{R}^3_{\mathbf{x}}; \mathbb{C}^4), \tag{1.1}$$

and the free Hamiltonian for the Dirac particle is the free Dirac operator $\boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + m\beta$ acting on $\mathcal{H}_{\mathrm{p}}$, where $\hat{\mathbf{p}} = -i\nabla_{\mathbf{x}}$ denotes the momentum for the Dirac particle. The Hilbert space for the radiation field is the Fock space:

$$\mathcal{F}_{\mathrm{rad}} := \bigoplus_{n=0}^{\infty} \bigotimes_{\mathrm{sym}}^{n} L^2(\mathbb{R}^3_{\mathbf{k}} \times \{1, 2\}), \tag{1.2}$$

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where $\otimes_{\rm sym}^n$ means the n-fold symmetric tensor product with $\otimes_{\rm sym}^0 L^2(\mathbb{R}^3_{\mathbf{k}}\times\{1,2\}) := \mathbb{C}$. The Hilbert space for the total system is defined by

$$\mathcal{H} := \mathcal{H}_{\rm p} \otimes \mathcal{F}_{\rm rad}. \tag{1.3}$$

In this paper, we consider the quantum system described by the Hamiltonian

$$H := \boldsymbol{\alpha}\cdot(\hat{\mathbf{p}} - q\mathbf{A}(\hat{\mathbf{x}})) + m\beta + H_f, \tag{1.4}$$

where $q \in \mathbb{R}$ is a coupling constant, $\mathbf{A}(\hat{\mathbf{x}})$ denotes the quantized magnetic vector potential in the Coulomb gauge and H_f denotes the free photon Hamiltonian. We impose an ultraviolet cutoff in the quantized vector potential. We call the quantum system defined by (1.4) the Dirac-Maxwell model. The Hamiltonian (1.4) was introduced and discussed in the early days in quantum theory(e.g., [He]). By an informal perturbation theory, the Klein-Nishina formula (which gives a differential cross section for the Compton scattering) can be derived from the Dirac-Maxwell model[He]. A mathematical analysis of the Dirac-Maxwell model was initiated by A. Arai [A1, A2]. In the paper [A3], A. Arai proved that a non-relativistic limit of the Dirac-Maxwell model converges to the Pauli-Fierz model(the non-relativistic QED). See also [A4]. The essential self-adjointness of the Hamiltonian (1.4) with an external potential was discussed by E. Stockmayer and H. Zenk [SZ].

Since the Hamiltonian H is translation invariant, the total momentum of the system conserved, i.e., the Hamiltonian of the system strongly commutes with the total momentum operator

$$\mathbf{P} := \hat{\mathbf{p}} + d\Gamma(\mathbf{k}), \tag{1.5}$$

where $d\Gamma(\mathbf{k})$ denotes the momentum operator of the radiation field. Hence the Hamiltonian can be decomposed as

$$H \cong \int_{\mathbb{R}^3}^{\oplus} H(\mathbf{p})d\mathbf{p}, \tag{1.6}$$

$$\mathbf{P} \cong \int_{\mathbb{R}^3}^{\oplus} \mathbf{p}d\mathbf{p}, \tag{1.7}$$

where the symbol $\cong$ means a unitary equivalence. In this paper, we mainly study the fiber Hamiltonian $H(\mathbf{p})$ which describes the dynamics of the relativistic particle dressed in photons with total momentum $\mathbf{p}$. We call the quantum system described by $H(\mathbf{p})$ the *Dirac polaron*. As shown in [A2, A1], for $\mathbf{p} \in \mathbb{R}^3$, $H(\mathbf{p})$ has the form

$$H(\mathbf{p}) = \boldsymbol{\alpha}\cdot\mathbf{p} + m\beta + H_f - \boldsymbol{\alpha}\cdot d\Gamma(\mathbf{k}) - q\boldsymbol{\alpha}\cdot\mathbf{A}, \tag{1.8}$$

which acts on $\mathbb{C}^4 \otimes \mathcal{F}_{\rm rad}$, where $\mathbf{A}$ denotes the quantized vector potential at the origin($= \mathbf{A}(\mathbf{0})$). The fourth term $-\boldsymbol{\alpha}\cdot d\Gamma(\mathbf{k})$ describes the reaction due to the radiation field, and the last term $-q\boldsymbol{\alpha} \cdot \mathbf{A}$ is the electromagnetic interaction. It should be noted that $-q\boldsymbol{\alpha} \cdot \mathbf{A}$ is *not* $H(\mathbf{p})|_{q=0}$-bounded for any nonzero q, because the reaction term $-\boldsymbol{\alpha} \cdot d\Gamma(\mathbf{k})$ is comparable to H_f and $-q\boldsymbol{\alpha} \cdot \mathbf{A}$ is unbounded. This fact implies that $-q\boldsymbol{\alpha} \cdot \mathbf{A}$ is not a small perturbation no matter how q is small. One of the important fact on the Dirac polaron is that $H(\mathbf{p})$ is bounded from below for all values of all constants: the total momentum $\mathbf{p}$, the mass m and the coupling constant q(see [S1]). Hence, one can define the lowest energy by

$$E(\mathbf{p}, m) := \inf \sigma(H(\mathbf{p})) > -\infty, \tag{1.9}$$

where $\sigma(A)$ denotes the spectrum of A. If $H(\mathbf{p})$ has an eigenvalue E for $q \neq 0$, we say that an dressed particle state exists and the corresponding eigenvector is called a dressed particle state. In Section 4, we show that a dressed particle state exists under suitable conditions including (i) infrared regularization and (ii) the inequality

$$E(\mathbf{p}, m) < E(\mathbf{p}, 0). \tag{1.10}$$

The condition (1.10) will be assumed in Theorem 4.1, 4.2 and 4.4 below. One can observe that there exist $m^* > 0$ such that (1.10) holds for all $|m| > m^*$. We expect that $m^* = 0$, but we don't have its proof. In Section 5, we study the angular momentum and degeneracy of eigenvalues of the Dirac polaron $H(\mathbf{p})$. We will show that the angular momentum of the $\mathbf{p}$-direction commutes with $H(\mathbf{p})$, and any eigenvalue of $H(\mathbf{p})$ has an even multiplicity(admit infinity). Therefore $E(\mathbf{p}, m)$ is degenerate if it is an eigenvalue of $H(\mathbf{p})$.

This paper has three appendices. In Appendix A, we show that all spectral properties of the Dirac-Maxwell model and the Dirac polarons are independent of the choice of polarization vectors. Namely, two Hamiltonians which are defined by different polarization vectors are unitarily equivalent to each other. The discussions in the Appendix A can be applicable for various QED models(e.g., Pauli-Fierz model). In Appendix B, we propose a general definition of the angular momentum. Although the spectral properties of the QED Hamiltonians are independent of the choice of the polarization vectors, the definition of the angular momentum depends on the polarization vectors.

In Appendix C, we show some properties of the lowest energy $E(\mathbf{p})$ which is used in proofs of Theorems 4.1-4.4.

§2. Definitions of the Model

In this paper, unless confusion arise, we omit the symbol "$\otimes$" between two operators, for example, we write $A \otimes I$ as A and $I \otimes B$ as B, where I denotes the identity operator. For a closable operator T on $L^2(\mathbb{R}^3_{\mathbf{k}} \times \{1,2\})$, we denote by $d\Gamma(T)$ and $\Gamma(T)$ the second quantization operators of T (see [RS2]), which acts on $\mathcal{F}_{\rm rad}$. For $f \in L^2(\mathbb{R}^3_{\mathbf{k}} \times \{1,2\})$, we denote by $a(f)$ and $a(f)^*$ the annihilation operator and the the creation operator, respectively(see [RS2]), which are closed operators acting on $\mathcal{F}_{\rm rad}$. Let $\mathbf{e}^{(\lambda)} : \mathbb{R}^3 \mapsto \mathbb{R}^3, \lambda = 1,2$, be polarization vectors:

$$\mathbf{e}^{(\lambda)}(\mathbf{k}) \cdot \mathbf{e}^{(\mu)}(\mathbf{k}) = \delta_{\lambda,\mu}, \quad \mathbf{e}^{(\lambda)}(\mathbf{k}) \cdot \mathbf{k} = \mathbf{0}, \quad \mathbf{k} \in \mathbb{R}^3,\ \lambda, \mu \in \{1,2\}.$$

We write as $\mathbf{e}^{(\lambda)}(\mathbf{k}) = (e_1^{(\lambda)}(\mathbf{k}), e_2^{(\lambda)}(\mathbf{k}), e_3^{(\lambda)}(\mathbf{k}))$, and we suppose that each component $e_j^{(\lambda)}(\mathbf{k})$ is a Borel measurable function in $\mathbf{k}$. For objects $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, we set $\mathbf{a} \cdot \mathbf{b} := \sum_{j=1}^3 a_j b_j$. For a linear $F(\cdot)$ we set $F(\mathbf{a}) := (F(a_1), F(a_2), F(a_3))$. Let ω be a multiplication by the function

$$\omega(\mathbf{k}) = |\mathbf{k}|. \tag{2.1}$$

We choose a function

$$\hat{\rho} \in L^2(\mathbb{R}^3_{\mathbf{k}}) \cap \mathrm{Dom}(\omega^{-1}), \tag{2.2}$$

where Dom means the operator domain. For $j = 1,2,3$ and $\mathbf{x} \in \mathbb{R}^3$, we set

$$g_j(\mathbf{k}, \lambda; \mathbf{x}) := |\mathbf{k}|^{-1/2} \hat{\rho}(\mathbf{k}) e_j^{(\lambda)}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}, \quad (\mathbf{k}, \lambda) \in \mathbb{R}^3_{\mathbf{k}} \times \{1,2\}.$$

For each fixed $\mathbf{x} \in \mathbb{R}^3$, the function $g_j(\mathbf{x})(\cdot) := g_j(\cdot\,; \mathbf{x})$ is a function in $L^2(\mathbb{R}^3_{\mathbf{k}} \times \{1,2\})$. The quantized magnetic vector potential at $\mathbf{x} \in \mathbb{R}^3$ is defined by

$$\begin{aligned} \mathbf{A}(\mathbf{x}) &:= (A_1(\mathbf{x})), A_2(\mathbf{x}), A_3(\mathbf{x})), \\ A_j(\mathbf{x}) &:= \frac{1}{\sqrt{2}} \overline{[a(g_j(\mathbf{x})) + a(g_j(\mathbf{x}))^*]}, \quad j = 1,2,3, \end{aligned}$$

where, for a closable operator T, $\bar{T}$ denotes its closure. For each $\mathbf{x} \in \mathbb{R}^3$, $A_j(\mathbf{x})$ is a self-adjoint operator on $\mathcal{F}_{\rm rad}$(see [RS2]). Since $\mathbf{e}^{(\lambda)}(\mathbf{k})$'s are perpendicular to $\mathbf{k}$, the operators $\mathbf{A}(\mathbf{x})$ satisfy the Coulomb gauge condition:

$$\mathrm{div}\, \mathbf{A}(\mathbf{x}) = \sum_{j=1}^3 \partial_{x_j} A_j(\mathbf{x}) = 0. \tag{2.3}$$

Remark 2.1. The function $\hat{\rho}$ is called an ultraviolet cutoff function. An typical example of $\hat{\rho}$ is the characteristic function of the region $\{\mathbf{k} \in \mathbb{R}^3 | \kappa \leq |\mathbf{k}| \leq \Lambda\}$, where κ and Λ are non-negative constants. Λ is called an ultraviolet cutoff. κ is called an infrared cutoff if it is strictly positive.

The Hilbert space $\mathcal{H}$ can be identified as

$$\mathcal{H} = L^2(\mathbb{R}^3_{\mathbf{x}}; \mathbb{C}^4 \otimes \mathcal{F}_{\rm rad}) = \int_{\mathbb{R}^3}^{\oplus} \mathbb{C}^4 \otimes \mathcal{F}_{\rm rad} d\mathbf{x}. \tag{2.4}$$

Under this identification, we define the quantized vector potential in the following way. Since $g_j(\mathbf{x}) \in L^2(\mathbb{R}^3_{\mathbf{k}} \times \{1,2\})$ is strongly continuous in $\mathbf{x} \in \mathbb{R}^3$, the map $\mathbf{x} \mapsto A_j(\mathbf{x})$ is a self-adjoint operator valued measurable function. Then we can define a self-adjoint operator on $\mathcal{H}$ by

$$A_j(\hat{\mathbf{x}}) := \int_{\mathbb{R}^3}^{\oplus} A_j(\mathbf{x}) d\mathbf{x}. \tag{2.5}$$

Namely, when we identify $\Psi \in D(A_j(\hat{\mathbf{x}}))$ as the $\mathcal{F}_{\mathrm{rad}}$-valued square integrable function and the operator, the action of the operator $A_j(\hat{\mathbf{x}})$ is given by $(A_j(\hat{\mathbf{x}})\Psi)(\mathbf{x}) = A_j(\mathbf{x})\Psi(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^3$. The operator valued vector

$$\mathbf{A}(\hat{\mathbf{x}}) := (A_1(\hat{\mathbf{x}}), A_2(\hat{\mathbf{x}}), A_3(\hat{\mathbf{x}})) \tag{2.6}$$

is also called the quantized vector potential.

The free photon Hamiltonian as the second quantization of ω:

$$H_f := d\Gamma(\omega). \tag{2.7}$$

The Dirac-Maxwell Hamiltonian is defined by

$$H := \boldsymbol{\alpha} \cdot (\hat{\mathbf{p}} - q\mathbf{A}(\hat{\mathbf{x}})) + m\beta + H_f, \tag{2.8}$$

where $\hat{\mathbf{p}} = -i\nabla_{\mathbf{x}}$ and $\nabla_{\mathbf{x}}$ is the gradient operator acting in $\mathcal{H}_{\mathrm{p}}$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and β are Dirac matrices satisfying $\alpha_1, \alpha_2, \alpha_3, \beta \in M_4(\mathbb{C})$ and

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk}, \tag{2.9}$$

$$\alpha_j \beta + \beta \alpha_j = 0, \tag{2.10}$$

$$\beta^2 = I_{\mathbb{C}^4}, \tag{2.11}$$

the constant $m \in \mathbb{R}$ is the rest mass of the Dirac particle, $q \in \mathbb{R}$ is a coupling constant. In the right hand side of (2.8), we omit the symbols $\otimes I$ and $I \otimes$, i.e., the expression (2.8) is an abbreviation for

$$H = (\boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + m\beta) \otimes I_{\mathcal{F}_{\mathrm{rad}}} - q \sum_{j=1}^{3} (\alpha_j \otimes I_{L^2(\mathbb{R}^3_{\mathbf{x}})}) \cdot A_j(\hat{\mathbf{x}}) + I_{\mathcal{H}_{\mathrm{p}}} \otimes H_f.$$

In this paper, we use the Weyl representation for the Dirac matrices. Since all representations of the Dirac matrices are unitarily equivalent to each other, this choice does not affect the spectral properties of H(see [T, Lemma 2.25]).

It is easy to see that H is symmetric. Although the essential self-adjointness of H was proven in [A1], we give a slightly improved result:

Proposition 2.2 (Essential self-adjointness). *$\bar{H}$ is a self-adjoint operator and essentially self-adjoint on any core for $\sqrt{-\triangle} + H_f$.*

Proof of Proposition 2.2. The proof is a simple application of Nelson's commutator theorem. Our choice of a comparison operator for Nelson's commutator theorem is $\sqrt{-\triangle} + H_f$. See [S2] for details. □

§3. Momentum Conservation and Fiber Hamiltonian $H(\mathbf{p})$

The total momentum operator is defined by

$$\mathbf{P} := \overline{\hat{\mathbf{p}} + d\Gamma(\mathbf{k})}. \tag{3.1}$$

The Hamiltonian H strongly commutes with $\mathbf{P}$ (see [A1]). To construct the fiber Hamiltonian, we define a self-adjoint operator

$$Q := \overline{\mathbf{x} \cdot d\Gamma(\mathbf{k})}. \tag{3.2}$$

Let U_F be the Fourier transform from $L^2(\mathbb{R}^3_{\mathbf{x}})$ to $L^2(\mathbb{R}^3_{\mathbf{p}})$. We set

$$U := (U_F \otimes I_{\mathbb{C}^4}) \exp(iQ). \tag{3.3}$$

Then we can identify $U\mathcal{H}$ as a constant fiber direct integral

$$U\mathcal{H} \cong \int_{\mathbb{R}^3}^{\oplus} \mathbb{C}^4 \otimes \mathcal{F}_{\mathrm{rad}} d\mathbf{p}. \tag{3.4}$$

For every $\mathbf{p} \in \mathbb{R}^3$, we define

$$H(\mathbf{p}) := \boldsymbol{\alpha} \cdot \mathbf{p} + m\beta + H_f - \boldsymbol{\alpha} \cdot d\Gamma(\mathbf{k}) - q\boldsymbol{\alpha} \cdot \mathbf{A}, \tag{3.5}$$

which acts on $\mathbb{C}^4 \otimes \mathcal{F}_{\mathrm{rad}}$, where $\mathbf{A} := \mathbf{A}(\mathbf{0})$.

Proposition 3.1. *For all* $\mathbf{p} \in \mathbb{R}^3$*,* $H(\mathbf{p})$ *is essentially self-adjoint and*

$$U\bar{H}U^* = \int_{\mathbb{R}^3}^{\oplus} \overline{H(\mathbf{p})} d\mathbf{p}, \tag{3.6}$$

$$U\mathbf{P}U^* = \int_{\mathbb{R}^3}^{\oplus} \mathbf{p} d\mathbf{p}. \tag{3.7}$$

hold, where $\int^{\oplus}(\cdots)$ *denotes fiber direct integral operator with respect to* (3.4)*.*

Proof. See [A2]. □

Remark 3.2. Physically $\overline{H(\mathbf{p})}$ is the Hamiltonian of the fixed total momentum $\mathbf{p} \in \mathbb{R}^3$. One can show that the spectral properties of $\overline{H(\mathbf{p})}$ is independent of the choice of polarization vectors, because the Hamiltonians with different polarization vectors are unitarily equivalent each other. See Appendix A.

Remark 3.3. We call $H(\mathbf{p})$ the Dirac polaron Hamiltonian, which was introduced in [A4]. It is expected that, as in the model of the H. Fröhlich polaron, the electromagnetic interaction forms a quasiparticle where the bare Dirac particle is surrounded by the photon clouds. Such a quasiparticle with momentum $\mathbf{p} \in \mathbb{R}^3$ is considered as the ground state of $\overline{H(\mathbf{p})}$, if it exist. The existence of ground state of $\overline{H(\mathbf{p})}$ is the main subject of our paper.

Remark 3.4. Note that $\mathrm{Dom}(\boldsymbol{\alpha}\cdot d\Gamma(\mathbf{k})) \subset \mathrm{Dom}(H_f)$. Hence $\mathrm{Dom}(H_f) = \mathrm{Dom}(H(\mathbf{p}))$ and $H(\mathbf{p})$ is essentially self-adjoint on $\mathrm{Dom}(H_f)$.

One of the most important fact of $\overline{H(\mathbf{p})}$ is the semi-boundedness:

Theorem 3.5. *([S1]) For any* $\mathbf{p}$, $\overline{H(\mathbf{p})}$ *is bounded from below. Moreover* $H(\mathbf{p})$ *is essentially self-adjoint on any core for* H_f.

Proof. The first statement was shown in [S1], where it is assumed the condition $\hat{\rho} \in \mathrm{Dom}(\omega^{1/2})$. But one can remove this condition in the following procedure. In [S1, ineq. (24)], it is shown that $H(\mathbf{p})$ is bounded from below and the lower bound is a function of $\|\omega^{1/2}\mathbf{g}\|_{L^2(\mathbb{R}^3)}$ but $\|\omega\mathbf{g}\|_{L^2(\mathbb{R}^3)}$. Therefore, firstly, we regularize $\hat{\rho}$ as $\hat{\rho}_\lambda(\mathbf{k}) := \hat{\rho}(\mathbf{k})\chi_{|\mathbf{k}|\leq\lambda}$, then we obtain the lower bound of the regularized Hamiltonian $H_\lambda(\mathbf{p}) \geq C_\lambda$. Since C_λ converges as $\lambda \to \infty$ and $H_\lambda(\mathbf{p})$ converges to $H(\mathbf{p})$ on a finite particle subspace, we get $H(\mathbf{p}) \geq \lim_{\epsilon\to+0} C_\epsilon > -\infty$. The second statement follows from the Wüst's Theorem([RS2]) and the bound

$$\|\boldsymbol{\alpha}\cdot(d\Gamma(\mathbf{k}) - q\mathbf{A})\Psi\|^2 \leq \|(H_f + E)\Psi\|^2, \quad \Psi \in \mathrm{Dom}(H_f) \tag{3.8}$$

for some $E > 0$. The bound (3.8) was given in [S1]. □

Thus we can define the lowest energy of the Dirac polaron with total momentum $\mathbf{p}$ by:

$$E(\mathbf{p}, m) := \inf \sigma(\overline{H(\mathbf{p})}). \tag{3.9}$$

The energy $E(\mathbf{p}, m)$ depends on all parameters $(\mathbf{p}, m, q) \in \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$. When m dependence in $E(\mathbf{p}, m)$ is not important, we write $E(\mathbf{p}, m)$ as $E(\mathbf{p})$.

§4. Existence of a Ground State

For a self-adjoint operator bounded below, T, we say that T has a ground state if $\inf \sigma(T)$ is an eigenvalue of T. In this section, we give criteria for $\overline{H(\mathbf{p})}$ to have a ground state.

Theorem 4.1. *Suppose that $\hat{\rho}$ is spherically symmetric and the bound*

$$\int_{\mathbb{R}^3} \frac{q^2}{(E(\mathbf{p}-\mathbf{k})-E(\mathbf{p})+|\mathbf{k}|)^2} \frac{|\hat{\rho}(\mathbf{k})|^2}{|\mathbf{k}|} d\mathbf{k} < 1 \tag{4.1}$$

holds. Assume that $E(\mathbf{p},m) < E(\mathbf{p},0)$. Then the Dirac polaron Hamiltonian $\overline{H(\mathbf{p})}$ has a ground state.

Using the lower bound on $E(\mathbf{p}-\mathbf{k})-E(\mathbf{p})+|\mathbf{k}|$, which is proved in Theorem Appendix C.10 below, we obtain the following result:

Theorem 4.2. *Assume that $\hat{\rho}$ be spherically symmetric and that $E(\mathbf{p},m) < E(\mathbf{p},0)$. Assume the infrared regular condition $\hat{\rho} \in \mathrm{Dom}(\omega^{-3/2})$. Then there exists a constant $q_0 > 0$ such that for all q with $|q| < q_0$, $\overline{H(\mathbf{p})}$ has a ground state.*

Remark 4.3. Since $E(\mathbf{p},m)$ is concave in m(Proposition Appendix C.1) and $\lim_{m\to\infty} E(\mathbf{p},m) = -\infty$, there exist $m^* \geq 0$ such that $E(\mathbf{p},m) < E(\mathbf{p},0)$ for all $|m| > m^*$.

A proof of Theorem 4.1 is based on the estimates of a photon number bound. The condition (4.1) can be considered as a restriction on the coupling constant q. There are two ways to remove this restriction. The first one is the method discovered by C. Gérard in [Ge] and another one is the photon derivative bound developed in [GLL]. In this paper, we use the photon derivative bound. We need the additional assumptions:

(Λ) (i) $\hat{\rho}$ is a spherically symmetric function. (ii) There is an open set $S \subset \mathbb{R}^3$ such that $\bar{S} = \mathrm{supp}\,\hat{\rho}$ and $\hat{\rho}$ is continuously differentiable on S. (iii) For all $R > 0$, the bounded region $S_R := \{\mathbf{k} \in S \,|\, |\mathbf{k}| < R\}$ has the cone property(see [LL] for the definition).

The theorem below proves the existence of ground state of the Dirac polaron for all values of coupling constant q:

Theorem 4.4. *Assume the condition (Λ). Moreover we assume that*

$$\hat{\rho} \in \mathrm{Dom}(\omega^{-3/2}), \quad |\mathbf{k}|^{-5/2}\hat{\rho}(\mathbf{k}) \in L^p(S_R), \quad |\mathbf{k}|^{-3/2}|\nabla\hat{\rho}(\mathbf{k})| \in L^p(S_R), \tag{4.2}$$

for all $p \in [1,2)$ and $R > 0$. Suppose that $E(\mathbf{p},m) < E(\mathbf{p},0)$. Then $\overline{H(\mathbf{p})}$ has a ground state.

Remark 4.5. The followings are examples Let $\chi_{\kappa,\Lambda}(\mathbf{k})$ be a characteristic function of the region $\{\mathbf{k} \in \mathbb{R}^3 | \kappa < |\mathbf{k}| < \Lambda\}$. For all $\kappa > 0$ and $\Lambda < \infty$, the cutoff function $\hat{\rho} = \chi_{\kappa,\Lambda}$ satisfies the conditions (Λ) and (4.2). The function $\hat{\rho}(\mathbf{k}) = |\mathbf{k}| \exp(-\lambda|\mathbf{k}|)$ ($\lambda > 0$) also satisfies the conditions (Λ) and (4.2).

Remark 4.6. It is known that, in non-relativistic QED, the existence of a dressed particle requires the restriction $|\mathbf{p}|/m \leq 1$ (see [C]). On the other hand, Theorems 4.1-4.4 does not require restriction on $|\mathbf{p}|/m$. This fact is a crucial difference between relativistic and non-relativistic dynamics. This result can be interpreted as follows. In general, the velocity operator is defined by $i = \sqrt{-1}$ times the commutator of the energy Hamiltonian with the position. Hence, the velocity operators of the non-relativistic particle and Dirac particle are defined by

$$\hat{\mathbf{p}}/m = i[\hat{\mathbf{p}}^2/2m, \mathbf{x}], \tag{4.3}$$

$$\boldsymbol{\alpha} = [\boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + m\beta, \mathbf{x}], \tag{4.4}$$

respectively. Hence the non-relativistic particle can move faster than the light, and the particle with velocity $|\mathbf{p}|/m > 1$ makes a shock wave of light and lose their kinetic energy. Therefore such a non-relativistic particle is unstable in the presence of the electromagnetic interaction. On the other hand, since the speed of the Dirac particle is smaller than that of light, $\|\boldsymbol{\alpha}\| \leq 1$, this kind of catastrophe does not occur, and the dressed electron state is stable for all $|\mathbf{p}|$.

Remark 4.7. It is easy to see that the Hermitian matrix $\boldsymbol{\alpha} \cdot \mathbf{p} + m\beta$ has two eigenvalues $\pm\sqrt{\mathbf{p}^2 + m^2}$, each of which is two-fold degenerate. Let $u_i^{(\pm)} \in \mathbb{C}^4$, $i = 1, 2$ be the corresponding normalized eigenvectors:

$$(\boldsymbol{\alpha} \cdot \mathbf{p} + m\beta)u_i^{(\pm)} = \pm\sqrt{\mathbf{p}^2 + m^2}u_i^{(\pm)}, \quad i = 1, 2.$$

Let $\Omega := (1, 0, 0, \ldots) \in \mathcal{F}_{\mathrm{rad}}$ be the vacuum. Ω is the unique eigenvector of both H_f and $d\Gamma(k_j)$, $j = 1, 2, 3$. We set $\Phi_i^{(\pm)} := u_i^{(\pm)} \otimes \Omega$, $j = 1, 2$. Clearly,

$$H(\mathbf{p})|_{q=0}\Phi_i^{(\pm)} = \pm\sqrt{\mathbf{p}^2 + m^2}\Phi_i^{(\pm)}, \quad i = 1, 2.$$

Thus, in the case $q = 0$, $H(\mathbf{p})|_{q=0}$ has two eigenvalues $\pm\sqrt{\mathbf{p}^2 + m^2}$. These eigenvectors $\Phi_i^{(+)}$, $i = 1, 2$ (resp. $\Phi_i^{(-)}$, $i = 1, 2$) describe states of a freely moving positive(resp. negative) energy particle with momentum $\mathbf{p}$. Hence, if photons and the Dirac particle are decoupled, a Dirac particle associated with a positive eigenvalue exists and the positive eigenvalue is embedded. We are interested in the fate of these eigenvalues when the interaction is switched on. As is shown in Fig.1, the lowest energy $E(\mathbf{p}, m)$ converges to $-\sqrt{\mathbf{p}^2 + m^2}$ as $q \to 0$. As is written in textbooks of physics(e.g. [B, He]), it is expected that any positive energy electron falls down to a negative energy states by a spontaneous emission of photons. Hence it is expected that the eigenvalue $+\sqrt{\mathbf{p}^2 + m^2}$ is unstable under the perturbation $q\boldsymbol{\alpha} \cdot \mathbf{A}$. Theorems 4.1-4.4 ensure that the negative energy dressed electron exists under some conditions. But the instability of $\sqrt{\mathbf{p}^2 + m^2}$ has not been proved yet.

Figure 1. Spectrum of $H(\mathbf{p})|_{q=0}$ and $H(\mathbf{p})$.

§5. Angular Momentum and Degeneracy of Eigenvalues

In this section we show that the angular momentum around $\mathbf{j}$-axis ($\mathbf{j} \in \mathbb{R}^3\backslash\{0\}$) of the Dirac polaron is conserved if $\mathbf{p}$ is parallel to $\mathbf{j}$ and $\hat{\rho}(\mathbf{k})$ has axial symmetry around $\mathbf{j}$. Let $(\overline{H(\mathbf{p})}, \mathbf{e})$ be a Dirac polaron model with an arbitrarily given polarization vectors $\mathbf{e} = (\mathbf{e}^{(1)}, \mathbf{e}^{(2)})$. The total angular momentum around $\mathbf{j}$-axis in the system $(\overline{H(\mathbf{p})}, \mathbf{e})$ is defined by

$$J_{\mathbf{j}}(\mathbf{e}) := S_{\mathbf{j}} + L_{\mathbf{j}}(\mathbf{e}),$$

where $S_{\mathbf{j}} := \oplus^2(\mathbf{j} \cdot \vec{\sigma})/2$, $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices, and $L_{\mathbf{j}}(\mathbf{e})$ is a angular momentum for the radiation field, which is defined in Appendix B.

Proposition 5.1. *The spectrum of $J_{\mathbf{j}}(\mathbf{e})$ is the set of half-integers:*

$$\sigma(J_{\mathbf{j}}(\mathbf{e})) = \mathbb{Z}_{1/2} := \{\pm 1/2, \pm 3/2, \pm 5/2, \cdots\}.$$

In particular, $J_{\mathbf{j}}(\mathbf{e})$ is decomposable as

$$J_{\mathbf{j}}(\mathbf{e}) \cong \bigoplus_{z \in \mathbb{Z}_{1/2}} z. \tag{5.1}$$

with respect to the identification/

$$\mathbb{C}^4 \otimes \mathcal{F}_{\mathrm{rad}} \cong \bigoplus_{z \in \mathbb{Z}_{1/2}} \mathcal{F}(z).$$

The conclusion in this section is the following:

Theorem 5.2. *Let $\mathbf{j}$ be a unit vector being parallel with $\mathbf{p}$. Assume that $\hat{\rho}(\mathbf{k}) = \hat{\rho}(R\mathbf{k}), \mathbf{k} \in \mathbb{R}^3$, for all $R \in O(3)$ with $R\mathbf{j} = \mathbf{j}$. Then $\overline{H(\mathbf{p})}$ strongly commutes with $J_{\mathbf{j}}(\mathbf{e})$. In particular, $\overline{H(\mathbf{p})}$ is decomposable as*

$$\overline{H(\mathbf{p})} \cong \bigoplus_{z \in \mathbb{Z}_{1/2}} H(\mathbf{p} : z),$$

corresponding to the decomposition (5.1). Moreover, for all $z \in \mathbb{Z}_{1/2}$*,* $H(\mathbf{p} : z)$ *is unitarily equivalent to* $H(\mathbf{p} : -z)$*, and the multiplicity of any eigenvalue of* $\overline{H(\mathbf{p})}$ *is even.*

Remark 5.3. In the paper [Hi], F. Hiroshima defines an angular momentum in QED, which differs from our definition.

§6. Proof of Theorems 4.1 - 4.4

For a constant $\nu \geq 0$, we define a regularized Hamiltonian to avoid the risk of infrared divergence:

$$H_\nu(\mathbf{p}) := \boldsymbol{\alpha} \cdot \mathbf{p} + m\beta + H_f(\nu) - \boldsymbol{\alpha} \cdot d\Gamma(\mathbf{k}) - q\boldsymbol{\alpha} \cdot \mathbf{A}, \tag{6.1}$$

where

$$H_f(\nu) := d\Gamma(\omega_\nu), \quad \omega_\nu(\mathbf{k}) = (1+\nu)|\mathbf{k}| + \nu. \tag{6.2}$$

Let $N_f := d\Gamma(1)$ be the photon number operator. Note that $H_f(\nu) = H_f + \nu(H_f + N_f)$ and $H_0(\mathbf{p}) = H(\mathbf{p})$. By the Kato-Rellich theorem, one can easily show that, for all $\nu > 0$, $H_\nu(\mathbf{p})$ is self-adjoint on $\mathrm{Dom}(H_f(\nu))$, and essentially self-adjoint on any core for $H_f(\nu)$. Since $H_\nu(\mathbf{p}) \geq H(\mathbf{p})$, $H_\nu(\mathbf{p})$ is also bounded from below. We set $\mathcal{D} := \mathrm{Dom}(H_f) \cap \mathrm{Dom}(N_f)$. Then $\mathcal{D}$ is a common core for $\overline{H_\nu(\mathbf{p})}$, $(\nu \geq 0)$. We set

$$E_\nu(\mathbf{p}) := \inf \sigma(\overline{H_\nu(\mathbf{p})}). \tag{6.3}$$

For $\nu > 0$, the massive Hamiltonian $H_\nu(\mathbf{p})$ was studied in [A1, A2], in which A. Arai showed that $H_\nu(\mathbf{p})$ has a ground state for all $\nu > 0$.

Lemma 6.1 (Existence of ground state for $\nu > 0$). *Assume that* $\nu > 0$*. Then*

$$\inf \sigma_{\mathrm{ess}}(H_\nu(\mathbf{p})) - E_\nu(\mathbf{p}) \geq \nu. \tag{6.4}$$

In particular, $H_\nu(\mathbf{p})$ *has a ground state.*

Proof. See [A2]. □

By Lemma 6.1, for all $\nu > 0$, $H_\nu(\mathbf{p})$ has a normalized ground state $\Phi_\nu(\mathbf{p}) \in \mathrm{Dom}(H_f(\nu))$. In the following, we construct a ground state of $H_0(\mathbf{p})$ as suitable limits of $\Phi_\nu(\mathbf{p})$. Since $\Phi_\nu(\mathbf{p})$ is normalized, there exists a sequence $\{\Phi_{\nu_j}(\mathbf{p})\}_{j=1}^\infty$ with $\lim_{j\to\infty} \nu_j = 0$ such that $\{\Phi_{\nu_j}\}_j$ has a weak limit.

Lemma 6.2. *Let* $\{\nu_j\}_{j=1}^\infty$ *be a sequence such that* Φ_{ν_j} *has a weak limit* $\Phi_0(\mathbf{p}) := \text{w-lim}_{j\to\infty} \Phi_{\nu_j}$*. Assume* $\Phi_0 \neq 0$*. Then* $\Phi_0 \in \mathrm{Dom}(\overline{H(\mathbf{p})})$ *and* Φ_0 *is a ground state of* $\overline{H(\mathbf{p})}$*.*

Proof. For all $\Psi \in \mathcal{D}$, one has

(6.5)
$$\langle H(\mathbf{p})\Psi, \Phi_0\rangle = \lim_{j\to\infty} \left\langle \Psi, H(\mathbf{p})\Phi_{\nu_j}\right\rangle = \lim_{j\to\infty} \left\langle \Psi, \{E_{\nu_j}(\mathbf{p}) - \nu_j(H_f + N_f)\}\Phi_{\nu_j}\right\rangle .$$

By Proposition Appendix C.9, we have $E_{\nu_j}(\mathbf{p}) \to E_0(\mathbf{p})$ as $j \to \infty$. By assumption (2), we have

$$\lim_{j\to\infty} \nu_j | \left\langle \Psi, (H_f + N_f)\Phi_{\nu_j}\right\rangle | \leq \lim_{j\to\infty} \nu_j \|(H_f + N_f)\Psi\| \cdot \|\Phi_{\nu_j}\| = 0. \tag{6.6}$$

Hence $\langle H(\mathbf{p})\Psi, \Phi_0\rangle = \langle \Psi, E(\mathbf{p})\Phi_0\rangle$ for all $\Psi \in \mathcal{D}$. Since $\mathcal{D}$ is a core for $\overline{H(\mathbf{p})}$, $\Phi_0 \in \mathrm{Dom}(\overline{H(\mathbf{p})})$ and $\overline{H(\mathbf{p})}\Phi_0 = E(\mathbf{p})\Phi_0$ holds. □

$H_\nu(\mathbf{p})$ and $E_\nu(\mathbf{p})$ depend on $\mathbf{p}, m, \nu$, etc. When we need to indicate its dependence, we write $E_\nu(\mathbf{p}, m, \cdots)$ and $H_\nu(\mathbf{p}, m, q, \cdots)$ for $E_\nu(\mathbf{p})$ and $H_\nu(\mathbf{p})$, respectively.

In this section, we use the following identification

$$\mathbb{C}^4 \otimes \mathcal{F}_{\mathrm{rad}} = \bigoplus_{n=0}^{\infty} \mathbb{C}^4 \otimes \mathcal{F}^{(n)}, \quad \mathcal{F}^{(n)} := \overset{n}{\underset{\mathrm{s}}{\otimes}} L^2(\mathbb{R}^3_{\mathbf{k}} \times \{1,2\}),$$

and each vector $\Psi^{(n)} \in \mathbb{C}^4 \otimes \mathcal{F}^{(n)}$ is identified with a Hilbert space valued function $\Psi^{(n)}(\mathbf{k}, \lambda; \cdot) : \mathbb{R}^3_{\mathbf{k}} \times \{1,2\} \mapsto \mathbb{C}^4 \otimes \mathcal{F}^{(n-1)}$. For all$(\mathbf{k}, \lambda) \in \mathbb{R}^3 \times \{1,2\}$, we define a map

(6.7)
$$a_\lambda(\mathbf{k}) : \mathbb{C}^4 \otimes \mathcal{F}_{\mathrm{rad}} \to \prod_{n=0}^{\infty} \mathbb{C}^4 \otimes \mathcal{F}^{(n)} := \{(\Phi^{(n)})_{n=0}^{\infty} | \Phi^{(n)} \in \mathbb{C}^4 \otimes \mathcal{F}^{(n)}\}$$

(6.8)
$$a_\lambda(\mathbf{k})\Psi := (\Psi^{(1)}(\mathbf{k}, \lambda), \sqrt{2}\Psi^{(2)}(\mathbf{k}, \lambda; \cdot), \ldots, \sqrt{n}\Psi^{(n)}(\mathbf{k}, \lambda; \cdot), \ldots) \in \prod_{n=0}^{\infty} \mathbb{C}^4 \otimes \mathcal{F}^{(n)}.$$

For almost every $(\mathbf{k}, \lambda)$, $a_\lambda(\mathbf{k})$ is well-defined as a linear map. The smeared annihilation operator $a(f)$ formally satisfies

$$a(f)\Psi = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} d\mathbf{k} f(\mathbf{k}, \lambda)^* a_\lambda(\mathbf{k})\Psi. \tag{6.9}$$

It is not necessary to consider that $a_\lambda(\mathbf{k})$ is an operator valued distribution. This definition of $a_\lambda(\mathbf{k})$ is useful for our purpose below(Proposition 6.3). In general, $a_\lambda(\mathbf{k})\Psi \notin \mathbb{C}^4 \otimes \mathcal{F}_{\mathrm{rad}}$, but one can show that $a_\lambda(\mathbf{k})\Psi \in \mathbb{C}^4 \otimes \mathcal{F}_{\mathrm{rad}}$ for a class of vectors $\Psi \in \mathbb{C}^4 \otimes \mathcal{F}_{\mathrm{rad}}$. Let $\mathrm{w} : \mathbb{R}^3 \to [0, \infty)$ be an almost positive Borel measurable function. Then, for any $\Psi \in \mathrm{Dom}(d\Gamma(\mathrm{w})^{1/2})$ and for almost every $(\mathbf{k}, \lambda) \in \mathbb{R}^3 \times$

$\{1,2\}$, the vector $a_\lambda(\mathbf{k})\Psi$ is a $\mathbb{C}^4\otimes\mathcal{F}_{\rm rad}$-valued function. Because, for any $\Psi\in \mathrm{Dom}(d\Gamma(\mathrm{w})^{1/2})$, one has

$$\|d\Gamma(\mathrm{w})^{1/2}\Psi\|^2=\sum_{n=1}^{\infty}\sum_{\lambda=1,2}\int_{\mathbb{R}^3}d\mathbf{k}\mathrm{w}(\mathbf{k})n\|\Psi^{(n)}(\mathbf{k},\lambda;\cdot)\|^2_{\mathbb{C}^4\otimes\mathcal{F}^{(n-1)}}<\infty, \tag{6.10}$$

and hence $\sum_{n=1}^{\infty}n\|\Psi^{(n)}(\mathbf{k},\lambda;\cdot)\|^2_{\mathbb{C}^4\otimes\mathcal{F}^{(n-1)}}<\infty$ for almost every $(\mathbf{k},\lambda)$.

We set $\mathbf{g}(\mathbf{k},\lambda):=\mathbf{g}(\mathbf{k},\lambda;0)$.

Proposition 6.3. *Let $\nu>0$. Then $a_\lambda(\mathbf{k})\Phi_\nu(\mathbf{p})\in\mathrm{Dom}(H_\nu(\mathbf{p}))$ and*

$$a_\lambda(\mathbf{k})\Phi_\nu(\mathbf{p})=\frac{q}{\sqrt{2}}(H_\nu(\mathbf{p}-\mathbf{k})-E_\nu(\mathbf{p})+\omega_\nu(\mathbf{k}))^{-1}\boldsymbol{\alpha}\cdot\mathbf{g}(\mathbf{k},\lambda)\Phi_\nu(\mathbf{p}), \tag{6.11}$$

for almost every $(\mathbf{k},\lambda)\in\mathbb{R}^3\times\{1,2\}$.

Proof. For all $f\in\mathrm{Dom}(\omega_\nu)$ and $\Psi\in\mathcal{D}$, we have

$$\langle(H_\nu(\mathbf{p})-E_\nu(\mathbf{p}))\Psi,a(f)\Phi_\nu(\mathbf{p})\rangle=\left\langle\Psi,\left\{-a(\omega_\nu f)+\boldsymbol{\alpha}\cdot a(\mathbf{k}f)+\tfrac{q}{\sqrt{2}}\boldsymbol{\alpha}\cdot\langle f,\mathbf{g}\rangle\right\}\Phi_\nu(\mathbf{p})\right\rangle.$$

Hence

$$\sum_{\lambda=1,2}\int_{\mathbb{R}^3}d\mathbf{k}f(\mathbf{k},\lambda)^*\langle(H_\nu(\mathbf{p})-E_\nu(\mathbf{p}))\Psi,a_\lambda(\mathbf{k})\Phi_\nu(\mathbf{p})\rangle=$$
$$\sum_{\lambda=1,2}\int_{\mathbb{R}^3}d\mathbf{k}f(\mathbf{k},\lambda)^*\left\langle\Psi,-\omega_\nu(\mathbf{k})a_\lambda(\mathbf{k})\Phi_\nu(\mathbf{p})+\boldsymbol{\alpha}\cdot\mathbf{k}a_\lambda(\mathbf{k})\Phi_\nu(\mathbf{p})+\tfrac{q}{\sqrt{2}}\boldsymbol{\alpha}\cdot\mathbf{g}(\mathbf{k},\lambda)\Phi_\nu(\mathbf{p})\right\rangle.$$

Since $\mathrm{Dom}(\omega_\nu)$ is dense in $L^2(\mathbb{R}^3_{\mathbf{k}}\times\{1,2\})$, we have

$$\begin{aligned}&\langle(H_\nu(\mathbf{p})-E_\nu(\mathbf{p}))\Psi,a_\lambda(\mathbf{k})\Phi_\nu(\mathbf{p})\rangle\\&=\langle\Psi,(-\omega_\nu(\mathbf{k})a_\lambda(\mathbf{k})+\boldsymbol{\alpha}\cdot\mathbf{k}a_\lambda(\mathbf{k})+\tfrac{q}{\sqrt{2}}\boldsymbol{\alpha}\cdot\mathbf{g}(\mathbf{k},\lambda))\Phi_\nu(\mathbf{p})\rangle,\end{aligned}$$

for almost every $(\mathbf{k},\lambda)\in\mathbb{R}^3\times\{1,2\}$, and all $\Psi\in\mathcal{D}$. This means that $a_\lambda(\mathbf{k})\Phi_\nu(\mathbf{p})\in D(H_\nu(\mathbf{p}))$ and

$$(H_\nu(\mathbf{p})-E_\nu(\mathbf{p})+\omega_\nu(\mathbf{k})-\boldsymbol{\alpha}\cdot\mathbf{k})a_\lambda(\mathbf{k})\Phi_\nu(\mathbf{p})=\frac{q}{\sqrt{2}}\boldsymbol{\alpha}\cdot\mathbf{g}(\mathbf{k},\lambda)\Phi_\nu(\mathbf{p}).$$

Hence (6.11) follows. □

Lemma 6.4. *Suppose that $\hat\rho$ is spherically symmetric and $\hat\rho\in\mathrm{Dom}(\omega^{-3/2})$. Assume that $E(\mathbf{p},m)<E(\mathbf{p},0)$. Then*

$$\limsup_{\nu\to0}\|N_f^{1/2}\Phi_\nu(\mathbf{p})\|^2\le\int_{\mathbb{R}^3}d\mathbf{k}\frac{q^2}{(E(\mathbf{p}-\mathbf{k})-E(\mathbf{p})+|\mathbf{k}|)^2}\frac{|\hat\rho(\mathbf{k})|^2}{|\mathbf{k}|}<\infty \tag{6.12}$$

$$\limsup_{\nu\to0}\|H_f^{1/2}\Phi_\nu(\mathbf{p})\|^2\le\int_{\mathbb{R}^3}d\mathbf{k}\frac{q^2}{(E(\mathbf{p}-\mathbf{k})-E(\mathbf{p})+|\mathbf{k}|)^2}|\hat\rho(\mathbf{k})|^2<\infty. \tag{6.13}$$

Proof. By Proposition 6.3 and (6.10) with w = 1, we have

$$\|N_f^{1/2}\Phi_\nu(\mathbf{p})\|^2 \le \sum_{\lambda=1}^{2}\int_{\mathbb{R}^3}\frac{q^2}{2}\frac{\|\boldsymbol{\alpha}\cdot\mathbf{g}(\mathbf{k},\lambda)\Phi_\nu(\mathbf{p})\|^2}{(E_\nu(\mathbf{p}-\mathbf{k})-E_\nu(\mathbf{p})+|\mathbf{k}|+\nu)^2}d\mathbf{k}$$
$$=\int_{\mathbb{R}^3}\frac{q^2}{(E_\nu(\mathbf{p}-\mathbf{k})-E_\nu(\mathbf{p})+|\mathbf{k}|+\nu)^2}\frac{|\hat\rho(\mathbf{k})|^2}{|\mathbf{k}|}d\mathbf{k}.$$

By Theorem Appendix C.10 and $\hat\rho \in \mathrm{Dom}(\omega^{-3/2})$, the right hand side of (6.12) is finite. Hence, by Proposition Appendix C.9 and the Lebesgue convergence theorem, one has (6.12). The proof of (6.13) is similar. The only thing we have to do is setting $\mathrm{w}(\mathbf{k}) = \omega(\mathbf{k})$. □

Proof of Theorem 4.1. By Proposition Appendix C.2 , we have

$$0 \le E(\mathbf{p}-\mathbf{k}) - E(\mathbf{p}) + |\mathbf{k}| \le 2|\mathbf{k}|.$$

Hence, by (4.1),

$$\frac{q^2}{4}\int_{\mathbb{R}^3}\frac{|\hat\rho(\mathbf{k})|^2}{|\mathbf{k}|^3}d\mathbf{k} \le \int_{\mathbb{R}^3}\frac{q^2}{(E(\mathbf{p}-\mathbf{k})-E(\mathbf{p})+|\mathbf{k}|)^2}\frac{|\hat\rho(\mathbf{k})|^2}{|\mathbf{k}|}d\mathbf{k} < 1,$$

which implies $\hat\rho \in \mathrm{Dom}(\omega^{-3/2})$. Hence (6.12) and (6.13) holds.

Since $\Phi_\nu(\mathbf{p})$ is a unit vector, there exists a subsequence ν_j such that $\nu_j \to 0$ as $j \to \infty$ and $\Phi_0(\mathbf{p}) := \text{w-lim}_{j\to\infty}\Phi_{\nu_j}(\mathbf{p})$ exists. Then, by (6.12) and (6.13), we have

$$\lim_{j\to\infty}\|N_f^{1/2}\Phi_{\nu_j}\| < 1, \qquad \lim_{j\to\infty}\|H_f^{1/2}\Phi_{\nu_j}\| < \infty,$$

which implies that $\Phi_0(\mathbf{p}) \in \mathrm{Dom}(N_f^{1/2}) \cap \mathrm{Dom}(H_f^{1/2})$. Hence $\Phi_0(\mathbf{p}) \in Q(\overline{H(\mathbf{p})})$, where Q denotes the form domain. For any $\varphi \in \mathrm{Dom}(H(\mathbf{p}))$, we have

$$\begin{aligned}\langle (H(\mathbf{p})-E(\mathbf{p}))\varphi, \Phi_0(\mathbf{p})\rangle &= \lim_{j\to\infty}\left\langle (H(\mathbf{p})-E(\mathbf{p}))\varphi, \Phi_{\nu_j}(\mathbf{p})\right\rangle \\ &= \lim_{j\to\infty}\left\langle \varphi, (E_{\nu_j}(\mathbf{p})-E(\mathbf{p})-\nu_j(H_f+N_f))\Phi_{\nu_j}(\mathbf{p})\right\rangle \\ &= 0.\end{aligned}$$

Thus $\Phi_0(\mathbf{p}) \in \mathrm{Dom}(\overline{H(\mathbf{p})})$ and $(\overline{H(\mathbf{p})} - E(\mathbf{p}))\Phi_0(\mathbf{p}) = 0$. Therefore, if $\Phi_0(\mathbf{p}) \neq 0$, then $\Phi_0(\mathbf{p})$ is a ground state of $\overline{H(\mathbf{p})}$. Since $\mathbb{C}^4$ is a finite dimensional space, the vacuum component $\Phi_{\nu_j}(\mathbf{p})^{(0)}$ strongly converges to $\Phi_0(\mathbf{p})^{(0)}$. Hence

$$\|\Phi_0(\mathbf{p})\|^2 \ge \|\Phi_0(\mathbf{p})^{(0)}\|^2 = \lim_{j\to\infty}\|\Phi_{\nu_j}(\mathbf{p})^{(0)}\|^2 = \lim_{j\to\infty}\left\langle \Phi_{\nu_j}(\mathbf{p}), P_\Omega\Phi_{\nu_j}(\mathbf{p})\right\rangle, \tag{6.14}$$

where P_Ω is the orthogonal projection on the vacuum $(1,0,0,\ldots) \in \mathcal{F}_{\mathrm{rad}}$. Thus, using (6.14) and $N_f \geq 1 - P_\Omega$, we have

$$\|\Phi_0(\mathbf{p})\|^2 \geq 1 - \lim_{j\to\infty} \|N_f^{1/2}\Phi_{\nu_j}(\mathbf{p})\|^2 > 0.$$

This means that $\Phi_0(\mathbf{p}) \neq 0$ and $\Phi_0(\mathbf{p})$ is a ground state of $\overline{H(\mathbf{p})}$. □

Proof of Theorem 4.2. Theorem 4.2 is immediately derived from Theorem 4.1 and Theorem Appendix C.10. □

Next, we prepare some lemmata for the proof of Theorem 4.4. For a Hilbert space $\mathcal{K}$, we denote by $\mathsf{B}(\mathcal{K})$ the set of all bounded operators on $\mathcal{K}$. The next lemma is followed by the second resolvent equation.

Lemma 6.5. *Let $\nu > 0$. For each $\mathbf{j} \in \mathbb{R}^3$ with $|\mathbf{j}| = 1$, the operator valued function $\mathbb{R}^3\backslash\{\mathbf{0}\} : \mathbf{k} \to (H_\nu(\mathbf{p}-\mathbf{k}) - E_\nu(\mathbf{p}) + \omega_\nu(\mathbf{k}))^{-1} \in \mathsf{B}(\mathbb{C}^4 \otimes \mathcal{F}_{\mathrm{rad}})$ is differentiable in the sense of operator norm, and*

$$\partial_{\mathbf{j}}(H_\nu(\mathbf{p}-\mathbf{k}) - E_\nu(\mathbf{p}) + \omega_\nu(\mathbf{k}))^{-1} =$$
$$(H_\nu(\mathbf{p}-\mathbf{k}) - E_\nu(\mathbf{p}) + \omega_\nu(\mathbf{k}))^{-1}\left(\boldsymbol{\alpha}\cdot\mathbf{j} - (1+\nu)\frac{\mathbf{k}\cdot\mathbf{j}}{|\mathbf{k}|}\right)(H_\nu(\mathbf{p}-\mathbf{k}) - E_\nu(\mathbf{p}) + \omega_\nu(\mathbf{k}))^{-1},$$

where $\partial_{\mathbf{j}}$ means the $\mathbf{j}$-direction derivative.

We fix the following polarization vectors in the rest of this section:

$$\mathbf{e}^{(1)}(\mathbf{k}) = \frac{(k_2, -k_1, 0)}{\sqrt{k_1^2 + k_2^2}}, \quad \mathbf{e}^{(2)}(\mathbf{k}) := \frac{\mathbf{k}}{|\mathbf{k}|} \wedge \mathbf{e}^{(1)}(\mathbf{k}). \tag{6.15}$$

Now, remember the definition of the set S (which is defined in condition (Λ)). We set $\mathsf{X} := S\backslash\{\mathbf{k} \in \mathbb{R}^3 | k_1 = k_2 = 0\}$, $\mathsf{X}_R := S_R \cap \mathsf{X}$. By Lemma 6.5 and (6.15), we obtain the following result:

Lemma 6.6. *Assume the same assumptions as in Theorem 4.4. Then $a_\lambda(\mathbf{k})\Phi_\nu(\mathbf{p})$ is strongly continuously differentiable in X and*

$$\begin{aligned}
&\partial_j a_\lambda(\mathbf{k})\Phi_\nu(\mathbf{p}) \\
&= \frac{q}{\sqrt{2}}(H_\nu(\mathbf{p}-\mathbf{k}) - E_\nu(\mathbf{p}) + \omega_\nu(\mathbf{k}))^{-1}\left(\alpha_j - (1+\nu)\frac{k_j}{|\mathbf{k}|}\right) \\
&\quad \times (H_\nu(\mathbf{p}-\mathbf{k}) - E_\nu(\mathbf{p}) + \omega_\nu(\mathbf{k}))^{-1}\boldsymbol{\alpha}\cdot\mathbf{g}(\mathbf{k},\lambda)\Phi_\nu(\mathbf{p}) \\
&\quad + \frac{q}{\sqrt{2}}(H_\nu(\mathbf{p}-\mathbf{k}) - E_\nu(\mathbf{p}) + \omega_\nu(\mathbf{k}))^{-1}\boldsymbol{\alpha}\cdot(\partial_j\mathbf{g}(\mathbf{k},\lambda))\Phi_\nu(\mathbf{p}),
\end{aligned}$$

where ∂_j denotes the strong derivative in $k_j, (j = 1,2,3)$.

We set

$$\Psi_j(\mathbf{k},\lambda) = (\Psi_j^{(n)}(\mathbf{k},\lambda;\cdot))_{n=0}^{\infty} := \partial_j a_\lambda(\mathbf{k})\Phi_\nu(\mathbf{p}).$$

Lemma 6.7. *Assume the same assumptions as in Theorem 4.4. Then*

$$\partial_j\Phi_\nu^{(n)}(\mathbf{p})(\mathbf{k},\lambda;X;k_2,\dots,k_n) = \frac{1}{\sqrt{n}}\Psi_j^{(n-1)}(\mathbf{k},\lambda;X;k_2,\dots,k_n), \quad k_\ell = (\mathbf{k}_\ell,\lambda_\ell),$$

for all $X \in \{1,2,3,4\}$, $\mathbf{k},\mathbf{k}_\ell \in \mathsf{X}$, $n\in\mathbb{N}$, $\lambda,\lambda_\ell = 1,2$ *and* $j=1,2,3$, *where* ∂_j *is the distributional derivative in* k_j.

Note that ∂_j in the left hand side is a distributional derivative and that in Ψ_j is a strong derivative.

Proof. In this proof, for simplicity, we do not indicate X,λ,λ_ℓ and $\mathbf{p}$. The operator δ_h is defined by $\delta_h f(\mathbf{k}) := f(\mathbf{k}+h\mathbf{j}) - f(\mathbf{k})$ for all functions $f(\mathbf{k})$. Let $\psi(\mathbf{k},\mathbf{k}_2,\dots,\mathbf{k}_n)\in C_0^\infty(\mathsf{X}^{n+1})$ be arbitrarily. Clearly, $(\partial_j\psi)(\mathbf{k},K) = \lim_{h\to 0} h^{-1}(\psi(\mathbf{k}+h\mathbf{j},K) - \psi(\mathbf{k},K))$ uniformly, where $K = (\mathbf{k}_2,\dots,\mathbf{k}_n)$ and $\mathbf{j}$ is the unit vector of j-th axis. By the definition of the distributional derivative, we have

$$\begin{aligned}\int_{\mathbb{R}^{3n}} d\mathbf{k}dK\psi(\mathbf{k},K)\partial_j\Phi_\nu^{(n)}(\mathbf{k},K) &= -\int_{\mathbb{R}^{3n}} d\mathbf{k}dK(\partial_j\psi)(\mathbf{k},K)\Phi_\nu^{(n)}(\mathbf{k},K)\\ &= -\lim_{h\to 0}\int_{\mathbb{R}^{3n}} d\mathbf{k}dK\frac{1}{-h}(\delta_{-h}\psi)(\mathbf{k},K)\Phi_\nu^{(n)}(\mathbf{k},K)\\ &= \lim_{h\to 0}\int_{\mathbb{R}^{3n}} d\mathbf{k}dK\psi(\mathbf{k},K)\frac{1}{h}(\delta_h\Phi_\nu^{(n)})(\mathbf{k},K).\end{aligned}$$

By Schwarz' inequality, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^3} d\mathbf{k}\left[\int_{\mathbb{R}^{3(n-1)}} dK\psi(\mathbf{k},K)\left\{\frac{1}{h}[\Phi_\nu^{(n)}(\mathbf{k}+h\mathbf{j},K) - \Phi_\nu^{(n)}(\mathbf{k},K)] - \frac{1}{\sqrt{n}}\Psi^{(n-1)}(\mathbf{k},K)\right\}\right]\right| \\ &\le \int_{\mathbb{R}^3} d\mathbf{k}\|\psi(\mathbf{k},\cdot)\|_{L^2(\mathbb{R}^{3(n-1)})}\left\|\frac{\delta_h}{h}\Phi_\nu^{(n)}(\mathbf{k},\cdot) - \frac{1}{\sqrt{n}}\Psi^{(n-1)}(\mathbf{k},\cdot)\right\|_{L^2(\mathbb{R}^{3(n-1)})}.\end{aligned} \tag{6.16}$$

Note that, for all $\mathbf{k}\in X$, $h^{-1}\delta_h\Phi_\nu^{(n)}(\mathbf{k},\cdot)$ strongly converges to $\frac{1}{\sqrt{n}}\Psi^{(n-1)}(\mathbf{k},\cdot)$ in $L^2(\mathsf{X}^{3(n-1)})$ by Lemma 6.6. Moreover, by Lemma 6.6 and the assumption that $\hat\rho$ is continuously differentiable, the function $\mathbf{k}\to\Psi^{(n-1)}(\mathbf{k},\cdot)$ is strongly continuous in X. Set D be the closure of $\{\mathbf{k}\in\mathbb{R}^3 \mid \|\psi(\mathbf{k},\cdot)\|_{L^2(\mathbb{R}^{3(n-1)})}\neq 0\}$. Note that $D\subset\mathsf{X}$ is a compact set and $d := \mathrm{dist}(D,\mathsf{X}^{\mathrm{c}}) > 0$.

For every $\mathbf{k}\in D$ and h with $|h|<d$, we have

$$\frac{\delta_h}{h}\Phi_\nu^{(n)}(\mathbf{k},\cdot) = \text{s-}\int_0^1 \frac{1}{\sqrt{n}}\Psi^{(n-1)}(\mathbf{k}+th\mathbf{j},\cdot)dt,$$

where s-$\int$ means the strong integral in $L^2(X^{3(n-1)})$. Since $\|\Psi^{(n-1)}(\mathbf{k},\cdot)\|_{L^2(\mathbb{R}^{3(n-1)})}$ is continuous in $\mathbf{k} \in \mathsf{X}$, it is bounded on the compact set D. For any $\mathbf{k} \in D$ and $|h| < d$, we have

$$
\begin{aligned}
&\Big\| \frac{\delta_h}{|h|} \Phi_\nu^{(n)}(\mathbf{k},\cdot) - \frac{1}{\sqrt{n}} \Psi^{(n-1)}(\mathbf{k},\cdot) \Big\|_{L^2(\mathbb{R}^{3(n-1)})} \\
&\le \sup_{|t|\le 1} \frac{1}{\sqrt{n}} \|\Psi^{(n-1)}(\mathbf{k}+th\mathbf{j},\cdot)\|_{L^2(\mathbb{R}^{3(n-1)})} + \frac{1}{\sqrt{n}} \|\Psi^{(n-1)}(\mathbf{k},\cdot)\|_{L^2(\mathbb{R}^{3(n-1)})} \\
&\le \text{const.},
\end{aligned}
$$

where "const" means the constant independent of $\mathbf{k}$ and h. Applying the Lebesgue dominated convergence theorem, we can see the right hand side of (6.16) converges to zero as $|h| \to 0$. □

By Lemmas 6.5-6.6 and direct calculations, we obtain the following inequality

Lemma 6.8. *Assume the same assumptions as in Theorem 4.4. Then*

$$
\begin{aligned}
&\|\partial_j a_\lambda(\mathbf{k})\Phi_\nu(\mathbf{p})\| \\
&\quad \le \frac{|q|}{\sqrt{2}}(2+\nu)(E_\nu(\mathbf{p}-\mathbf{k}) - E_\nu(\mathbf{p}) + \omega_\nu(\mathbf{k}))^{-2} \frac{|\hat{\rho}(\mathbf{k})|}{|\mathbf{k}|^{1/2}} \\
&\qquad + \frac{|q|}{\sqrt{2}}(E_\nu(\mathbf{p}-\mathbf{k}) - E_\nu(\mathbf{p}) + \omega_\nu(\mathbf{k}))^{-1} \frac{|\partial_j \hat{\rho}(\mathbf{k})|}{|\mathbf{k}|^{1/2}} \\
&\qquad + \frac{|q|}{\sqrt{2}}(E_\nu(\mathbf{p}-\mathbf{k}) - E_\nu(\mathbf{p}) + \omega_\nu(\mathbf{k}))^{-1} \frac{|\hat{\rho}(\mathbf{k})|}{|\mathbf{k}|^{3/2}} \\
&\qquad + \frac{|q|}{\sqrt{2}}(E_\nu(\mathbf{p}-\mathbf{k}) - E_\nu(\mathbf{p}) + \omega_\nu(\mathbf{k}))^{-1} \frac{|\hat{\rho}(\mathbf{k})|}{|\mathbf{k}|^{1/2}} |\partial_j \mathbf{e}^{(\lambda)}(\mathbf{k})|
\end{aligned}
$$

for all $\mathbf{k} \in \mathsf{X}$, $\lambda = 1, 2$, $j = 1, 2, 3$.

Our polarization vectors (6.15) satisfy that

$$
|\partial_j \mathbf{e}^{(\lambda)}(\mathbf{k})| \le \frac{2}{\sqrt{k_1^2 + k_2^2}}, \quad \text{for } \mathbf{k} \in \mathbb{R}^3 \backslash \{\mathbf{k}' \in \mathbb{R}^3 | k_1' = k_2' = 0\}. \tag{6.17}
$$

We set

$$f_\nu^{(1)}(\mathbf{k}) := (E_\nu(\mathbf{p}-\mathbf{k}) - E_\nu(\mathbf{p}) + \omega_\nu(\mathbf{k}))^{-2}\frac{|\hat{\rho}(\mathbf{k})|}{|\mathbf{k}|^{1/2}}$$
$$f_\nu^{(2)}(\mathbf{k}) := (E_\nu(\mathbf{p}-\mathbf{k}) - E_\nu(\mathbf{p}) + \omega_\nu(\mathbf{k}))^{-1}\frac{|\partial_j\hat{\rho}(\mathbf{k})|}{|\mathbf{k}|^{1/2}}$$
$$f_\nu^{(3)}(\mathbf{k}) := (E_\nu(\mathbf{p}-\mathbf{k}) - E_\nu(\mathbf{p}) + \omega_\nu(\mathbf{k}))^{-1}\frac{|\hat{\rho}(\mathbf{k})|}{|\mathbf{k}|^{3/2}}$$
$$f_\nu^{(4)}(\mathbf{k}) := (E_\nu(\mathbf{p}-\mathbf{k}) - E_\nu(\mathbf{p}) + \omega_\nu(\mathbf{k}))^{-1}\frac{|\hat{\rho}(\mathbf{k})|}{|\mathbf{k}|^{1/2}}|\partial_j\mathbf{e}^{(\lambda)}(\mathbf{k})|.$$

Lemma 6.9. *Assume the conditions in Theorem 4.4. Then*

$$\sup_{0<\nu\le 1}\|f_\nu^{(j)}\|_{L^p(S_R)} < \infty, \quad j = 1,2,3,4, \quad p \in [1,2). \tag{6.18}$$

Proof. First we consider the case $\mathbf{p} \neq \mathbf{0}$. Let $b_\nu(\mathbf{p})$ be the constant defined in Theorem Appendix C.10. Since $b_\nu(\mathbf{p})$ is continuous in ν for fixed $\mathbf{p}$, Theorem Appendix C.10 guarantees $\sup_{0\le\nu\le 1} b_\nu(\mathbf{p}) = \max_{0\le\nu\le 1} b_\nu(\mathbf{p}) < 1$. By Theorem Appendix C.10, we have

$$(E_\nu(\mathbf{p}-\mathbf{k}) - E_\nu(\mathbf{p}) + |\mathbf{k}|)^{-1} \le \frac{1}{1-b_\nu(\mathbf{p})}\max\left\{\frac{1}{|\mathbf{k}|}, \frac{1}{|\mathbf{p}|}\right\} \le C\max\left\{\frac{1}{|\mathbf{k}|}, \frac{1}{|\mathbf{p}|}\right\},$$

where

$$C := \sup_{0<\nu\le 1}\frac{1}{1-b_\nu(\mathbf{p})}$$

is a finite constant. Hence

$$f_\nu^{(1)}(\mathbf{k}) \le C^2\left\{\frac{1}{|\mathbf{k}|^2} + \frac{1}{|\mathbf{p}|^2}\right\}\frac{|\hat{\rho}(\mathbf{k})|}{|\mathbf{k}|^{1/2}}.$$

Since S_R is a bounded region, by the assumption $|\mathbf{k}|^{-5/2}|\hat{\rho}(\mathbf{k})| \in L^p(S_R)$, we obtain that

$$\sup_{0<\nu\le 1}\|f_\nu^{(1)}\|_{L^p(S_R)} < \infty.$$

Similarly, we obtain that

$$\sup_{0<\nu\le 1}\|f_\nu^{(j)}\|_{L^2(S_R)} < \infty, \quad j = 2,3.$$

By (6.17), we have

$$f_\nu^{(4)}(\mathbf{k}) \le C^2\left\{\frac{1}{|\mathbf{k}|} + \frac{1}{|\mathbf{p}|}\right\}\frac{1}{\sqrt{k_1^2+k_2^2}}\frac{|\hat{\rho}(\mathbf{k})|}{|\mathbf{k}|^{1/2}}.$$

By using the polar coordinate, we have

$$\begin{aligned}&\int_{S_R} |f_\nu^{(4)}(\mathbf{k})|^p d\mathbf{k}\\&\leq 2\pi C^{2p}\int_{[0,\pi)}\sin\theta d\theta\left[\frac{1}{\sin\theta}\right]^p\int_{[0,R)}|\mathbf{k}|^{2-p}\left(\frac{|\mathbf{k}|+|\mathbf{p}|}{|\mathbf{k}|\cdot|\mathbf{p}|}\right)^p\frac{|\hat{\rho}(\mathbf{k})|^p}{|\mathbf{k}|}d|\mathbf{k}|\\&<\infty.\end{aligned}$$

Next we consider the case $\mathbf{p}=0$. By (Appendix C.4) in Proposition Appendix C.10, we have

$$(E_\nu(-\mathbf{k})-E_\nu(\mathbf{0})+\omega_\nu(\mathbf{k}))^{-1}\leq\begin{cases}\dfrac{P}{a_\nu(P)|\mathbf{k}|}, & \text{if } |\mathbf{k}|\leq P\\ a_\nu(P)^{-1}, & \text{if } |\mathbf{k}|>P,\end{cases}$$

for any $P>0$. By the similar arguments as above, one can prove (6.18). This completes the proof. □

Let $W^{1,p}(\mathcal{X})$ be the Sobolev space on the configuration space $\mathcal{X}$, i.e., the set of all L^p-functions with its first derivatives are also in L^p.

Lemma 6.10. *Suppose the same assumptions as in Theorem 4.4. Then the n-th component of the massive ground state satisfies* $\Phi_\nu^{(n)}\in\oplus^4W^{1,p}((\mathsf{X}_R\times\{1,2\})^n)$ *for all* $p\in[1,2)$ *and all* $R>0$, *and*

$$\sup_{0<\nu<1}\|\Phi_\nu^{(n)}(\mathbf{p})\|_{\oplus^4W^{1,p}((\mathsf{X}_R\times\{1,2\})^n)}<\infty.$$

Proof. By Lemma 6.7, we have

$$\begin{aligned}&(\nabla_{\mathbf{k}}a_\lambda(\mathbf{k})\Phi_\nu(\mathbf{p}))^{(n-1)}(X;\mathbf{k}_1,\lambda_1;\dots;\mathbf{k}_{n-1},\lambda_{n-1})\\&=\sqrt{n}\nabla_{\mathbf{k}}\Phi_\nu^{(n)}(\mathbf{p};X;\mathbf{k},\lambda;\mathbf{k}_1,\lambda_1;\dots;\mathbf{k}_{n-1},\lambda_{n-1}).\end{aligned}$$

Using Hölder's inequality and making a change of variables, one has, for all $p<2$,

$$\begin{aligned}&\sum_{X=1}^4\sum_{\lambda_1,\cdots,\lambda_n\in\{1,2\}}\int_{(\mathsf{X}_R)^n}d\mathbf{k}_1\cdots d\mathbf{k}_n\sum_{i=1}^n\left|\nabla_{\mathbf{k}_i}\Phi_\nu^{(n)}(\mathbf{p};X;\mathbf{k}_1,\lambda_1;\cdots;\mathbf{k}_n,\lambda_n)\right|^p\\&\leq C\int_{\mathsf{X}_R}d\mathbf{k}\|\nabla_{\mathbf{k}}a_\lambda(\mathbf{k})\Phi_\nu(\mathbf{p})\|^p,\end{aligned}\tag{6.19}$$

where C is a constant independent of ν. By Lemma 6.8 and Lemma 6.9, the right hand side of (6.19) is finite uniformly in $\nu>0$. □

Proof of Theorem 4.4. As shown in the Proof of Theorem 4.1, there exists a sequence $\{\nu_j\}_{j=1}^{\infty}$ such that $\Phi_0(\mathbf{p}) := \text{w-lim}_{j\to\infty} \Phi_{\nu_j}(\mathbf{p})$ exists, and $\Phi_0(\mathbf{p}) \in \text{Dom}(H_f^{1/2}) \cap \text{Dom}(N_f^{1/2})$. Then, $\Phi_0 \in Q(H(\mathbf{p}))$. If $\Phi_0(\mathbf{p}) \neq 0$, then $\Phi_0(\mathbf{p})$ is a ground state of $H(\mathbf{p})$. In the following, we show that $\Phi_0(\mathbf{p}) \neq 0$.

Any vector $\Psi \in \oplus^4 \mathcal{F}^n = \mathbb{C}^4 \otimes \mathcal{F}^n$ is a function of the particle helicity $X \in \{1,2,3,4\}$, the n-photon wave number $(\mathbf{k}_1, \dots, \mathbf{k}_n) \in \mathbb{R}^{3n}$, and the photon polarization $\lambda_1, \dots, \lambda_n \in \{1,2\}$. For simplicity, we set

$$\Phi_j^{(n)}(\mathbf{k}_1, \dots, \mathbf{k}_n) := \Phi_{\nu_j}(\mathbf{p})^{(n)}(X; \mathbf{k}_1, \lambda_1; \cdots; \mathbf{k}_n, \lambda_n),$$
$$\Phi_0^{(n)}(\mathbf{k}_1, \dots, \mathbf{k}_n) := \Phi_0(\mathbf{p})^{(n)}(X; \mathbf{k}_1, \lambda_1; \dots; \mathbf{k}_n, \lambda_n).$$

for $X \in \{1,2,3,4\}$ and $\lambda_1, \dots, \lambda_n \in \{1,2\}$. Note that $\Phi_j^{(n)}, \Phi_0^{(n)} \in L^2(\mathbb{R}^{3n})$. We show that $\text{s-lim}_{j\to\infty} \Phi_j^{(n)} = \Phi_0^{(n)}$ for all $n \in \mathbb{N}$, $X \in \{1,2,3,4\}$ and $\lambda_1, \dots, \lambda_n \in \{1,2\}$.

By Lemma 6.10 and the Rellich-Kondrashov theorem, it holds that

$$\lim_{j\to\infty} \|\Phi_j^{(n)} - \Phi_0^{(n)}\|_{L^2(\mathsf{X}_R^n)} = 0 \tag{6.20}$$

for all $R > 0$(we refer [GLL, page 578] for details). We set $\Phi_j := (\Phi_j^{(n)})_{n=0}^{\infty}$, $\Phi_0 := (\Phi_0^{(n)})_{n=0}^{\infty} \in \oplus^4 \mathcal{F}_{\text{rad}}$. Let χ_R be the characteristic function of the ball $\{\mathbf{k} \in \mathbb{R}^3 | |\mathbf{k}| < R\}$. We denote the orthogonal projection onto $\oplus_{j=0}^n \mathbb{C}^4 \otimes \mathcal{F}^j$ by P_n. Then we have

$$\begin{aligned}\|\Gamma(\chi_R)(\Phi_j - \Phi_0)\|^2 &= \|P_n \Gamma(\chi_R)(\Phi_j - \Phi_0)\|^2 + \|(1 - P_n)\Gamma(\chi_R)(\Phi_j - \Phi_0)\|^2 \\ &\leq \|P_n \Gamma(\chi_R)(\Phi_j - \Phi_0)\|^2 + \frac{1}{n}\|N_f^{1/2}\Gamma(\chi_R)(\Phi_j - \Phi_0)\|^2.\end{aligned}$$

Since each component $(\Gamma(\chi_R)\Phi_j)^{(n)}$ converges to $(\Gamma(\chi_R)\Phi_0)^{(n)}$ strongly as $j \to \infty$, we have

$$\limsup_{j\to\infty} \|\Gamma(\chi_R)(\Phi_j - \Phi_0)\|^2 \leq \frac{1}{n} \limsup_{j\to\infty} \|N_f^{1/2}(\Phi_j - \Phi_0)\|^2$$

for all $n \in \mathbb{N}$. By Lemma 6.4, $\limsup_{j\to\infty} \|N_f^{1/2}(\Phi_j - \Phi_0)\|^2 < \infty$. Thus we obtain that

$$\underset{j\to\infty}{\text{s-lim}}\, \Gamma(\chi_R)\Phi_j = \Gamma(\chi_R)\Phi_0. \tag{6.21}$$

Therefore for all $R > 0$ we have

$$\begin{aligned}\|\Phi_j - \Phi_0\| &= \|\Gamma(\chi_R)(\Phi_j - \Phi_0)\| + \|(1 - P_0)(\Gamma(\chi_R) - 1)(\Phi_j - \Phi_0)\|^2 \\ &\leq \|\Gamma(\chi_R)(\Phi_j - \Phi_0)\| + \|(1 - P_0)(1 - \Gamma(\chi_R))H_f^{-1/2}\| \cdot \|H_f^{1/2}(\Phi_j - \Phi_0)\| \\ &\leq \|\Gamma(\chi_R)(\Phi_j - \Phi_0)\| + \frac{C}{R^{1/2}}\end{aligned}$$

where C is a constant independent of $R > 0$. By (6.21), we obtain

$$\operatorname*{s-lim}_{j\to\infty} \Phi_j = \Phi_0,$$

which implies that Φ_0 is a normalized ground state of $\overline{H(\mathbf{p})}$. □

§7. Proof of Theorem 5.2

In this section we assume the assumptions in Theorem 5.2. By Appendices A and B, it suffices to prove Theorem 5.2 in the case $\mathbf{e} = \bar{\mathbf{e}}$. Here $\bar{\mathbf{e}}$ is the polarization vector defined in (Appendix B.1). Note that $\bar{\mathbf{e}}$ depends on $\mathbf{j}$. By assumption, there exists a non-negative constant t such that $\mathbf{p} = t\mathbf{j}$. We choose a matrix $T \in SO(3)$ such that $T^{-1}\mathbf{p} = (0, 0, |\mathbf{p}|)$ and $T^{-1}\mathbf{j} = (0, 0, 1)$. Let U be the unitary operator defined in the proof of Proposition Appendix C.4. By (Appendix C.1), we obtain that

$$U\overline{H(\mathbf{p})}U^* = \overline{(|\mathbf{p}|\alpha_3 + m\beta + H_f - \boldsymbol{\alpha}\cdot d\Gamma(\mathbf{k}) - q\boldsymbol{\alpha}\cdot \Phi_{\mathrm{S}}(\vec{\lambda}))},$$

where

$$\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3) = \frac{\hat{\rho}(T\mathbf{k})}{|\mathbf{k}|^{1/2}}(T^{-1}\bar{\mathbf{e}}^{(1)}(T\mathbf{k}), T^{-1}\bar{\mathbf{e}}^{(2)}(T\mathbf{k})) \in (L^2(\mathbb{R}^3_{\mathbf{k}} \times \{1,2\}))^3.$$

Since $T \in SO(3)$, we have

$$T^{-1}\bar{\mathbf{e}}^{(1)}(T\mathbf{k}) = \frac{T^{-1}[(T\mathbf{k}) \wedge \mathbf{j}]}{|(T\mathbf{k}) \wedge \mathbf{j}|} = \frac{\mathbf{k} \wedge (0,0,1)}{|\mathbf{k} \wedge (0,0,1)|},$$
$$T^{-1}\bar{\mathbf{e}}^{(2)}(T\mathbf{k}) = \frac{\mathbf{k}}{|\mathbf{k}|} \wedge (T^{-1}\bar{\mathbf{e}}^{(1)}(T\mathbf{k})).$$

It is easy to see that $\hat{\rho}(TR'\mathbf{k}) = \hat{\rho}(T\mathbf{k}), \mathbf{k} \in \mathbb{R}^3$ for all $R' \in O(3)$ such that $R'(0,0,1) = (0,0,1)$. Since $\mathbf{S} = (i/4)\boldsymbol{\alpha} \wedge \boldsymbol{\alpha}$, we have

$$U(\mathbf{j}\cdot\mathbf{S})U^* = \frac{i}{4}\mathbf{j}\cdot[(T\boldsymbol{\alpha})\cdot(T\boldsymbol{\alpha})] = \frac{i}{4}\mathbf{j}\cdot[T(\boldsymbol{\alpha}\wedge\boldsymbol{\alpha})] = \frac{i}{4}(\boldsymbol{\alpha}\wedge\boldsymbol{\alpha})_3 = S_3.$$

Moreover, one can show that $U(\mathbf{j}\cdot d\Gamma(\vec{\ell}))U^* = d\Gamma(\ell_3)$. Therefore,

$$UJ_{\mathbf{j}}(\bar{\mathbf{e}})U^* = S_3 + d\Gamma(\ell_3),$$

and, hence, we conclude that it is sufficient to prove Theorem 5.2 in the case

$$\mathbf{p} = (0, 0, |\mathbf{p}|), \quad \mathbf{j} = (0, 0, 1). \tag{7.1}$$

Proof of Theorem 5.2. We assume (7.1) to the end of this proof. We put

$$\check{\mathbf{e}}^{(1)}(\mathbf{k}) := \frac{(k_2, -k_1, 0)}{\sqrt{k_1^2 + k_2^2}}, \quad \check{\mathbf{e}}^{(2)}(\mathbf{k}) := \frac{\mathbf{k}}{|\mathbf{k}|} \wedge \check{\mathbf{e}}^{(1)}(\mathbf{k}).$$

For a real parameter $\theta \in \mathbb{R}$, we set

$$W := \exp[i\theta J_{\mathbf{j}}(\check{\mathbf{e}})], \quad \Theta := \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then we obtain that

$$W\boldsymbol{\alpha}W^* = \Theta\boldsymbol{\alpha}, \quad W\beta W^* = \beta, \tag{7.2}$$

$$Wd\Gamma(\mathbf{k})W^* = \Theta d\Gamma(\mathbf{k}), \quad WH_f(m)W^* = H_f(m), \tag{7.3}$$

$$W\mathbf{A}W^* = \Theta\mathbf{A}. \tag{7.4}$$

Here, to show (7.4), we used the specific form of $\check{\mathbf{e}}$:

$$\check{\mathbf{e}}^{(\lambda)}(\Theta\mathbf{k}) = \Theta\check{\mathbf{e}}^{(\lambda)}(\mathbf{k}), \quad \lambda = 1, 2.$$

Since $\theta \in \mathbb{R}$ is arbitrary, (7.2),(7.3) and (7.4) imply that $\overline{H(\mathbf{p})}$ strongly commutes with $J_{\mathbf{j}}(\check{\mathbf{e}})$. Thus, $\overline{H(\mathbf{p})}$ is reduced by the projection onto the eigenspace of $J_{\mathbf{j}}(\check{\mathbf{e}})$. In other words, $\overline{H(\mathbf{p})}$ is decomposable as

$$\overline{H(\mathbf{p})} \cong \bigoplus_{z \in \mathbb{Z}_{1/2}} H(\mathbf{p} : z),$$

in the sense of (5.1). We furthermore define unitary operators η, τ and Υ by

$$(\eta f)(\mathbf{k}, \lambda) := \begin{cases} -f(k_1, -k_2, k_3, 1) & \text{if } \lambda = 1, \\ f(k_1, -k_2, k_3, 2) & \text{if } \lambda = 2, \end{cases} \quad f \in L^2(\mathbb{R}^3_{\mathbf{k}} \times \{1, 2\}),$$

$$\tau := \alpha_1\alpha_2\beta, \quad \Upsilon := \tau \cdot \Gamma(\eta).$$

It is easy to see that

$$\eta\ell_3\eta^* = -\ell_3, \quad \tau S_3\tau^* = -S_3,$$

$$\eta k_1\eta^* = k_1, \quad \eta k_2\eta^* = -k_2, \quad \eta k_3\eta^* = k_3,$$

$$\tau\alpha_1\tau^* = \alpha_1, \quad \tau\alpha_2\tau^* = -\alpha_2, \quad \tau\alpha_3\tau^* = \alpha_3, \quad \tau\beta\tau^* = \beta,$$

$$\eta\check{\mathbf{e}}^{(1)}(\mathbf{k})\eta^{-1} = \frac{(k_2, -(-k_1), 0)}{\sqrt{k_1^2 + k_2^2}}, \quad \eta\check{\mathbf{e}}^{(2)}(\mathbf{k})\eta^{-1} = \frac{(k_1k_3, -k_2k_3, -k_1^2 - k_2^2)}{|\mathbf{k}|\sqrt{k_1^2 + k_2^2}}.$$

Hence

$$\Upsilon\overline{H(\mathbf{p})}\Upsilon^* = \overline{H(\mathbf{p})}, \quad \Upsilon J_{\mathbf{j}}\Upsilon^* = -J_{\mathbf{j}}.$$

Let $E(z), z \in \mathbb{Z}_{1/2}$, be the orthogonal projection on $\ker(J_{\mathbf{j}} - z)$. Note that $\mathrm{Ran}(E(z)) = \mathcal{F}(z)$. $E(-z)\Upsilon E(z)$ is a unitary operator from $\mathrm{Ran}(E(z))$ to $\mathrm{Ran}(E(-z))$ and

$$\begin{aligned} E(-z)\Upsilon E(z)H(\mathbf{p} : z)E(z)\Upsilon^* E(-z) &= E(-z)\Upsilon E(z)\Upsilon^*\overline{H(\mathbf{p})}\Upsilon E(z)\Upsilon^* E(-z) \\ &= H(\mathbf{p} : -z). \end{aligned}$$

Therefore $H(\mathbf{p}:z)$ is unitarily equivalent to $H(\mathbf{p}:-z)$ for all $z \in \mathbb{Z}_{1/2}$. □

Appendix A. Remarks on the Polarization Vectors

In this appendix, we show that the quantum electrodynamics is independent of the choice of polarization vectors, i.e., the Hamiltonians defined by different polarization vectors are unitarily equivalent each other. We show the equivalence only for the Hamiltonians H and $H(\mathbf{p})$, but one can apply our proof to the Pauli-Fierz model and various QED models. The proof here is independent of the choice of $\hat{\rho}$ and ω.

We assume that the polarization vectors $\mathbf{e}^{(1)}(\mathbf{k})$, $\mathbf{e}^{(2)}(\mathbf{k})$ and $\mathbf{k}$ are a right-handed system;

$$\mathbf{k}\cdot\mathbf{e}^{(1)}(\mathbf{k}) = 0, \quad \|\mathbf{e}^{(1)}(\mathbf{k})\|_{\mathbb{R}^3} = 1, \quad \mathbf{e}^{(2)}(\mathbf{k}) = \frac{\mathbf{k}}{|\mathbf{k}|}\wedge \mathbf{e}^{(1)}(\mathbf{k}), \quad \mathbf{k}\in\mathbb{R}^3.$$

Next, we take any polarization vectors $\mathbf{e}'^{(1)}$, $\mathbf{e}'^{(2)}$:

$$\mathbf{k}\cdot\mathbf{e}'^{(\lambda)}(\mathbf{k}) = 0, \quad \mathbf{e}'^{(\lambda)}(\mathbf{k})\cdot\mathbf{e}'^{(\mu)}(\mathbf{k}) = \delta_{\lambda,\mu}, \quad \mathbf{k}\in\mathbb{R}^3,\ \lambda,\mu\in\{1,2\}.$$

Let H' and $H'(\mathbf{p})$ be the Hamiltonians H and $H(\mathbf{p})$ with $\mathbf{e}^{(\lambda)}$ replaced by $\mathbf{e}'^{(\lambda)}$, $\lambda = 1,2$, respectively.

Theorem Appendix A.1. *Assume that H is essentially self-adjoint. Then H' is essentially self-adjoint and $\bar{H}$ is unitarily equivalent to $\bar{H}'$ by a unitary operator $U(\mathbf{e}\leftarrow\mathbf{e}')$:*

$$U(\mathbf{e}\leftarrow\mathbf{e}')\bar{H}'U(\mathbf{e}\leftarrow\mathbf{e}')^* = \bar{H}.$$

Theorem Appendix A.2. *Assume that $H(\mathbf{p})$ is essentially self-adjoint. Then $H'(\mathbf{p})$ is essentially self-adjoint and $\overline{H(\mathbf{p})}$ is unitarily equivalent to $\overline{H'(\mathbf{p})}$:*

$$U(\mathbf{e}\leftarrow\mathbf{e}')\overline{H'(\mathbf{p})}U(\mathbf{e}\leftarrow\mathbf{e}')^* = \overline{H(\mathbf{p})}.$$

Remark Appendix A.3. The unitary operators $U(\mathbf{e}\leftarrow\mathbf{e}')$ defined below satisfy the chain-rule:

$$U(\mathbf{e}\leftarrow\mathbf{e}') = U(\mathbf{e}\leftarrow\mathbf{e}'')U(\mathbf{e}''\leftarrow\mathbf{e}')$$
$$U(\mathbf{e}\leftarrow\mathbf{e}')^* = U(\mathbf{e}'\leftarrow\mathbf{e}).$$

Proofs of Theorem Appendix A.1 and Appendix A.2. By the definition of polarization vectors, for each $\mathbf{k}\in\mathbb{R}^3$ it holds that $\mathbf{e}'^{(2)}(\mathbf{k}) = \frac{\mathbf{k}}{|\mathbf{k}|}\wedge\mathbf{e}'^{(1)}(\mathbf{k})$ or $\mathbf{e}'^{(2)}(\mathbf{k}) = -\frac{\mathbf{k}}{|\mathbf{k}|}\wedge\mathbf{e}'^{(1)}(\mathbf{k})$. Let $O\subset\mathbb{R}^3$ be the set such that $\mathbf{e}'^{(2)}(\mathbf{k}) = -\frac{\mathbf{k}}{|\mathbf{k}|}\wedge\mathbf{e}'^{(1)}(\mathbf{k})$, $\mathbf{k}\in O$, holds. We define

$$\mathbf{e}''^{(1)}(\mathbf{k}) := \mathbf{e}'^{(1)}(\mathbf{k}), \quad \mathbf{e}''^{(2)}(\mathbf{k}) := \begin{cases} \mathbf{e}'^{(2)}(\mathbf{k}), & \mathbf{k} \in \mathbb{R}^3 \backslash O, \\ -\mathbf{e}'^{(2)}(\mathbf{k}), & \mathbf{k} \in O. \end{cases}$$

We define an operator H'' by H with $\mathbf{e}^{(\lambda)}$ replaced by $\mathbf{e}''^{(\lambda)}$, $\lambda = 1, 2$. Let

$$\mathbf{g}'(\mathbf{k}, \lambda; \mathbf{x}) := \frac{\hat{\rho}(\mathbf{k})}{|\mathbf{k}|^{1/2}} \mathbf{e}'^{(\lambda)}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}, \quad \mathbf{g}''(\mathbf{k}, \lambda; \mathbf{x}) := \frac{\hat{\rho}(\mathbf{k})}{|\mathbf{k}|^{1/2}} \mathbf{e}''^{(\lambda)}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}},$$

and we set

$$\mathbf{A}^\sharp(\hat{\mathbf{x}}) := \frac{1}{\sqrt{2}} \int_{\mathbb{R}^3}^{\oplus} \overline{[a(\mathbf{g}^\sharp(\cdot, \mathbf{x})) + a(\mathbf{g}^\sharp(\cdot, \mathbf{x}))^*]} d\mathbf{x},$$

where $\sharp$ stands for $'$ and $''$. Since $(\mathbf{e}''^{(1)}(\mathbf{k}), \mathbf{e}''^{(2)}(\mathbf{k}), \mathbf{k})$ are right-handed vectors, i.e., $\mathbf{k} \cdot \mathbf{e}''^{(1)}(\mathbf{k}) = 0$, $\mathbf{e}''^{(2)}(\mathbf{k}) = \frac{\mathbf{k}}{|\mathbf{k}|} \wedge \mathbf{e}''^{(1)}(\mathbf{k})$, there exists $\theta(\mathbf{k}) \in [0, 2\pi)$ such that

$$\begin{bmatrix} \mathbf{e}^{(1)}(\mathbf{k}) \\ \mathbf{e}^{(2)}(\mathbf{k}) \end{bmatrix} = \begin{bmatrix} \cos\theta(\mathbf{k}) & -\sin\theta(\mathbf{k}) \\ \sin\theta(\mathbf{k}) & \cos\theta(\mathbf{k}) \end{bmatrix} \begin{bmatrix} \mathbf{e}''^{(1)}(\mathbf{k}) \\ \mathbf{e}''^{(2)}(\mathbf{k}) \end{bmatrix}.$$

We define a unitary operator u_1 on $L^2(\mathbb{R}^3_{\mathbf{k}} \times \{1, 2\})$ by

$$\begin{bmatrix} (u_1 f)(\mathbf{k}, 1) \\ (u_1 f)(\mathbf{k}, 2) \end{bmatrix} := \begin{bmatrix} \cos\theta(\mathbf{k}) & -\sin\theta(\mathbf{k}) \\ \sin\theta(\mathbf{k}) & \cos\theta(\mathbf{k}) \end{bmatrix} \begin{bmatrix} f(\mathbf{k}, 1) \\ f(\mathbf{k}, 2) \end{bmatrix}, \quad \mathbf{k} \in \mathbb{R}^3.$$

The operator $U(\mathbf{e} \leftarrow \mathbf{e}'') := \Gamma(u_1)$ is a unitary operator on $\mathcal{F}_{\mathrm{rad}}$. It is clear that

$$U(\mathbf{e} \leftarrow \mathbf{e}'') d\Gamma(\omega) U(\mathbf{e} \leftarrow \mathbf{e}'')^* = d\Gamma(\omega).$$

By the equality $u_1 \mathbf{g}''(\cdot, \mathbf{x}) = \mathbf{g}(\cdot, \mathbf{x})$, we have $U(\mathbf{e} \leftarrow \mathbf{e}'')\mathbf{A}''(\hat{\mathbf{x}})U(\mathbf{e} \leftarrow \mathbf{e}'')^* = \mathbf{A}(\hat{\mathbf{x}})$. Therefore we get

$$U(\mathbf{e} \leftarrow \mathbf{e}'') \overline{H''} U(\mathbf{e} \leftarrow \mathbf{e}'')^* = \overline{U(\mathbf{e} \leftarrow \mathbf{e}'') H'' U(\mathbf{e} \leftarrow \mathbf{e}'')^*} = \overline{H}.$$

This means that the operator H'' is essentially self-adjoint and $\overline{H''}$ is unitarily equivalent to $\bar{H}$. Next we show that $\overline{H''}$ is unitarily equivalent to $\overline{H'}$. Let u_2 be a unitary operator on $L^2(\mathbb{R}^3_{\mathbf{k}} \times \{1, 2\})$ such that

$$(u_2 f)(\mathbf{k}, \lambda) := \begin{cases} -f(\mathbf{k}, 2), & \mathbf{k} \in O, \\ f(\mathbf{k}, \lambda), & \text{otherwise}. \end{cases}$$

It is easy to see that $u_1 g'_j(\cdot, \mathbf{x}) = g''_j(\cdot, \mathbf{x})$, $j = 1, 2, 3$. Then $U(\mathbf{e}'' \leftarrow \mathbf{e}') := \Gamma(u_2)$ is a unitary transformation on $\mathcal{F}_{\mathrm{rad}}$, and

$$U(\mathbf{e}'' \leftarrow \mathbf{e}') d\Gamma(\omega) U(\mathbf{e}'' \leftarrow \mathbf{e}')^* = d\Gamma(\omega).$$

By the definition of u_2, the equality $U(\mathbf{e}'' \leftarrow \mathbf{e}')\mathbf{A}'(\hat{\mathbf{x}})U(\mathbf{e}'' \leftarrow \mathbf{e}')^* = \mathbf{A}''(\hat{\mathbf{x}})$ holds. Hence we have

$$U(\mathbf{e}'' \leftarrow \mathbf{e}')\overline{H'}U(\mathbf{e}'' \leftarrow \mathbf{e}')^* = \overline{U(\mathbf{e}'' \leftarrow \mathbf{e}')H'U(\mathbf{e}'' \leftarrow \mathbf{e}')^*} = \overline{H''},$$

which implies that H' is essentially self-adjoint and $\overline{H'}$ is unitarily equivalent to $\overline{H''}$. We set

$$U(\mathbf{e} \leftarrow \mathbf{e}') := U(\mathbf{e} \leftarrow \mathbf{e}'')U(\mathbf{e}'' \leftarrow \mathbf{e}').$$

Then $U(\mathbf{e} \leftarrow \mathbf{e}')\overline{H'}U(\mathbf{e} \leftarrow \mathbf{e}')^* = \bar{H}$. Therefore Theorem Appendix A.1 is proved. The proof of Theorem Appendix A.2 is similar to the proof of Theorem Appendix A.1. □

Appendix B. Remarks on the Angular Momentum

As is shown in Appendix A, spectral properties of QED models are independent of the choice of polarization vectors. Hence, in the definition of QED models, usually we do not need to specify the choice of the polarization vectors. However, the angular momentum of the electromagnetic field depends on a choice of the polarization vectors, since the angular momentum does not commute with $U(\mathbf{e} \leftarrow \mathbf{e}')$. Therefore, when we discuss an angular momentum, we take care of specifying the choice of polarization vectors. One can find the definition of an angular momentum for the electromagnetic field in the textbook [Sp, Section 13.5](see also [Hi]). In this appendix, we propose an alternate definition of angular momentum in the electromagnetic field.

Let $(H, \mathbf{e})$ be the pair of a Hamiltonian and polarization vectors.

For each unit vector $\mathbf{j} \in \mathbb{R}^3$, we can define a specific polarization vectors $\bar{\mathbf{e}} = (\bar{\mathbf{e}}^{(1)}, \bar{\mathbf{e}}^{(2)})$ by

$$\bar{\mathbf{e}}^{(1)}(\mathbf{k}) := \frac{\mathbf{k} \wedge \mathbf{j}}{|\mathbf{k} \wedge \mathbf{j}|}, \quad \bar{\mathbf{e}}^{(2)}(\mathbf{k}) := \frac{\mathbf{k}}{|\mathbf{k}|} \wedge \bar{\mathbf{e}}^{(1)}(\mathbf{k}). \tag{Appendix B.1}$$

For a Dirac-Maxwell model $(H, \bar{\mathbf{e}})$, we define the angular momentum around $\mathbf{j}$-axis by

$$L_{\mathbf{j}}(\bar{\mathbf{e}}) := d\Gamma(\overline{\mathbf{j} \cdot \vec{\ell}}),$$

where

$$\vec{\ell} := (\ell_1, \ell_2, \ell_3) := i(\nabla_{\mathbf{k}} \wedge \mathbf{k}),$$

is a triplet of self-adjoint operators acting on $L^2(\mathbb{R}^3_{\mathbf{k}} \times \{1, 2\})$.

Let $\mathbf{e} = (\mathbf{e}^{(1)}, \mathbf{e}^{(2)})$ be any polarization vectors. The angular momentum around $\mathbf{j}$-axis in the Dirac-Maxwell model $(H, \mathbf{e})$ is defined by

$$L_{\mathbf{j}}(\mathbf{e}) := U(\mathbf{e} \leftarrow \bar{\mathbf{e}})L_{\mathbf{j}}(\bar{\mathbf{e}})U(\mathbf{e} \leftarrow \bar{\mathbf{e}})^*,$$

where $U(\bar{\mathbf{e}} \leftarrow \mathbf{e})$ is a unitary operator defined in Appendix A. By the chain-rule of $U(\mathbf{e} \leftarrow \mathbf{e}')$, the angular momentums transformed as

$$L_{\mathbf{j}}(\mathbf{e}) = U(\mathbf{e} \leftarrow \mathbf{e}')L_{\mathbf{j}}(\mathbf{e}')U(\mathbf{e} \leftarrow \mathbf{e}')^*,$$

where $\mathbf{e}$ and $\mathbf{e}'$ are arbitrary polarization vectors.

Appendix C. Some Properties of the Lowest Energy

In Appendix C, we show some properties of $E_\nu(\mathbf{p})$ which are used in proofs of Theorems 4.1-4.4.

Proposition Appendix C.1 (Concavity). *$E_\nu(\mathbf{p})$ is concave in $(\mathbf{p}, m, q) \in \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$.*

Proof. See [A2]. □

Proposition Appendix C.2 (Continuity). *$E_\nu(\mathbf{p}, m)$ is Lipschitz continuous in $(\mathbf{p}, m)$, i.e.,*

$$|E_\nu(\mathbf{p}, m) - E_\nu(\mathbf{p}', m')| \leq \sqrt{|\mathbf{p} - \mathbf{p}'|^2 + |m - m'|^2}, \quad \mathbf{p}, \mathbf{p}' \in \mathbb{R}^3,\ m, m' \in \mathbb{R}.$$

Proof. See [A2] □

Proposition Appendix C.3 (Reflection symmetry in m). *The Hamiltonian $\overline{H_\nu(\mathbf{p}, m)}$ is unitarily equivalent to $\overline{H_\nu(\mathbf{p}, -m)}$. In particular*

$$E_\nu(\mathbf{p}, m) = E_\nu(\mathbf{p}, -m), \quad E_\nu(\mathbf{p}, m) \leq E_\nu(\mathbf{p}, 0).$$

Proof. Let $\gamma_5 := -i\alpha_1\alpha_2\alpha_3$. Then γ_5 is a unitary operator and $\gamma_5\overline{H_\nu(\mathbf{p}, m)}\gamma_5^* = \overline{H_\nu(\mathbf{p}, -m)}$. Therefore $E_\nu(\mathbf{p}, m) = E_\nu(\mathbf{p}, -m)$. By Proposition Appendix C.1, $m \mapsto E_\nu(\mathbf{p}, m)$ is concave. Hence $E_\nu(\mathbf{p}, 0) = E_\nu(\mathbf{p}, \frac{1}{2}m - \frac{1}{2}m) \geq E_\nu(\mathbf{p}, m)$. □

Proposition Appendix C.4 (Rotation invariance of the total momentum). *Let $T \in O(3)$ be an orthogonal matrix. Assume that $|\hat{\rho}(\mathbf{k})| = |\hat{\rho}(T\mathbf{k})|$ a.e. $\mathbf{k} \in \mathbb{R}^3$. Then $\overline{H_\nu(\mathbf{p})}$ is unitarily equivalent to $\overline{H_\nu(T\mathbf{p})}$. In particular, $E_\nu(\mathbf{p}) = E_\nu(T\mathbf{p})$ follows.*

Proof. For matrix $T \in O(3)$, we define four 4×4 matrices by

$$\beta' := \beta, \quad \alpha'_j := \sum_{l=1}^{3} T_{j,l}\alpha_l, \quad j = 1, 2, 3,$$

which obeys $\{\alpha'_j, \beta'\} = 0$, $\{\alpha'_j, \alpha'_l\} = 2\delta_{j,l}$, $j, l = 1, 2, 3$. Then there exists a 4×4 unitary matrix u_T such that (see [T, Lemma 2.25])

$$u_T\alpha_j u_T^{-1} = \sum_{k=1}^{3} T_{j,k}\alpha_k, \quad u_T\beta u_T^{-1} = \beta.$$

Therefore $u_T\boldsymbol{\alpha}\cdot\mathbf{p}u_T^{-1} = \sum_{k,l=1}^3 T_{l,k}\alpha_k p_l = \sum_{k,l=1}^3 \alpha_k (T^{-1})_{k,l} p_l = \boldsymbol{\alpha}\cdot(T^{-1}\mathbf{p})$. Similarly, we have

$$u_T(\boldsymbol{\alpha}\cdot d\Gamma(\mathbf{k}))u_T^{-1} = \boldsymbol{\alpha}\cdot(T^{-1}d\Gamma(\mathbf{k})), \quad u_T\boldsymbol{\alpha}\cdot\mathbf{A}u_T^{-1} = \boldsymbol{\alpha}\cdot(T^{-1}\mathbf{A}) = (T\boldsymbol{\alpha})\cdot\mathbf{A}.$$

We define rotation operator $\hat{T}$ of photon momentum, $\hat{T}$, by

$$(\hat{T}f)(\mathbf{k},\lambda) = f(T^{-1}\mathbf{k},\lambda), \quad (\mathbf{k},\lambda)\in\mathbb{R}^3_{\mathbf{k}}\times\{1,2\}, \quad f\in L^2(\mathbb{R}^3_{\mathbf{k}}\times\{1,2\}).$$

Then for all $f\in \mathrm{Dom}(k_j\hat{T})$

$$\hat{T}^{-1}k_j\hat{T}f(\mathbf{k},\lambda) = (k_j\hat{T}f)(T\mathbf{k},\lambda) = (T\mathbf{k})_j(\hat{T}f)(T\mathbf{k},\lambda) = (T\mathbf{k})_j f(\mathbf{k},\lambda).$$

Hence we obtain the operator equality $\hat{T}^{-1}k_j\hat{T} = (T\mathbf{k})_j$, $j=1,2,3$. Thus

$$\begin{aligned}
&\Gamma(\hat{T}^{-1})d\Gamma(k_j)\Gamma(\hat{T}) = d\Gamma((T\mathbf{k})_j) = (T\cdot d\Gamma(\mathbf{k}))_j,\\
&\Gamma(\hat{T}^{-1})H_f(\nu)\Gamma(\hat{T}) = H_f(\nu)\\
&\Gamma(\hat{T}^{-1})A_j\Gamma(\hat{T}), = \Phi_S(\hat{T}^{-1}g_j), \quad j=1,2,3,
\end{aligned}$$

where $\Phi_S(\cdot)$ is the Segal field operator(see[RS2, Page 209]) and $g_j(\cdot) := g_j(\cdot,\mathbf{x}=\mathbf{0}) \in L^2(\mathbb{R}^3_{\mathbf{k}}\times\{1,2\})$. The operator $U := u_T\otimes\Gamma(\hat{T}^{-1})$ is a unitary operator on $\mathbb{C}^4\otimes\mathcal{F}_{\mathrm{rad}}$ and

(Appendix C.1)
$$U\overline{H_\nu(\mathbf{p})}U^{-1} = \overline{(\boldsymbol{\alpha}\cdot(T^{-1}\mathbf{p}) + m\beta + H_f(\nu) - \boldsymbol{\alpha}\cdot d\Gamma(\mathbf{k}) - q(T\boldsymbol{\alpha})\cdot\Phi_S(\hat{T}^{-1}\mathbf{g}))}.$$

Note that T is a 3×3-matrix and $\hat{T}$ is unitary on $L^2(\mathbb{R}^3_{\mathbf{k}}\times\{1,2\})$. Since $T\in O(3)$, we have $(T\boldsymbol{\alpha})\cdot\Phi_S(\hat{T}^{-1}\mathbf{g}) = \boldsymbol{\alpha}\cdot T^{-1}\Phi_S(\hat{T}^{-1}\mathbf{g})$, i.e.,

(Appendix C.2)
$$(T^{-1}\Phi_S(\hat{T}^{-1}\mathbf{g}))_j = \sum_{l=1}^3 (T^{-1})_{j,l}\Phi_S(\hat{T}^{-1}g_l), \quad j=1,2,3.$$

We define functions

$$\mathbf{e}'^{(\lambda)}(\mathbf{k}) = T^{-1}\mathbf{e}^{(\lambda)}(T\mathbf{k}), \quad (\mathbf{k},\lambda)\in\mathbb{R}^3\times\{1,2\}.$$

Then $\mathbf{e}'^{(1)}$ and $\mathbf{e}'^{(2)}$ are polarization vectors: $\mathbf{k}\cdot\mathbf{e}'^{(\lambda)}(\mathbf{k}) = 0,\ \mathbf{e}'^{(\lambda)}(\mathbf{k})\cdot\mathbf{e}'^{(\mu)}(\mathbf{k}) = \delta_{\lambda,\mu}$. Since $|\hat{\rho}(\mathbf{k})| = |\hat{\rho}(T\mathbf{k})|$, there exists a Borel measurable function $\mathbf{k}\mapsto\kappa(\mathbf{k})\in\mathbb{R}$ such that $\hat{\rho}(T\mathbf{k}) = e^{i\kappa(\mathbf{k})}\hat{\rho}(\mathbf{k})$, a.e. $\mathbf{k}\in\mathbb{R}^3$. Therefore, we have

(Appendix C.3)
$$\sum_{l=1}^3 (T^{-1})_{j,l}g_l(T\mathbf{k},\lambda) = \frac{e^{i\kappa(\mathbf{k})}\hat{\rho}(\mathbf{k})}{|\mathbf{k}|^{1/2}}e_j'^{(\lambda)}(\mathbf{k}).$$

Let $H'_\nu(\mathbf{p})$ be defined by $H_\nu(\mathbf{p})$ with $\mathbf{e}^{(\lambda)}$ replaced by $\mathbf{e}'^{(\lambda)}$. By (Appendix C.1),(Appendix C.2) and (Appendix C.3), we have

$$U\overline{H_\nu(\mathbf{p})}U^* = V\overline{H'_\nu(T^{-1}\mathbf{p})}V^*,$$

where $V := \Gamma(e^{i\kappa(\cdot)})$. By Theorem Appendix A.2, $\overline{H'_\nu(T^{-1}\mathbf{p})}$ is unitarily equivalent to $\overline{H_\nu(T^{-1}\mathbf{p})}$. Therefore, $\overline{H(\mathbf{p})}$ is unitarily equivalent to $\overline{H_\nu(T^{-1}\mathbf{p})}$. Since $\mathbf{p} \in \mathbb{R}^3$ is arbitrary, $\overline{H_\nu(\mathbf{p})}$ is unitarily equivalent to $\overline{H_\nu(T\mathbf{p})}$, and $E_\nu(\mathbf{p}) = E_\nu(T\mathbf{p})$. □

If the cutoff function $|\hat{\rho}(\mathbf{k})|$ has the reflection symmetry at the origin, the following important inequality holds.

Proposition Appendix C.5. *Assume that* $|\hat{\rho}(\mathbf{k})| = |\hat{\rho}(-\mathbf{k})|$ *for almost every* $\mathbf{k} \in \mathbb{R}^3$. *Then the inequality*

$$E_\nu(\mathbf{p}) \leq E_\nu(\mathbf{0}), \quad \mathbf{p} \in \mathbb{R}^3 \backslash \{\mathbf{0}\}$$

holds.

Proof. By the assumption $\hat{\rho}(\mathbf{k}) = \hat{\rho}(-\mathbf{k})$ a.e. $\mathbf{k} \in \mathbb{R}^3$ and Proposition Appendix C.4, we have $E_\nu(\mathbf{p}) = E_\nu(-\mathbf{p})$, $\mathbf{p} \in \mathbb{R}^3$. Using the concavity of $E_\nu(\mathbf{p})$ with respect to $\mathbf{p}$. we obtain

$$E_\nu(\mathbf{0}) = E_\nu(\tfrac{1}{2}\mathbf{p} - \tfrac{1}{2}\mathbf{p}) \geq \frac{1}{2}E_\nu(\mathbf{p}) + \frac{1}{2}E_\nu(-\mathbf{p}) = E_\nu(\mathbf{p})$$

for all $\mathbf{p} \in \mathbb{R}^3$. □

Assuming that $H_\nu(\mathbf{0})$ has a ground state, we can obtain the following strict inverse energy inequality:

Proposition Appendix C.6. *Assume that* $|\hat{\rho}(\mathbf{k})| = |\hat{\rho}(-\mathbf{k})|$ a.e. $\mathbf{k} \in \mathbb{R}^3$. *If* $\overline{H_\nu(\mathbf{0})}$ *has a ground state, then*

$$E_\nu(\mathbf{p}) < E_\nu(\mathbf{0}) \quad \textit{for all } \mathbf{p} \neq \mathbf{0}.$$

Remark Appendix C.7. When $\nu > 0$, the massive Hamiltonian $H_\nu(\mathbf{0})$ has a ground state (Lemma 6.1). In the massless case $\nu = 0$, $H(\mathbf{0})$ has a ground state under suitable conditions(see Theorems 4.1, 4.2 and 4.4.)

Proof of Proposition Appendix C.6. We assume the equality $E_\nu(\mathbf{p}) = E_\nu(\mathbf{0})$ for a nonzero vector $\mathbf{p} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$. Let $\Phi_\nu(\mathbf{0})$ be a normalized ground state of $H_\nu(\mathbf{0})$. For $t = 1, -1$, we have

$$E_\nu(\mathbf{p}) = E_\nu(t\mathbf{p}) \leq \langle \Phi_\nu(\mathbf{0}), H_\nu(t\mathbf{p})\Phi_\nu(\mathbf{0})\rangle = t\,\langle \Phi_\nu(\mathbf{0}), \boldsymbol{\alpha} \cdot \mathbf{p}\Phi_\nu(\mathbf{0})\rangle + E_\nu(\mathbf{0}).$$

Therefore $\langle \Phi_\nu(\mathbf{0}), \boldsymbol{\alpha}\cdot\mathbf{p}\Phi_\nu(\mathbf{0})\rangle = 0$, and hence $\langle \Phi_\nu(\mathbf{0}), H_\nu(\mathbf{p})\Phi_\nu(\mathbf{0})\rangle = E_\nu(\mathbf{0}) = E_\nu(\mathbf{p})$, which implies $\|(H_\nu(\mathbf{p}) - E_\nu(\mathbf{p}))^{1/2}\Phi_\nu(\mathbf{0})\| = 0$, and therefore, $\Phi_\nu(\mathbf{0})$ is a ground state of $H_\nu(\mathbf{p})$. Thus $\boldsymbol{\alpha}\cdot\mathbf{p}\Phi_\nu(\mathbf{0}) = 0$, and we get a contradiction $|\mathbf{p}|^2\Phi_\nu(\mathbf{0}) = 0$. □

If the cutoff function $\hat{\rho}$ is spherically symmetric, the spectral properties of $\overline{H_\nu(\mathbf{p})}$ is independent of the direction of $\mathbf{p}$. The first part of the following proposition immediately follows from Proposition Appendix C.4, and thus, the last part from Proposition Appendix C.1.

Proposition Appendix C.8 (Spherical symmetry in the total momentum). *Assume that $|\hat{\rho}(\mathbf{k})|$ is a spherically symmetric function. Then $\overline{H_\nu(\mathbf{p})}$ is unitarily equivalent to $\overline{H_\nu(\mathbf{p}')}$ for all $\mathbf{p}' \in \mathbb{R}^3$ with $|\mathbf{p}| = |\mathbf{p}'|$. In particular $E_\nu(\mathbf{p})$ is spherically symmetric with respect to $\mathbf{p}$, and $E_\nu(\mathbf{p}) \geq E_\nu(\mathbf{p}')$ if $|\mathbf{p}| \leq |\mathbf{p}'|$.*

Proposition Appendix C.9 (Massless limit). *$E_\nu(\mathbf{p})$ is monotonously non-decreasing in $\nu \geq 0$ and*

$$\lim_{\nu\to+0} E_\nu(\mathbf{p}) = E_0(\mathbf{p}).$$

Proof. Let $\nu \geq \nu' \geq 0$. Then we have $H_\nu(\mathbf{p}) \geq H_{\nu'}(\mathbf{p})$ in the sense of quadratic form on $\mathcal{D} := \mathrm{Dom}(H_f) \cap \mathrm{Dom}(N_f)$. Therefore $\nu \mapsto E_\nu(\mathbf{p})$ is non-decreasing: $E_\nu(\mathbf{p}) \geq E_{\nu'}(\mathbf{p})$. It is easy to see that for all $\Psi \in \mathcal{D}$, $H_\nu(\mathbf{p})\Psi \to H(\mathbf{p})\Psi$ as $\nu \to 0$. Since $\mathcal{D}$ is a common core for all $H_\nu(\mathbf{p})$, $H_\nu(\mathbf{p}) \to H(\mathbf{p})$ in the strong resolvent sense (see [RS1, Theorem VIII. 25]). Using a fact about a strongly convergent operators[RS1, Theorem VIII. 24], we obtain that $E_\nu(\mathbf{p}) \to E(\mathbf{p})$ as $\nu \to +0$. □

By Proposition Appendix C.2, the following inequality holds:

$$0 \leq E_\nu(\mathbf{p}-\mathbf{k}) - E_\nu(\mathbf{p}) + |\mathbf{k}|, \quad \mathbf{p}, \mathbf{k} \in \mathbb{R}^3.$$

The function $\mathbf{k} \to E_\nu(\mathbf{p}-\mathbf{k}) - E_\nu(\mathbf{p}) + |\mathbf{k}|$ plays the role of a dispersion relation in the low-energy Dirac polaron.

Theorem Appendix C.10. *Let $\nu \geq 0$. Assume that $\hat{\rho}$ is spherically symmetric. Suppose that $E_\nu(\mathbf{p}, m) < E_\nu(\mathbf{p}, 0)$. Then, for $\mathbf{p} \neq \mathbf{0}$, the following estimate holds:*

$$E_\nu(\mathbf{p}-\mathbf{k}, m) - E_\nu(\mathbf{p}, m) + |\mathbf{k}| \geq \begin{cases} |\mathbf{k}| & \text{if} \quad |\mathbf{p}-\mathbf{k}| \leq |\mathbf{p}|, \\ (1 - b_\nu(\mathbf{p}))|\mathbf{k}| & \text{if} \quad |\mathbf{p}| \leq |\mathbf{p}-\mathbf{k}| \leq 2|\mathbf{p}|, \\ (1 - b_\nu(\mathbf{p}))|\mathbf{p}| & \text{if} \quad 2|\mathbf{p}| \leq |\mathbf{p}-\mathbf{k}|, \end{cases}$$

where

$$b_\nu(\mathbf{p}) := \frac{E_\nu(\mathbf{p}, m) - E_\nu(2\mathbf{p}, m)}{|\mathbf{p}|} < 1.$$

In the case $\mathbf{p} = \mathbf{0}$*, for all constant* $P > 0$ *the following estimate holds:*

$$E_\nu(\mathbf{k}, m) - E_\nu(\mathbf{0}, m) + |\mathbf{k}| \geq \begin{cases} \frac{a_\nu(P)}{P}|\mathbf{k}|, & \text{if } |\mathbf{k}| \leq P \\ a_\nu(P), & \text{if } |\mathbf{k}| > P, \end{cases} \tag{Appendix C.4}$$

where

$$a_\nu(P) := (E_\nu(\mathbf{k}, m) - E_\nu(\mathbf{0}, m) + |\mathbf{k}|)\Big|_{|\mathbf{k}|=P}$$

is a strictly positive constant.

Remark Appendix C.11. The idea of the proof of Theorem Appendix C.10 was developed in [LMS].

Proof of Theorem Appendix C.10. Before proving Theorem Appendix C.10, we prove the next lemma:

Lemma Appendix C.12. *Let* $\nu \geq 0$*. Assume that* $E_\nu(\mathbf{p}, m) < E_\nu(\mathbf{p}, 0)$*. Then*

$$E_\nu(\mathbf{p} - \mathbf{k}, m) - E_\nu(\mathbf{p}, m) + |\mathbf{k}| > 0, \quad \mathbf{k} \in \mathbb{R}^3 \backslash \{\mathbf{0}\}. \tag{Appendix C.5}$$

Proof. First we prove (Appendix C.5) for positive $\nu > 0$. We fix $m \neq 0$ and $\mathbf{p} \in \mathbb{R}^3$. Suppose that

$$E_\nu(\mathbf{p} - \mathbf{k}) - E_\nu(\mathbf{p}) + |\mathbf{k}| = 0, \tag{Appendix C.6}$$

for some $\mathbf{k} \in \mathbb{R}^3 \backslash \{\mathbf{0}\}$. Let $\Phi_\nu(\mathbf{p}-\mathbf{k})$ be a normalized ground state of $H_\nu(\mathbf{p}-\mathbf{k})$(see Lemma 6.1). Then

$$\begin{aligned} E_\nu(\mathbf{p} - \mathbf{k}) &= \left\langle \Phi_\nu(\mathbf{p} - \mathbf{k}), \overline{H_\nu(\mathbf{p} - \mathbf{k})} \Phi_\nu(\mathbf{p} - \mathbf{k}) \right\rangle \\ &= \left\langle \Phi_\nu(\mathbf{p} - \mathbf{k}), \overline{H_\nu(\mathbf{p})} \Phi_\nu(\mathbf{p} - \mathbf{k}) \right\rangle - \langle \Phi_\nu(\mathbf{p} - \mathbf{k}), \boldsymbol{\alpha} \cdot \mathbf{k} \Phi_\nu(\mathbf{p} - \mathbf{k}) \rangle \\ &\geq E_\nu(\mathbf{p}) - |\mathbf{k}|. \end{aligned}$$

Hence, by assumption (Appendix C.6) we have $\left\langle \Phi_\nu(\mathbf{p} - \mathbf{k}), \overline{H_\nu(\mathbf{p})} \Phi_\nu(\mathbf{p} - \mathbf{k}) \right\rangle = E_\nu(\mathbf{p})$ and $\langle \Phi_\nu(\mathbf{p} - \mathbf{k}), \boldsymbol{\alpha} \cdot \mathbf{k} \Phi_\nu(\mathbf{p} - \mathbf{k}) \rangle = |\mathbf{k}|$, which implies that $\Phi_\nu(\mathbf{p} - \mathbf{k})$ is a ground state of both $\overline{H_\nu(\mathbf{p})}$ and $-\boldsymbol{\alpha} \cdot \mathbf{k}$. Since $\mathbf{k} \neq \mathbf{0}$, we have $\langle \Phi_\nu(\mathbf{p} - \mathbf{k}), \beta \Phi_\nu(\mathbf{p} - \mathbf{k}) \rangle = 0$, because $\boldsymbol{\alpha} \cdot \mathbf{k} \beta = -\beta \boldsymbol{\alpha} \cdot \mathbf{k}$. In what follows, to emphasize m-dependence, we write $H_\nu(\mathbf{p} - \mathbf{k}, m)$ and $\Phi_\nu(\mathbf{p} - \mathbf{k}, m)$ for $H_\nu(\mathbf{p} - \mathbf{k})$ and $\Phi_\nu(\mathbf{p} - \mathbf{k})$, respectively. By using the above facts, we have

$$E_\nu(\mathbf{p}, m) = \left\langle \Phi_\nu(\mathbf{p} - \mathbf{k}, m), \overline{H_\nu(\mathbf{p}, 0)} \Phi_\nu(\mathbf{p} - \mathbf{k}, m) \right\rangle \geq E_\nu(\mathbf{p}, 0),$$

which contradicts the inequality $E_\nu(\mathbf{p}, m) < E_\nu(\mathbf{p}, 0)$. Next, we prove the case $\nu = 0$. Suppose that there exist a vector $\mathbf{k} \in \mathbb{R}^3 \backslash \{\mathbf{0}\}$ such that $E(\mathbf{p} - \mathbf{k}, m) -$

$E(\mathbf{p},m)+|\mathbf{k}|=0$ holds. It is not difficult to see that

$$\lim_{\nu\to+0}\left\langle \Phi_\nu(\mathbf{p}-\mathbf{k},m),\overline{H(\mathbf{p}-\mathbf{k},m)}\Phi_\nu(\mathbf{p}-\mathbf{k},m)\right\rangle = E(\mathbf{p}-\mathbf{k},m).$$

By these equations, we have

$$\lim_{\nu\to+0}\langle \Phi_\nu(\mathbf{p}-\mathbf{k},m),\boldsymbol{\alpha}\cdot\mathbf{k}\Phi_\nu(\mathbf{p}-\mathbf{k},m)\rangle = |\mathbf{k}|, \tag{Appendix C.7}$$

$$\lim_{\nu\to+0}\left\langle \Phi_\nu(\mathbf{p}-\mathbf{k},m),\overline{H(\mathbf{p},m)}\Phi_\nu(\mathbf{p}-\mathbf{k},m)\right\rangle = E(\mathbf{p},m). \tag{Appendix C.8}$$

Equation (Appendix C.7) implies that

$$\lim_{\nu\to+0}(|\mathbf{k}|-\boldsymbol{\alpha}\cdot\mathbf{k})\Phi_\nu(\mathbf{p}-\mathbf{k},m)=0.$$

Therefore $\lim_{\nu\to+0}\langle \Phi_\nu(\mathbf{p}-\mathbf{k},m),\beta\Phi_\nu(\mathbf{p}-\mathbf{k},m)\rangle = 0$. This fact and equation (Appendix C.8) imply $E(\mathbf{p},m)=E(\mathbf{p},0)$, which contradicts $E(\mathbf{p},m)<E(\mathbf{p},0)$. □

We fix a vector $\mathbf{p}$ such that $E_\nu(\mathbf{p},m)<E_\nu(\mathbf{p},0)$. Since $\hat{\rho}$ is spherically symmetric, by Proposition Appendix C.8, the function

$$G_\nu(|\mathbf{k}|):=E_\nu(\mathbf{0})-E_\nu(\mathbf{k}),\quad \mathbf{k}\in\mathbb{R}^3,$$

is monotonously non-decreasing, convex with respect to $|\mathbf{k}|$, and the following inequality holds

$$0\le G_\nu(|\mathbf{k}|)\le|\mathbf{k}|,\quad \mathbf{k}\in\mathbb{R}^3. \tag{Appendix C.9}$$

Since $G_\nu(s)$ is convex, $G_\nu(s)$ has a right derivative ${G_\nu^+}'(s)$:

$${G_\nu^+}'(s):=\lim_{h\to+0}[G_\nu(s+h)-G_\nu(s)]/h.$$

First we show that

$${G_\nu^+}'(s)<1,\quad 0\le s\le|\mathbf{p}|. \tag{Appendix C.10}$$

Since $G_\nu(s)$ is convex and $0\le G_\nu(s)\le s$, ${G_\nu^+}'(s)$ is a monotonously non-decreasing function of s. If ${G_\nu^+}'(s_0)>1$ for a constant $s_0\ge0$, then ${G_\nu^+}'(s)>1$ for all $s\ge s_0$ and

$$G_\nu(s)=\int_{s_0}^{s}{G_\nu^+}'(t)dt+\int_0^{s_0}{G_\nu^+}'(t)dt\ge(s-s_0){G_\nu^+}'(s_0)+\int_0^{s_0}{G_\nu^+}'(t)dt,$$

holds for all $s>s_0$. It contradicts (Appendix C.9). Thus, ${G_\nu^+}'(s)\le1$ for all $s\ge0$. Let $s_1\ge0$ be a point such that ${G_\nu^+}'(s_1)=1$ and ${G_\nu^+}'(s_1-\epsilon)<1$ for all $0<\epsilon\le s_1$.

If $|\mathbf{p}| < s_1$, (Appendix C.10) is trivial. Thus we consider the case $|\mathbf{p}| \geq s_1$. Note that ${G_\nu^+}'(s) = 1$ for all $s \geq s_1$. Hence $G_\nu(s)$ is a linear function of s if $s \geq s_1$:

$$G_\nu(s) = s + C, \quad s \geq s_1,$$

where C is a negative constant. By this equality, we have that

$$E_\nu(\mathbf{p}-\mathbf{k}) - E_\nu(\mathbf{p}) + |\mathbf{k}| = -|\mathbf{p}-\mathbf{k}| + |\mathbf{p}| + |\mathbf{k}|,$$

for all $\mathbf{p}$ and $\mathbf{k}$ such that $|\mathbf{p}-\mathbf{k}| \geq s_1$ and $|\mathbf{p}| \geq s_1$. We choose $\mathbf{k} = -C\mathbf{p}$ for a constant $C > s_1/|\mathbf{p}|$. Then

$$E_\nu(\mathbf{p}-\mathbf{k}) - E_\nu(\mathbf{p}) + |\mathbf{k}| = 0.$$

It contradicts Lemma Appendix C.12. Therefore ${G_\nu^+}'(s) < 1$ holds for all $0 \leq s \leq |\mathbf{p}|$.

Next, by using this inequality, we prove Theorem Appendix C.10. By (Appendix C.10) and convexity of G_ν, it holds that

$$c_\nu(\mathbf{p}) := \frac{G_\nu(|\mathbf{p}|)}{|\mathbf{p}|} \leq b_\nu(\mathbf{p}) < 1.$$

We define a set of functions:

$$\begin{aligned}\mathcal{C} := \{J : \mathbb{R}_+ \to \mathbb{R}_+ | J \text{ is convex}, \ \ 0 \leq J(s) \leq s, \ (s \geq 0) \\ J(|\mathbf{p}|) = G_\nu(|\mathbf{p}|), J(2|\mathbf{p}|) = G_\nu(2|\mathbf{p}|)\}\end{aligned}$$

Then we have

$$\begin{aligned}E_\nu(\mathbf{p}-\mathbf{k}) - E_\nu(\mathbf{p}) + |\mathbf{k}| &= |\mathbf{k}| + G_\nu(\mathbf{p}) - G_\nu(\mathbf{p}-\mathbf{k}) \\ &\geq |\mathbf{k}| + G_\nu(\mathbf{p}) - \sup_{J \in \mathcal{C}} J(\mathbf{p}-\mathbf{k}) := I.\end{aligned}$$

The maximal function in $\mathcal{C}$ is given by the following linear interpolation:

$$J_{\max}(s) := \begin{cases} c_\nu(\mathbf{p})s & \text{if} \ \ s \leq |\mathbf{p}|, \\ b_\nu(\mathbf{p})(s - |\mathbf{p}|) + G_\nu(|\mathbf{p}|) & \text{if} \ \ |\mathbf{p}| \leq s \leq 2|\mathbf{p}|, \\ s - 2|\mathbf{p}| + G_\nu(2|\mathbf{p}|) & \text{if} \ \ 2|\mathbf{p}| \leq |\mathbf{p}-\mathbf{k}|. \end{cases}$$

Hence

$$I \geq |\mathbf{k}| + G_\nu(|\mathbf{p}|) - \begin{cases} c_\nu(\mathbf{p})|\mathbf{p}-\mathbf{k}| & \text{if } \ |\mathbf{p}-\mathbf{k}| \leq |\mathbf{p}|, \\ b_\nu(\mathbf{p})(|\mathbf{p}-\mathbf{k}| - |\mathbf{p}|) + G_\nu(|\mathbf{p}|) & \text{if } \ |\mathbf{p}| \leq |\mathbf{p}-\mathbf{k}| \leq 2|\mathbf{p}|, \\ |\mathbf{p}-\mathbf{k}| - 2|\mathbf{p}| + G_\nu(2|\mathbf{p}|) & \text{if } \ 2|\mathbf{p}| \leq |\mathbf{p}-\mathbf{k}|. \end{cases}$$

$$= \begin{cases} |\mathbf{k}| + c_\nu(\mathbf{p})(|\mathbf{p}| - |\mathbf{p}-\mathbf{k}|) & \text{if } \ |\mathbf{p}-\mathbf{k}| \leq |\mathbf{p}|, \\ |\mathbf{k}| - b_\nu(\mathbf{p})(|\mathbf{p}-\mathbf{k}| - |\mathbf{p}|) & \text{if } \ |\mathbf{p}| \leq |\mathbf{p}-\mathbf{k}| \leq 2|\mathbf{p}|, \\ |\mathbf{k}| - |\mathbf{p}-\mathbf{k}| + (2 - b_\nu(\mathbf{p}))|\mathbf{p}| & \text{if } \ 2|\mathbf{p}| \leq |\mathbf{p}-\mathbf{k}|. \end{cases}$$

Using the triangle inequality, one can obtain the desired estimate. Finally we prove (Appendix C.4). Since ${G_\nu^+}'(0) < 1$ and G_ν is convex, the constant $a_\nu(P)$ is strictly positive for all $P > 0$. It is easy to see that

$${G_\nu^+}'(s) \leq \frac{G_\nu(P)}{P} = \frac{-a_\nu(P) + P}{P}, \quad s \leq P.$$

Hence

$$E_\nu(\mathbf{k}) - E_\nu(\mathbf{0}) + |\mathbf{k}| = |\mathbf{k}| - G_\nu(|\mathbf{k}|) = \int_0^{|\mathbf{k}|} (1 - {G_\nu^+}'(s))ds$$

$$\geq \begin{cases} \displaystyle\int_0^{|\mathbf{k}|} \Big(1 - \frac{G_\nu(P)}{P}\Big) ds & \text{if } \ |\mathbf{k}| \leq P. \\ \displaystyle\int_0^{P} \Big(1 - \frac{G_\nu(P)}{P}\Big) ds + \int_P^{|\mathbf{k}|} (1 - {G_\nu^+}'(s))ds & \text{if } \ |\mathbf{k}| > P. \end{cases}$$

$$\geq \begin{cases} (a_\nu(P)/P)|\mathbf{k}|, & \text{if } \ |\mathbf{k}| \leq P. \\ a_\nu(P), & \text{if } \ |\mathbf{k}| > P. \end{cases}$$

This completes the proof. □

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