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Definition of Flat Poset and Existence Theorems for Recursive Call

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Summary. This text includes the definition and basic notions of product of posets, chain-complete and flat posets, flattening operation, and the existence theorems of recursive call using the flattening operator. First part of the article, devoted to product and flat posets has a purely mathematical quality. Definition 3 allows to construct a flat poset from arbitrary non-empty set [12] in order to provide formal apparatus which eanbles to work with recursive calls within the Mizar langauge. To achieve this we extensively use technical Mizar functors like `BaseFunc` or `RecFunc`. The remaining part builds the background for information engineering approach for lists, namely recursive call for posets [21]. We formalized some facts from Chapter 8 of this book as an introduction to the next two sections where we concentrate on binary product of posets rather than on a more general case.

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The notation and terminology used in this paper have been introduced in the following articles: [1], [17], [11], [6], [7], [8], [2], [13], [19], [14], [4], [9], [15], [22], [23], [20], [5], [16], and [10].

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1. Preliminaries from Poset Theory

From now on a, Z_1, Z_2, Z_3 denote sets, x, y, z denote objects, and k denotes a natural number.

Now we state the propositions:

(1) Let us consider a lower-bounded non empty poset P and an element p of P. If $p \leqslant$ the carrier of P, then $p = \bot_P$.

(2) Let us consider a chain-complete non empty poset P, a non empty chain L of P, and an element p of P. If $p \in L$, then $p \leqslant \sup L$.

(3) Let us consider a chain-complete non empty poset P, a non empty chain L of P, and an element p_1 of P. Suppose an element p of P. If $p \in L$, then $p \leqslant p_1$. Then $\sup L \leqslant p_1$.

2. On the Product of Posets

Now we state the proposition:

(4) Let us consider non empty relational structures P, Q and an object x. Then x is an element of $P \times Q$ if and only if there exists an element p of P and there exists an element q of Q such that $x = \langle p, q \rangle$.

Let P, Q be non empty posets and L be a non empty chain of $P \times Q$. The functors: $\pi_1(L)$ and $\pi_2(L)$ yield non empty chains of P. Let P, Q_1, Q_2 be non empty posets, f_1 be a monotone function from P into Q_1, and f_2 be a monotone function from P into Q_2. One can verify that $\langle f_1, f_2 \rangle$ is monotone as a function from P into $Q_1 \times Q_2$.

Let P, Q be chain-complete non empty posets. Observe that $P \times Q$ is chain-complete.

Now we state the proposition:

(5) Let us consider chain-complete non empty posets P, Q and a non empty chain L of $P \times Q$. Then $\sup L = \langle \sup \pi_1(L), \sup \pi_2(L) \rangle$.

Let P, Q_1, Q_2 be strict chain-complete non empty posets, f_1 be a continuous function from P into Q_1, and f_2 be a continuous function from P into Q_2. Note that $\langle f_1, f_2 \rangle$ is continuous as a function from P into $Q_1 \times Q_2$.

3. Definition of Flat Poset and Poset Flattening

Let I_3 be a relational structure. We say that I_3 is flat if and only if

(Def. 1) There exists an element a of I_3 such that for every elements x, y of I_3, $x \leqslant y$ iff $x = a$ or $x = y$.

One can verify that every non empty relational structure which is discrete is also reflexive and every discrete non empty relational structure which is trivial is also flat and there exists a poset which is strict, non empty, and flat and every relational structure which is flat is also reflexive transitive and antisymmetric and every non empty poset which is flat is also lower-bounded.

In the sequel S denotes a relational structure, P, Q denote non empty flat posets, p, p_1, p_2 denote elements of P, and K denotes a non empty chain of P.

Now we state the proposition:

(6) Let us consider a non empty flat poset P and a non empty chain K of P. Then there exists an element a of P such that $K = \{a\}$ or $K = \{\bot_P, a\}$.

Let us consider a function f from P into Q. Now we state the propositions:

(7) There exists an element a of P such that $K = \{a\}$ and $f^\circ K = \{f(a)\}$ or $K = \{\bot_P, a\}$ and $f^\circ K = \{f(\bot_P), f(a)\}$. The theorem is a consequence of (6).

(8) If $f(\bot_P) = \bot_Q$, then f is monotone.

Now we state the proposition:

(9) If $K = \{\bot_P, p\}$, then $\sup K = p$.

One can verify that there exists a poset which is strict, non empty, flat, and chain-complete and every poset which is non empty and flat is also chain-complete.

Now we state the proposition:

(10) Let us consider strict non empty chain-complete flat posets P, Q and a function f from P into Q. If $f(\bot_P) = \bot_Q$, then f is continuous. Proof: For every non empty chain K of P, $f(\sup K) \leqslant \sup(f^\circ K)$ by [15, (1)], (7), [5, (39)], (9). □

4. Primaries for Existence Theorems of Recursive Call Using Flattening

In the sequel X, Y denote non empty sets.

Let X be a non empty set. The functor $\mathrm{FlatRelat}\, X$ yielding a relation between $\mathrm{succ}\, X$ and $\mathrm{succ}\, X$ is defined by the term

(Def. 2) $(\{\langle X, X\rangle\} \cup \{X\} \times X) \cup \mathrm{id}_X$.

Now we state the proposition:

(11) Let us consider elements x, y of succ X. Then $\langle x, y\rangle \in$ FlatRelat X if and only if $x = X$ or $x = y$.

Let X be a non empty set. The functor FlatPoset X yielding a strict non empty chain-complete flat poset is defined by the term

(Def. 3) $\langle \text{succ}\, X, \text{FlatRelat}\, X\rangle$.

Now we state the propositions:

(12) Let us consider elements x, y of FlatPoset X. Then $x \leqslant y$ if and only if $x = X$ or $x = y$.

(13) X is an element of FlatPoset X.

Let us consider X. Let us observe that $\perp_{\text{FlatPoset}\, X}$ reduces to X.

Let x be an object, X, Y be non empty sets, and f be a function from X into Y. The functor Flatten(f, x) yielding a set is defined by the term

(Def. 4) $\begin{cases} f(x), & \textbf{if } x \in X, \\ Y, & \textbf{otherwise}. \end{cases}$

The functor Flatten(f) yielding a function from FlatPoset X into FlatPoset Y is defined by

(Def. 5) (i) $it(X) = Y$, and

(ii) for every element x of FlatPoset X such that $x \neq X$ holds $it(x) = f(x)$.

Let us observe that Flatten(f) is continuous.

Now we state the proposition:

(14) Let us consider a function f from X into Y. If $x \in X$, then $(\text{Flatten}(f))(x) \in Y$.

Let us consider X and Y. The functor FlatConF(X, Y) yielding a strict chain-complete non empty poset is defined by the term

(Def. 6) ConPoset(FlatPoset X, FlatPoset Y).

Let L be a flat poset. One can verify that every chain of L is finite and there exists a lattice which is non empty, flat, and lower-bounded.

Now we state the propositions:

(15) Let us consider a non empty lattice L, an element x of L, and an x-chain A of x. Then $\overline{\overline{A}} = 1$. Proof: For every element z of L such that $z \in A$ holds $z \in \{x\}$ by [19, (2)]. □

(16) Let us consider a non empty flat lower-bounded lattice L, an element x of L, and a $\perp_L$-chain A of x. Then $\overline{\overline{A}} \leqslant 2$. The theorem is a consequence of (6) and (15).

(17) Let us consider a finite lower-bounded antisymmetric non empty lattice L. Then L is flat if and only if for every element x of L, height $x \leqslant 2$. Proof: There exists an element a of L such that for every elements x, y of L, $x \leqslant y$ iff $x = a$ or $x = y$ by [5, (44)], [13, (2), (6)], [3, (13)]. □

5. Existence Theorem of Recursive Call for Single-equation

From now on D denotes a subset of X, I denotes a function from X into Y, J denotes a function from $X \times Y$ into Y, and E denotes a function from X into X.

Let X be a non empty set, D be a subset of X, and E be a function from X into X. We say that E is well founded with minimal set D if and only if

(Def. 7) There exists a function l from X into $\mathbb{N}$ such that for every element x of X, if $l(x) \leqslant 0$, then $x \in D$ and if $x \notin D$, then $l(E(x)) < l(x)$.

Let X, Y be non empty sets. Let I be a function from X into Y, J be a function from $X \times Y$ into Y, and x, y be objects. The functor $\mathrm{BaseFunc01}(x, y, I, J, D)$ yielding a set is defined by the term

(Def. 8) $\begin{cases} I(x), & \textbf{if } x \in D, \\ J(\langle x, y \rangle), & \textbf{if } x \notin D \text{ and } x \in X \text{ and } y \in Y, \\ Y, & \textbf{otherwise}. \end{cases}$

Let E be a function from X into X and h be an object. Assume h is a continuous function from FlatPoset X into FlatPoset Y.

The functor $\mathrm{RecFunc01}(h, E, I, J, D)$ yielding a continuous function from FlatPoset X into FlatPoset Y is defined by

(Def. 9) Let us consider an element x of FlatPoset X and a continuous function f from FlatPoset X into FlatPoset Y. Suppose $h = f$. Then $it(x) = \mathrm{BaseFunc01}(x, f((\mathrm{Flatten}(E))(x)), I, J, D)$.

Now we state the propositions:

(18) There exists a continuous function W from $\mathrm{FlatConF}(X, Y)$ into $\mathrm{FlatConF}(X, Y)$ such that for every element f of ConFuncs(FlatPoset X, FlatPoset Y), $W(f) = \mathrm{RecFunc01}(f, E, I, J, D)$. Proof: Set $F_1 =$ FlatPoset X. Set $F_2 =$ FlatPoset Y. Set $F_3 = \mathrm{FlatConF}(X, Y)$. Set $C_1 = \mathrm{ConFuncs}(F_1, F_2)$. Define $\mathcal{H}(\text{object}) = \mathrm{RecFunc01}(\$_1, E, I, J, D)$. For every continuous function h from F_1 into F_2, $h \in C_1$ by [7, (8)]. For every set h such that $h \in C_1$ holds h is a continuous function from F_1 into F_2. There exists a function W from C_1 into C_1 such that for every object f such that $f \in C_1$ holds $W(f) = \mathcal{H}(f)$ from [7, Sch. 2]. Consider I_3 being a function from C_1 into C_1 such that for every object f such that $f \in C_1$ holds $I_3(f) = \mathcal{H}(f)$. I_3 is a continuous function from F_3 into F_3 by [7, (5)], (12), [24, (9)], [15, (1), (11)]. □

(19) There exists a set f such that

(i) $f \in$ ConFuncs(FlatPoset X, FlatPoset Y), and

(ii) $f = \mathrm{RecFunc01}(f, E, I, J, D)$.

The theorem is a consequence of (18).

Let us assume that E is well founded with minimal set D. Now we state the propositions:

(20) There exists a continuous function f from FlatPoset X into FlatPoset Y such that for every element x of X, $f(x) \in Y$ and $f(x) = \mathrm{BaseFunc01}(x, f(E(x)), I, J, D)$. Proof: Consider f being a set such that $f \in$ ConFuncs (FlatPoset X, FlatPoset Y) and $f = \mathrm{RecFunc01}(f, E, I, J, D)$. Consider l being a function from X into $\mathbb{N}$ such that for every element x_0 of X, if $l(x_0) \leqslant 0$, then $x_0 \in D$ and if $x_0 \notin D$, then $l(E(x_0)) < l(x_0)$. Define $\mathcal{P}$[natural number] $\equiv$ for every element x_0 of X such that $l(x_0) \leqslant \$_1$ holds $f(x_0) \in Y$ and $f(x_0) = \mathrm{BaseFunc01}(x_0, f(E(x_0)), I, J, D)$. $\mathcal{P}[0]$ by [7, (5)]. For every k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [7, (5)], [3, (13)]. For every natural number k, $\mathcal{P}[k]$ from [3, Sch. 2]. For every element x of X, $f(x) \in Y$ and $f(x) = \mathrm{BaseFunc01}(x, f(E(x)), I, J, D)$. $\square$

(21) There exists a function f from X into Y such that for every element x of X, if $x \in D$, then $f(x) = I(x)$ and if $x \notin D$, then $f(x) = J(\langle x, f(E(x))\rangle)$.

Now we state the proposition:

(22) Let us consider functions f_1, f_2 from X into Y. Suppose

(i) E is well founded with minimal set D, and

(ii) for every element x of X, if $x \in D$, then $f_1(x) = I(x)$ and if $x \notin D$, then $f_1(x) = J(\langle x, f_1(E(x))\rangle)$, and

(iii) for every element x of X, if $x \in D$, then $f_2(x) = I(x)$ and if $x \notin D$, then $f_2(x) = J(\langle x, f_2(E(x))\rangle)$.

Then $f_1 = f_2$. Proof: Consider l being a function from X into $\mathbb{N}$ such that for every element x of X, if $l(x) \leqslant 0$, then $x \in D$ and if $x \notin D$, then $l(E(x)) < l(x)$. Define $\mathcal{P}$[natural number] $\equiv$ for every element x of X such that $l(x) \leqslant \$_1$ holds $f_1(x) = f_2(x)$. $\mathcal{P}[0]$. For every k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [3, (13)]. For every natural number k, $\mathcal{P}[k]$ from [3, Sch. 2]. For every element x of X, $f_1(x) = f_2(x)$. $\square$

6. Existence Theorem of Recursive Calls for 2-equations

From now on D denotes a subset of X, I, I_1, I_2 denote functions from X into Y, J, J_1, J_2 denote functions from $X \times Y \times Y$ into Y, and E_1, E_2 denote functions from X into X.

Let X be a non empty set, D be a subset of X, and E_1, E_2 be functions from X into X. We say that (E_1,E_2) is well founded with minimal set D if and only if

(Def. 10) There exists a function l from X into $\mathbb{N}$ such that for every element x of X, if $l(x) \leqslant 0$, then $x \in D$ and if $x \notin D$, then $l(E_1(x)) < l(x)$ and $l(E_2(x)) < l(x)$.

Let X, Y be non empty sets. Let I be a function from X into Y, J be a function from $X \times Y \times Y$ into Y, and x, y_1, y_2 be objects. The functor $\mathrm{BaseFunc02}(x, y_1, y_2, I, J, D)$ yielding a set is defined by the term

(Def. 11) $$\begin{cases} I(x), & \textbf{if } x \in D, \\ J(\langle x, y_1, y_2 \rangle), & \textbf{if } x \notin D \text{ and } x \in X \text{ and } \mathrm{y}_1, y_2 \in Y, \\ Y, & \textbf{otherwise}. \end{cases}$$

Let E_1, E_2 be functions from X into X and h_1, h_2 be objects. Assume h_1 is a continuous function from $\mathrm{FlatPoset}\,X$ into $\mathrm{FlatPoset}\,Y$ and h_2 is a continuous function from $\mathrm{FlatPoset}\,X$ into $\mathrm{FlatPoset}\,Y$. The functor $\mathrm{RecFunc02}(h_1, h_2, E_1, E_2, I, J, D)$ yielding a continuous function from $\mathrm{FlatPoset}\,X$ into $\mathrm{FlatPoset}\,Y$ is defined by

(Def. 12) Let us consider an element x of $\mathrm{FlatPoset}\,X$ and continuous functions f_1, f_2 from $\mathrm{FlatPoset}\,X$ into $\mathrm{FlatPoset}\,Y$. Suppose

(i) $h_1 = f_1$, and

(ii) $h_2 = f_2$.

Then $it(x) =$
$\mathrm{BaseFunc02}(x, f_1((\mathrm{Flatten}(E_1))(x)), f_2((\mathrm{Flatten}(E_2))(x)), I, J, D)$.

Now we state the propositions:

(23) There exists a continuous function W from $\mathrm{FlatConF}(X, Y) \times \mathrm{FlatConF}(X, Y)$ into $\mathrm{FlatConF}(X, Y)$ such that for every set f such that $f \in \mathrm{ConFuncs}(\mathrm{FlatPoset}\,X, \mathrm{FlatPoset}\,Y) \times \mathrm{ConFuncs}(\mathrm{FlatPoset}\,X, \mathrm{FlatPoset}\,Y)$ holds $W(f) = \mathrm{RecFunc02}(f_{\mathbf{1}}, f_{\mathbf{2}}, E_1, E_2, I, J, D)$. Proof: Set $F_1 = \mathrm{FlatPoset}\,X$. Set $F_2 = \mathrm{FlatPoset}\,Y$. Set $F_3 = \mathrm{FlatConF}(X, Y)$. Set $C_1 = \mathrm{ConFuncs}(F_1, F_2)$. Set $F_4 = F_3 \times F_3$. Set $C_2 = C_1 \times C_1$. Define $\mathcal{H}(\text{object}) = \mathrm{RecFunc02}(\$_{1\mathbf{1}}, \$_{1\mathbf{2}}, E_1, E_2, I, J, D)$. For every continuous function h from F_1 into F_2, $h \in C_1$ by [7, (8)]. For every set h such that $h \in C_1$ holds h is a continuous function from F_1 into F_2. For every element h of F_4, there exist continuous functions h_1, h_2 from F_1 into F_2 such that $h = \langle h_1, h_2 \rangle$. There exists a function W from C_2 into C_1 such that for every object f such that $f \in C_2$ holds $W(f) = \mathcal{H}(f)$ from [7, Sch. 2]. Consider I_3 being a function from C_2 into C_1 such that for every object f such that $f \in C_2$ holds $I_3(f) = \mathcal{H}(f)$. I_3 is a continuous function from F_4 into F_3 by [7, (5)], [16, (12)], (12), [24, (9)]. □

(24) There exist sets f, g such that

(i) $f, g \in \mathrm{ConFuncs}(\mathrm{FlatPoset}\,X, \mathrm{FlatPoset}\,Y)$, and

(ii) $f = \mathrm{RecFunc02}(f, g, E_1, E_2, I_1, J_1, D)$, and

(iii) $g = \mathrm{RecFunc02}(f, g, E_1, E_2, I_2, J_2, D)$.

The theorem is a consequence of (23) and (4).

Let us assume that (E_1,E_2) is well founded with minimal set D. Now we state the propositions:

(25) There exist continuous functions f, g from FlatPoset X into FlatPoset Y such that for every element x of X, $f(x) \in Y$ and $f(x) = \mathrm{BaseFunc02}(x, f(E_1(x)), g(E_2(x)), I_1, J_1, D)$ and $g(x) \in Y$ and $g(x) = \mathrm{BaseFunc02}(x, f(E_1(x)), g(E_2(x)), I_2, J_2, D)$. PROOF: Consider f, g being sets such that f, $g \in \mathrm{ConFuncs}(\mathrm{FlatPoset}\,X, \mathrm{FlatPoset}\,Y)$ and $f = \mathrm{RecFunc02}(f, g, E_1, E_2, I_1, J_1, D)$ and $g = \mathrm{RecFunc02}(f, g, E_1, E_2, I_2, J_2, D)$. Consider l being a function from X into $\mathbb{N}$ such that for every element x_0 of X, if $l(x_0) \leqslant 0$, then $x_0 \in D$ and if $x_0 \notin D$, then $l(E_1(x_0)) < l(x_0)$ and $l(E_2(x_0)) < l(x_0)$. Define $\mathcal{P}[\text{natural number}] \equiv$ for every elements x_1, x_2 of X such that $l(x_1) \leqslant \$_1$ and $l(x_2) \leqslant \$_1$ holds $f(x_1) \in Y$ and $f(x_1) = \mathrm{BaseFunc02}(x_1, f(E_1(x_1)), g(E_2(x_1)), I_1, J_1, D)$ and $g(x_2) \in Y$ and $g(x_2) = \mathrm{BaseFunc02}(x_2, f(E_1(x_2)), g(E_2(x_2)), I_2, J_2, D)$. $\mathcal{P}[0]$ by [7, (5)]. For every k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [7, (5)], [3, (13)], [18, (69)]. For every natural number k, $\mathcal{P}[k]$ from [3, Sch. 2]. For every elements x_1, x_2 of X, $f(x_1) \in Y$ and $f(x_1) = \mathrm{BaseFunc02}(x_1, f(E_1(x_1)), g(E_2(x_1)), I_1, J_1, D)$ and $g(x_2) \in Y$ and $g(x_2) = \mathrm{BaseFunc02}(x_2, f(E_1(x_2)), g(E_2(x_2)), I_2, J_2, D)$ by [3, (11)]. $\square$

(26) There exist functions f, g from X into Y such that for every element x of X, if $x \in D$, then $f(x) = I_1(x)$ and $g(x) = I_2(x)$ and if $x \notin D$, then $f(x) = J_1(\langle x, f(E_1(x)), g(E_2(x))\rangle)$ and $g(x) = J_2(\langle x, f(E_1(x)), g(E_2(x))\rangle)$.

Now we state the propositions:

(27) Let us consider functions f_1, g_1, f_2, g_2 from X into Y. Suppose

(i) (E_1,E_2) is well founded with minimal set D, and

(ii) for every element x of X, if $x \in D$, then $f_1(x) = I_1(x)$ and $g_1(x) = I_2(x)$ and if $x \notin D$, then $f_1(x) = J_1(\langle x, f_1(E_1(x)), g_1(E_2(x))\rangle)$ and $g_1(x) = J_2(\langle x, f_1(E_1(x)), g_1(E_2(x))\rangle)$, and

(iii) for every element x of X, if $x \in D$, then $f_2(x) = I_1(x)$ and $g_2(x) = I_2(x)$ and if $x \notin D$, then $f_2(x) = J_1(\langle x, f_2(E_1(x)), g_2(E_2(x))\rangle)$ and $g_2(x) = J_2(\langle x, f_2(E_1(x)), g_2(E_2(x))\rangle)$.

Then

(iv) $f_1 = f_2$, and

(v) $g_1 = g_2$.

PROOF: Consider l being a function from X into $\mathbb{N}$ such that for every element x of X, if $l(x) \leqslant 0$, then $x \in D$ and if $x \notin D$, then $l(E_1(x)) < l(x)$ and $l(E_2(x)) < l(x)$. Define $\mathcal{P}[\text{natural number}] \equiv$ for every element x of X such that $l(x) \leqslant \$_1$ holds $f_1(x) = f_2(x)$ and $g_1(x) = g_2(x)$. $\mathcal{P}[0]$. For every k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [3, (13)]. For every natural number

k, $\mathcal{P}[k]$ from [3, Sch. 2]. For every element x of X, $f_1(x) = f_2(x)$ and $g_1(x) = g_2(x)$. □

(28) Suppose (E_1,E_2) is well founded with minimal set D. Then there exists a function f from X into Y such that for every element x of X, if $x \in D$, then $f(x) = I(x)$ and if $x \notin D$, then $f(x) = J(\langle x, f(E_1(x)), f(E_2(x))\rangle)$. The theorem is a consequence of (26).

(29) Let us consider functions f_1, f_2 from X into Y. Suppose

(i) (E_1,E_2) is well founded with minimal set D, and

(ii) for every element x of X, if $x \in D$, then $f_1(x) = I(x)$ and if $x \notin D$, then $f_1(x) = J(\langle x, f_1(E_1(x)), f_1(E_2(x))\rangle)$, and

(iii) for every element x of X, if $x \in D$, then $f_2(x) = I(x)$ and if $x \notin D$, then $f_2(x) = J(\langle x, f_2(E_1(x)), f_2(E_2(x))\rangle)$.

Then $f_1 = f_2$. PROOF: Consider l being a function from X into $\mathbb{N}$ such that for every element x of X, if $l(x) \leqslant 0$, then $x \in D$ and if $x \notin D$, then $l(E_1(x)) < l(x)$ and $l(E_2(x)) < l(x)$. Define $\mathcal{P}[\text{natural number}] \equiv$ for every element x of X such that $l(x) \leqslant \$_1$ holds $f_1(x) = f_2(x)$. $\mathcal{P}[0]$. For every k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [3, (13)]. For every natural number k, $\mathcal{P}[k]$ from [3, Sch. 2]. For every element x of X, $f_1(x) = f_2(x)$. □

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