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Submodule of free $\mathbb{Z}$-module[1]

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Summary. In this article, we formalize a free $\mathbb{Z}$-module and its property. In particular, we formalize the vector space of rational field corresponding to a free $\mathbb{Z}$-module and prove formally that submodules of a free $\mathbb{Z}$-module are free. $\mathbb{Z}$-module is necassary for lattice problems - LLL (Lenstra, Lenstra and Lovász) base reduction algorithm and cryptographic systems with lattice [20]. Some theorems in this article are described by translating theorems in [11] into theorems of $\mathbb{Z}$-module, however their proofs are different.

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The notation and terminology used in this paper have been introduced in the following articles: [6], [1], [24], [22], [5], [12], [7], [8], [16], [25], [19], [23], [21], [3], [4], [9], [17], [30], [32], [31], [26], [29], [18], [27], [28], [33], [10], [13], [14], and [15].

1. Vector Space of Rational Field Generated by a Free $\mathbb{Z}$-module

From now on V denotes a $\mathbb{Z}$-module and W, W_1, W_2 denote submodules of V.

Let us consider a $\mathbb{Z}$-module V, submodules W_1, W_2 of V, and submodules W_5, W_6 of $W_1 + W_2$. Now we state the propositions:

(1) If $W_5 = W_1$ and $W_6 = W_2$, then $W_1 + W_2 = W_5 + W_6$.

(2) If $W_5 = W_1$ and $W_6 = W_2$, then $W_1 \cap W_2 = W_5 \cap W_6$.

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Let V be a $\mathbb{Z}$-module. Note that (the carrier of V) $\times$ ($\mathbb{Z} \setminus \{0\}$) is non empty.

Assume V is cancelable on multiplication. The functor EQRZM(V) yielding an equivalence relation of (the carrier of V) $\times$ ($\mathbb{Z} \setminus \{0\}$) is defined by

(Def. 1) Let us consider elements S, T. Then $\langle S, T\rangle \in it$ if and only if $S, T \in$ (the carrier of V) $\times$ ($\mathbb{Z} \setminus \{0\}$) and there exist elements z_1, z_2 of V and there exist integers i_1, i_2 such that $S = \langle z_1, i_1\rangle$ and $T = \langle z_2, i_2\rangle$ and $i_1 \neq 0$ and $i_2 \neq 0$ and $i_2 \cdot z_1 = i_1 \cdot z_2$.

Now we state the proposition:

(3) Let us consider a $\mathbb{Z}$-module V, elements z_1, z_2 of V, and integers i_1, i_2. Suppose V is cancelable on multiplication. Then $\langle\langle z_1, i_1\rangle, \langle z_2, i_2\rangle\rangle \in$ EQRZM(V) if and only if $i_1 \neq 0$ and $i_2 \neq 0$ and $i_2 \cdot z_1 = i_1 \cdot z_2$.

Let V be a $\mathbb{Z}$-module. Assume V is cancelable on multiplication. The functor addCoset V yielding a binary operation on Classes EQRZM(V) is defined by

(Def. 2) Let us consider elements A, B. Suppose $A, B \in$ Classes EQRZM(V). Let us consider elements z_1, z_2 of V and integers i_1, i_2. Suppose

- (i) $i_1 \neq 0$, and
- (ii) $i_2 \neq 0$, and
- (iii) $A = [\langle z_1, i_1\rangle]_{\mathrm{EQRZM}(V)}$, and
- (iv) $B = [\langle z_2, i_2\rangle]_{\mathrm{EQRZM}(V)}$.

Then $it(A, B) = [\langle i_2 \cdot z_1 + i_1 \cdot z_2, i_1 \cdot i_2\rangle]_{\mathrm{EQRZM}(V)}$.

Assume V is cancelable on multiplication. The functor zeroCoset V yielding an element of Classes EQRZM(V) is defined by

(Def. 3) Let us consider an integer i. Suppose $i \neq 0$. Then $it = [\langle 0_V, i\rangle]_{\mathrm{EQRZM}(V)}$.

Assume V is cancelable on multiplication. The functor lmultCoset V yielding a function from (the carrier of $\mathbb{F}_{\mathbb{Q}}$) $\times$ Classes EQRZM(V) into Classes EQRZM(V) is defined by

(Def. 4) Let us consider an element q and an element A. Suppose

- (i) $q \in \mathbb{Q}$, and
- (ii) $A \in$ Classes EQRZM(V).

Let us consider integers m, n, i and an element z of V. Suppose

- (iii) $n \neq 0$, and
- (iv) $q = \frac{m}{n}$, and
- (v) $i \neq 0$, and
- (vi) $A = [\langle z, i\rangle]_{\mathrm{EQRZM}(V)}$.

Then $it(q, A) = [\langle m \cdot z, n \cdot i\rangle]_{\mathrm{EQRZM}(V)}$.

Now we state the propositions:

(4) Let us consider a $\mathbb{Z}$-module V, an element z of V, and integers i, n. Suppose

(i) $i \neq 0$, and

(ii) $n \neq 0$, and

(iii) V is cancelable on multiplication.

Then $[\langle z, i\rangle]_{\mathrm{EQRZM}(V)} = [\langle n \cdot z, n \cdot i\rangle]_{\mathrm{EQRZM}(V)}$. The theorem is a consequence of (3).

(5) Let us consider a $\mathbb{Z}$-module V and an element v of $\langle \mathrm{Classes\,EQRZM}(V), \mathrm{addCoset}\,V, \mathrm{zeroCoset}\,V, \mathrm{lmultCoset}\,V\rangle$. Suppose V is cancelable on multiplication. Then there exists an integer i and there exists an element z of V such that $i \neq 0$ and $v = [\langle z, i\rangle]_{\mathrm{EQRZM}(V)}$.

Let V be a $\mathbb{Z}$-module. Assume V is cancelable on multiplication. The functor $\mathrm{ZMQVectSp}(V)$ yielding a vector space over $\mathbb{F}_{\mathbb{Q}}$ is defined by the term

(Def. 5) $\langle \mathrm{Classes\,EQRZM}(V), \mathrm{addCoset}\,V, \mathrm{zeroCoset}\,V, \mathrm{lmultCoset}\,V\rangle$.

Assume V is cancelable on multiplication. The functor $\mathrm{MorphsZQ}(V)$ yielding a function from V into $\mathrm{ZMQVectSp}(V)$ is defined by

(Def. 6) (i) *it* is one-to-one, and

(ii) for every element v of V, $it(v) = [\langle v, 1\rangle]_{\mathrm{EQRZM}(V)}$, and

(iii) for every elements v, w of V, $it(v + w) = it(v) + it(w)$, and

(iv) for every element v of V and for every integer i and for every element q of $\mathbb{F}_{\mathbb{Q}}$ such that $i = q$ holds $it(i \cdot v) = q \cdot it(v)$, and

(v) $it(0_V) = 0_{\mathrm{ZMQVectSp}(V)}$.

Now we state the propositions:

(6) Let us consider a $\mathbb{Z}$-module V. Suppose V is cancelable on multiplication. Let us consider a finite sequence s of elements of V and a finite sequence t of elements of $\mathrm{ZMQVectSp}(V)$. Suppose

(i) $\mathrm{len}\,s = \mathrm{len}\,t$, and

(ii) for every element i of $\mathbb{N}$ such that $i \in \mathrm{dom}\,s$ there exists a vector s_1 of V such that $s_1 = s(i)$ and $t(i) = (\mathrm{MorphsZQ}(V))(s_1)$.

Then $\sum t = (\mathrm{MorphsZQ}(V))(\sum s)$. Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite sequence s of elements of V for every finite sequence t of elements of $\mathrm{ZMQVectSp}(V)$ such that $\mathrm{len}\,s = \$_1$ and $\mathrm{len}\,s = \mathrm{len}\,t$ and for every element i of $\mathbb{N}$ such that $i \in \mathrm{dom}\,s$ there exists a vector s_1 of V such that $s_1 = s(i)$ and $t(i) = (\mathrm{MorphsZQ}(V))(s_1)$ holds $\sum t = (\mathrm{MorphsZQ}(V))(\sum s)$. $\mathcal{P}[0]$ by [26, (43)]. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [5, (59)], [3, (11)], [5, (4)]. For every natural number k, $\mathcal{P}[k]$ from [3, Sch. 2]. $\square$

(7) Let us consider a $\mathbb{Z}$-module V, a subset I of V, a subset I_6 of ZMQVectSp (V), a z linear combination l of I, and a linear combination l_5 of I_6. Suppose

(i) V is cancelable on multiplication, and

(ii) $I_6 = (\text{MorphsZQ}(V))^\circ I$, and

(iii) $l = l_5 \cdot \text{MorphsZQ}(V)$.

Then $\sum l_5 = (\text{MorphsZQ}(V))(\sum l)$. The theorem is a consequence of (6).

(8) Let us consider a $\mathbb{Z}$-module V, a subset I_6 of ZMQVectSp(V), and a linear combination l_5 of I_6. Then there exists an integer m and there exists an element a of $\mathbb{F}_{\mathbb{Q}}$ such that $m \neq 0$ and $m = a$ and $\text{rng}(a \cdot l_5) \subseteq \mathbb{Z}$. PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every linear combination l_5 of I_6 such that $\overline{\overline{\text{the support of } l_5}} = \$_1$ there exists an integer m and there exists an element a of $\mathbb{F}_{\mathbb{Q}}$ such that $m \neq 0$ and $m = a$ and $\text{rng}(a \cdot l_5) \subseteq \mathbb{Z}$. $\mathcal{P}[0]$ by [27, (28)], [8, (113)], [27, (3)]. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [2, (44)], [10, (31)], [2, (42)]. For every natural number n, $\mathcal{P}[n]$ from [3, Sch. 2]. □

(9) Let us consider a $\mathbb{Z}$-module V, a subset I of V, a subset I_6 of ZMQVectSp(V), and a linear combination l_5 of I_6. Suppose

(i) V is cancelable on multiplication, and

(ii) $I_6 = (\text{MorphsZQ}(V))^\circ I$.

Then there exists an integer m and there exists an element a of $\mathbb{F}_{\mathbb{Q}}$ and there exists a z linear combination l of I such that $m \neq 0$ and $m = a$ and $l = (a \cdot l_5) \cdot \text{MorphsZQ}(V)$ and $(\text{MorphsZQ}(V))^{-1}$(the support of l_5) = the support of l. The theorem is a consequence of (8). PROOF: Consider m being an integer, a being an element of $\mathbb{F}_{\mathbb{Q}}$ such that $m \neq 0$ and $m = a$ and $\text{rng}(a \cdot l_5) \subseteq \mathbb{Z}$. Reconsider $l = (a \cdot l_5) \cdot \text{MorphsZQ}(V)$ as an element of $\mathbb{Z}^{\text{the carrier of } V}$. Set $T = \{v$, where v is an element of $V : l(v) \neq 0\}$. Set $F = \text{MorphsZQ}(V)$. $T \subseteq F^{-1}$(the support of l_5) by [7, (13)], [8, (38)]. F^{-1}(the support of l_5) $\subseteq T$ by [8, (38)], [7, (13)]. □

(10) Let us consider a $\mathbb{Z}$-module V, a subset I of V, a subset I_6 of ZMQVectSp(V), a linear combination l_5 of I_6, an integer m, an element a of $\mathbb{F}_{\mathbb{Q}}$, and a z linear combination l of I. Suppose

(i) V is cancelable on multiplication, and

(ii) $I_6 = (\text{MorphsZQ}(V))^\circ I$, and

(iii) $m \neq 0$, and

(iv) $m = a$, and

(v) $l = (a \cdot l_5) \cdot \text{MorphsZQ}(V)$.

Then $a \cdot \sum l_5 = (\text{MorphsZQ}(V))(\sum l)$. The theorem is a consequence of (7).

(11) Let us consider a $\mathbb{Z}$-module V, a subset I of V, and a subset I_6 of ZMQVectSp(V). Suppose

(i) V is cancelable on multiplication, and

(ii) $I_6 = (\text{MorphsZQ}(V))^\circ I$, and

(iii) I is linearly independent.

Then I_6 is linearly independent. The theorem is a consequence of (9) and (10).

(12) Let us consider a $\mathbb{Z}$-module V, a subset I of V, a z linear combination l of I, and a subset I_6 of ZMQVectSp(V). Suppose

(i) V is cancelable on multiplication, and

(ii) $I_6 = (\text{MorphsZQ}(V))^\circ I$.

Then there exists a linear combination l_5 of I_6 such that

(iii) $l = l_5 \cdot \text{MorphsZQ}(V)$, and

(iv) the support of $l_5 = (\text{MorphsZQ}(V))^\circ$the support of l.

PROOF: Reconsider $I_0 =$ the support of l as a finite subset of V. Reconsider $I_7 = (\text{MorphsZQ}(V))^\circ I_0$ as a finite subset of ZMQVectSp(V). Define $\mathcal{P}[\text{element}, \text{element}] \equiv \$_1 \in I_7$ and there exists an element v of V such that $v \in I_0$ and $\$_1 = (\text{MorphsZQ}(V))(v)$ and $\$_2 = l(v)$ or $\$_1 \notin I_7$ and $\$_2 = 0_{\mathbb{F}_\mathbb{Q}}$. For every element x such that $x \in$ the carrier of ZMQVectSp(V) there exists an element y such that $y \in \mathbb{Q}$ and $\mathcal{P}[x, y]$ by [8, (64)] . Consider l_5 being a function from the carrier of ZMQVectSp(V) into $\mathbb{Q}$ such that for every element x such that $x \in$ the carrier of ZMQVectSp(V) holds $\mathcal{P}[x, l_5(x)]$ from [8, Sch. 1]. The support of $l_5 \subseteq I_7$. For every element x such that $x \in \operatorname{dom} l$ holds $l(x) = (l_5 \cdot \text{MorphsZQ}(V))(x)$ by [8, (35), (19)], [7, (12)]. $I_7 \subseteq$ the support of l_5 by [8, (64)], [7, (12)], [14, (8)]. □

(13) Let us consider a free $\mathbb{Z}$-module V, a subset I of V, a subset I_6 of ZMQVectSp(V), a z linear combination l of I, and an integer i. Suppose

(i) $i \neq 0$, and

(ii) $I_6 = (\text{MorphsZQ}(V))^\circ I$.

Then $[\langle \sum l, i \rangle]_{\text{EQRZM}(V)} \in \text{Lin}(I_6)$. The theorem is a consequence of (12) and (7).

Let us consider a free $\mathbb{Z}$-module V, a subset I of V, and a subset I_6 of ZMQVectSp(V). Now we state the propositions:

(14) If $I_6 = (\text{MorphsZQ}(V))^\circ I$, then $\overline{\overline{I}} = \overline{\overline{I_6}}$.

(15) If $I_6 = (\text{MorphsZQ}(V))^\circ I$ and I is a basis of V, then I_6 is a basis of ZMQVectSp(V).

Let V be a finite-rank free $\mathbb{Z}$-module. Note that ZMQVectSp(V) is finite dimensional.

Now we state the propositions:

(16) Let us consider a finite-rank free $\mathbb{Z}$-module V. Then $\operatorname{rank} V = \dim(\text{ZMQVectSp}(V))$. The theorem is a consequence of (15) and (14).

(17) Let us consider a free $\mathbb{Z}$-module V and finite subsets I, A of V. Suppose

(i) I is a basis of V, and

(ii) $\overline{\overline{I}} + 1 = \overline{\overline{A}}$.

Then A is linearly dependent. The theorem is a consequence of (15), (11), and (14).

(18) Let us consider a free $\mathbb{Z}$-module V and subsets A, B of V. If A is linearly dependent and $A \subseteq B$, then B is linearly dependent.

(19) Let us consider a free $\mathbb{Z}$-module V and subsets D, A of V. Suppose

(i) D is basis of V and finite, and

(ii) $\overline{\overline{D}} \subset \overline{\overline{A}}$.

Then A is linearly dependent. The theorem is a consequence of (17) and (18).

(20) Let us consider a free $\mathbb{Z}$-module V and subsets I, A of V. Suppose

(i) I is basis of V and finite, and

(ii) A is linearly independent.

Then $\overline{\overline{A}} \subseteq \overline{\overline{I}}$.

2. Submodule of Free $\mathbb{Z}$-module

Now we state the proposition:

(21) Let us consider a $\mathbb{Z}$-module V. If Ω_V is free, then V is free.

Let us consider a $\mathbb{Z}$-module V, submodules W_1, W_2 of V, and strict submodules W_3, W_4 of V. Now we state the propositions:

(22) If $W_3 = \Omega_{W_1}$ and $W_4 = \Omega_{W_2}$, then $W_3 + W_4 = W_1 + W_2$.

(23) If $W_3 = \Omega_{W_1}$ and $W_4 = \Omega_{W_2}$, then $W_3 \cap W_4 = W_1 \cap W_2$.

Now we state the propositions:

(24) Let us consider a $\mathbb{Z}$-module V and a strict submodule W of V. Suppose $W \neq \mathbf{0}_V$. Then there exists a vector v of V such that

(i) $v \in W$, and

(ii) $v \neq 0_V$.

(25) Let us consider a subset A of V and z linear combinations l_1, l_2 of A. Suppose (the support of l_1) $\cap$ (the support of l_2) $= \emptyset$. Then the support of $l_1 + l_2 =$ (the support of l_1) $\cup$ (the support of l_2). Proof: (The support of l_1) $\cup$ (the support of l_2) $\subseteq$ the support of $l_1 + l_2$ by [14, (8)]. □

(26) Let us consider subsets A_1, A_2 of V and a z linear combination l of $A_1 \cup A_2$. Suppose $A_1 \cap A_2 = \emptyset$. Then there exists a z linear combination l_1 of A_1 and there exists a z linear combination l_2 of A_2 such that $l = l_1 + l_2$. Proof: Define $\mathcal{P}[\text{element}, \text{element}] \equiv$ if $\$_1$ is a vector of V, then $\$_1 \in A_1$ and $\$_2 = l(\$_1)$ or $\$_1 \notin A_1$ and $\$_2 = 0$. For every element x such that $x \in$ the carrier of V there exists an element y such that $y \in \mathbb{Z}$ and $\mathcal{P}[x, y]$. There exists a function l_1 from the carrier of V into $\mathbb{Z}$ such that for every element x such that $x \in$ the carrier of V holds $\mathcal{P}[x, l_1(x)]$ from [8, Sch. 1]. Consider l_1 being a function from the carrier of V into $\mathbb{Z}$ such that for every element x such that $x \in$ the carrier of V holds $\mathcal{P}[x, l_1(x)]$. For every element x such that $x \in$ the support of l_1 holds $x \in A_1$ by [14, (8)]. Define $\mathcal{Q}[\text{element}, \text{element}] \equiv$ if $\$_1$ is a vector of V, then $\$_1 \in A_2$ and $\$_2 = l(\$_1)$ or $\$_1 \notin A_2$ and $\$_2 = 0$. For every element x such that $x \in$ the carrier of V there exists an element y such that $y \in \mathbb{Z}$ and $\mathcal{Q}[x, y]$. There exists a function l_2 from the carrier of V into $\mathbb{Z}$ such that for every element x such that $x \in$ the carrier of V holds $\mathcal{Q}[x, l_2(x)]$ from [8, Sch. 1]. Consider l_2 being a function from the carrier of V into $\mathbb{Z}$ such that for every element x such that $x \in$ the carrier of V holds $\mathcal{Q}[x, l_2(x)]$. For every element x such that $x \in$ the support of l_2 holds $x \in A_2$ by [14, (8)]. For every vector v of V, $l(v) = (l_1 + l_2)(v)$. □

(27) Let us consider a $\mathbb{Z}$-module V, free submodules W_1, W_2 of V, a basis I_1 of W_1, and a basis I_2 of W_2. If V is the direct sum of W_1 and W_2, then $I_1 \cap I_2 = \emptyset$.

Let us consider a $\mathbb{Z}$-module V, free submodules W_1, W_2 of V, a basis I_1 of W_1, a basis I_2 of W_2, and a subset I of V. Now we state the propositions:

(28) If V is the direct sum of W_1 and W_2 and $I = I_1 \cup I_2$, then $\mathrm{Lin}(I) =$ the $\mathbb{Z}$-module structure of V.

(29) If V is the direct sum of W_1 and W_2 and $I = I_1 \cup I_2$, then I is linearly independent.

Let us consider a $\mathbb{Z}$-module V and free submodules W_1, W_2 of V. Now we state the propositions:

(30) If V is the direct sum of W_1 and W_2, then V is free.

(31) If $W_1 \cap W_2 = \mathbf{0}_V$, then $W_1 + W_2$ is free.

Let us consider a free $\mathbb{Z}$-module V, a basis I of V, and a vector v of V. Now we state the propositions:

(32) If $v \in I$, then $\mathrm{Lin}(I \setminus \{v\})$ is free and $\mathrm{Lin}(\{v\})$ is free.

(33) If $v \in I$, then V is the direct sum of $\mathrm{Lin}(I \setminus \{v\})$ and $\mathrm{Lin}(\{v\})$.

Let V be a finite-rank free $\mathbb{Z}$-module. One can verify that every submodule of V is free.

Now we state the propositions:

(34) Let us consider a $\mathbb{Z}$-module V, a submodule W of V, and free submodules W_1, W_2 of V. Suppose

(i) $W_1 \cap W_2 = \mathbf{0}_V$, and

(ii) the $\mathbb{Z}$-module structure of $W = W_1 + W_2$.

Then W is free. The theorem is a consequence of (31).

(35) Let us consider a prime number p and a free $\mathbb{Z}$-module V. If $\mathrm{Z_MQ_VectSp}(V, p)$ is finite dimensional, then V is finite-rank.

(36) Let us consider a prime number p, a $\mathbb{Z}$-module V, an element s of V, an integer a, and an element b of $\mathrm{GF}(p)$. Suppose $b = a \bmod p$. Then $b \cdot \mathrm{ZMtoMQV}(V, p, s) = \mathrm{ZMtoMQV}(V, p, a \cdot s)$.

(37) Let us consider a prime number p, a free $\mathbb{Z}$-module V, a subset I of V, a subset I_6 of $\mathrm{Z_MQ_VectSp}(V, p)$, and a z linear combination l of I. Suppose $I_6 = \{\mathrm{ZMtoMQV}(V, p, u),$ where u is a vector of $V : u \in I\}$. Then $\mathrm{ZMtoMQV}(V, p, \sum l) \in \mathrm{Lin}(I_6)$.

(38) Let us consider a prime number p, a free $\mathbb{Z}$-module V, a subset I of V, and a subset I_6 of $\mathrm{Z_MQ_VectSp}(V, p)$. Suppose

(i) $\mathrm{Lin}(I) =$ the $\mathbb{Z}$-module structure of V, and

(ii) $I_6 = \{\mathrm{ZMtoMQV}(V, p, u),$ where u is a vector of $V : u \in I\}$.

Then $\mathrm{Lin}(I_6) =$ the vector space structure of $\mathrm{Z_MQ_VectSp}(V, p)$. The theorem is a consequence of (37). PROOF: For every element v_3 of $\mathrm{Z_MQ_VectSp}(V, p)$, $v_3 \in \mathrm{Lin}(I_6)$ by [15, (22)], [14, (64)]. □

(39) Let us consider a finitely-generated free $\mathbb{Z}$-module V. Then there exists a finite subset A of V such that A is a basis of V. The theorem is a consequence of (38). PROOF: Set $p =$ the prime number. Consider B being a finite subset of V such that $\mathrm{Lin}(B) =$ the $\mathbb{Z}$-module structure of V. Set $B_1 = \{\mathrm{ZMtoMQV}(V, p, u),$ where u is a vector of $V : u \in B\}$. Define $\mathcal{F}(\text{element of } V) = \mathrm{ZMtoMQV}(V, p, \$_1)$. Consider f being a function from the carrier of V into $\mathrm{Z_MQ_VectSp}(V, p)$ such that for every element x of V, $f(x) = \mathcal{F}(x)$ from [8, Sch. 4]. For every element y such that $y \in B_1$ there exists an element x such that $x \in \mathrm{dom}(f \restriction B)$ and $y = (f \restriction B)(x)$ by [30, (62)], [7, (47)]. Consider I_6 being a basis of $\mathrm{Z_MQ_VectSp}(V, p)$ such that $I_6 \subseteq B_1$. □

One can verify that every finitely-generated free $\mathbb{Z}$-module is finite-rank and every finite-rank free $\mathbb{Z}$-module is finitely-generated.

Now we state the proposition:

(40) Let us consider a finite-rank free $\mathbb{Z}$-module V and a subset A of V. If A is linearly independent, then A is finite. The theorem is a consequence of (19).

Let V be a $\mathbb{Z}$-module and W_1, W_2 be finite-rank free submodules of V. One can check that $W_1 \cap W_2$ is free.

Note that $W_1 \cap W_2$ is finite-rank.

Let V be a finite-rank free $\mathbb{Z}$-module. Note that every submodule of V is finite-rank.

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