

# Double Sequences and Limits<sup>1</sup>

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**Summary.** Double sequences are important extension of the ordinary notion of a sequence. In this article we formalized three types of limits of double sequences and the theory of these limits.

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The notation and terminology used in this paper have been introduced in the following articles: [3], [4], [13], [5], [15], [6], [7], [16], [10], [1], [2], [8], [11], [18], [12], [17], and [9].

In this paper  $R$ ,  $R_1$ ,  $R_2$  denote functions from  $\mathbb{N} \times \mathbb{N}$  into  $\mathbb{R}$ ,  $r_1$ ,  $r_2$  denote convergent sequences of real numbers,  $n$ ,  $m$ ,  $N$ ,  $M$  denote natural numbers, and  $e$ ,  $r$  denote real numbers.

Let us consider  $R$ . We say that  $R$  is  $p$ -convergent if and only if

(Def. 1) There exists a real number  $p$  such that for every real number  $e$  such that  $0 < e$  there exists a natural number  $N$  such that for every natural numbers  $n$ ,  $m$  such that  $n \geq N$  and  $m \geq N$  holds  $|R(n, m) - p| < e$ .

Assume  $R$  is  $p$ -convergent. The functor  $P\text{-lim } R$  yielding a real number is defined by

(Def. 2) Let us consider a real number  $e$ . Suppose  $0 < e$ . Then there exists a natural number  $N$  such that for every natural numbers  $n$ ,  $m$  such that  $n \geq N$  and  $m \geq N$  holds  $|R(n, m) - it| < e$ .

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We say that  $R$  is convergent in the first coordinate if and only if

(Def. 3) Let us consider an element  $m$  of  $\mathbb{N}$ . Then  $\text{curry}'(R, m)$  is convergent.

We say that  $R$  is convergent in the second coordinate if and only if

(Def. 4) Let us consider an element  $n$  of  $\mathbb{N}$ . Then  $\text{curry}(R, n)$  is convergent.

The lim in the first coordinate of  $R$  yielding a function from  $\mathbb{N}$  into  $\mathbb{R}$  is defined by

(Def. 5) Let us consider an element  $m$  of  $\mathbb{N}$ . Then  $it(m) = \lim \text{curry}'(R, m)$ .

The lim in the second coordinate of  $R$  yielding a function from  $\mathbb{N}$  into  $\mathbb{R}$  is defined by

(Def. 6) Let us consider an element  $n$  of  $\mathbb{N}$ . Then  $it(n) = \lim \text{curry}(R, n)$ .

Assume the lim in the first coordinate of  $R$  is convergent. The first coordinate major iterated lim of  $R$  yielding a real number is defined by

(Def. 7) Let us consider a real number  $e$ . Suppose  $0 < e$ . Then there exists a natural number  $M$  such that for every natural number  $m$  such that  $m \geq M$  holds  $|(the\ lim\ in\ the\ first\ coordinate\ of\ R)(m) - it| < e$ .

Assume the lim in the second coordinate of  $R$  is convergent. The second coordinate major iterated lim of  $R$  yielding a real number is defined by

(Def. 8) Let us consider a real number  $e$ . Suppose  $0 < e$ . Then there exists a natural number  $N$  such that for every natural number  $n$  such that  $n \geq N$  holds  $|(the\ lim\ in\ the\ second\ coordinate\ of\ R)(n) - it| < e$ .

Let  $R$  be a function from  $\mathbb{N} \times \mathbb{N}$  into  $\mathbb{R}$ . We say that  $R$  is uniformly convergent in the first coordinate if and only if

(Def. 9) (i)  $R$  is convergent in the first coordinate, and  
(ii) for every real number  $e$  such that  $e > 0$  there exists a natural number  $M$  such that for every natural number  $m$  such that  $m \geq M$  for every natural number  $n$ ,  $|R(n, m) - (the\ lim\ in\ the\ first\ coordinate\ of\ R)(n)| < e$ .

We say that  $R$  is uniformly convergent in the second coordinate if and only if

(Def. 10) (i)  $R$  is convergent in the second coordinate, and  
(ii) for every real number  $e$  such that  $e > 0$  there exists a natural number  $N$  such that for every natural number  $n$  such that  $n \geq N$  for every natural number  $m$ ,  $|R(n, m) - (the\ lim\ in\ the\ second\ coordinate\ of\ R)(m)| < e$ .

Let us consider  $R$ . We say that  $R$  is non-decreasing if and only if

(Def. 11) Let us consider natural numbers  $n_1, m_1, n_2, m_2$ . If  $n_1 \geq n_2$  and  $m_1 \geq m_2$ , then  $R(n_1, m_1) \geq R(n_2, m_2)$ .

We say that  $R$  is non-increasing if and only if

(Def. 12) Let us consider natural numbers  $n_1, m_1, n_2, m_2$ . If  $n_1 \geq n_2$  and  $m_1 \geq m_2$ , then  $R(n_1, m_1) \leq R(n_2, m_2)$ .

Now we state the proposition:

- (1) Let us consider real numbers  $a, b, c$ . If  $a \leq b \leq c$ , then  $|b| \leq |a|$  or  $|b| \leq |c|$ .

Note that every function from  $\mathbb{N} \times \mathbb{N}$  into  $\mathbb{R}$  which is non-decreasing and p-convergent is also lower bounded and upper bounded and every function from  $\mathbb{N} \times \mathbb{N}$  into  $\mathbb{R}$  which is non-increasing and p-convergent is also lower bounded and upper bounded.

Let  $r$  be an element of  $\mathbb{R}$ . Let us note that  $\mathbb{N} \times \mathbb{N} \mapsto r$  is p-convergent convergent in the first coordinate and convergent in the second coordinate as a function from  $\mathbb{N} \times \mathbb{N}$  into  $\mathbb{R}$ .

Now we state the proposition:

- (2) Let us consider an element  $r$  of  $\mathbb{R}$ . Then  $P\text{-lim}(\mathbb{N} \times \mathbb{N} \mapsto r) = r$ . PROOF: Set  $R = \mathbb{N} \times \mathbb{N} \mapsto r$ . For every natural numbers  $n, m$ ,  $R(n, m) = r$  by [15, (70)].  $\square$

Note that there exists a function from  $\mathbb{N} \times \mathbb{N}$  into  $\mathbb{R}$  which is p-convergent, convergent in the first coordinate, and convergent in the second coordinate.

In this paper  $P_1$  denotes a p-convergent function from  $\mathbb{N} \times \mathbb{N}$  into  $\mathbb{R}$ .

Let  $P_4$  be a p-convergent convergent in the second coordinate function from  $\mathbb{N} \times \mathbb{N}$  into  $\mathbb{R}$ . Note that the lim in the second coordinate of  $P_4$  is convergent.

Now we state the proposition:

- (3) Suppose  $R$  is p-convergent and convergent in the second coordinate. Then  $P\text{-lim } R =$  the second coordinate major iterated lim of  $R$ . PROOF: Consider  $z$  being a real number such that for every  $e$  such that  $0 < e$  there exists a natural number  $N_1$  such that for every  $n$  and  $m$  such that  $n \geq N_1$  and  $m \geq N_1$  holds  $|R(n, m) - z| < e$ . For every  $e$  such that  $0 < e$  there exists  $N$  such that for every  $n$  such that  $n \geq N$  holds  $|(\text{the lim in the second coordinate of } R)(n) - z| < e$  by [4, (63), (60)]. For every  $e$  such that  $0 < e$  there exists  $N$  such that for every  $n$  such that  $n \geq N$  holds  $|(\text{the lim in the second coordinate of } R)(n) - P\text{-lim } R| < e$  by [4, (60), (63)].  $\square$

Let  $P_3$  be a p-convergent convergent in the first coordinate function from  $\mathbb{N} \times \mathbb{N}$  into  $\mathbb{R}$ . Let us note that the lim in the first coordinate of  $P_3$  is convergent.

Now we state the proposition:

- (4) Suppose  $R$  is p-convergent and convergent in the first coordinate. Then  $P\text{-lim } R =$  the first coordinate major iterated lim of  $R$ . PROOF: Consider  $z$  being a real number such that for every  $e$  such that  $0 < e$  there exists a natural number  $N_1$  such that for every  $n$  and  $m$  such that  $n \geq N_1$  and  $m \geq N_1$  holds  $|R(n, m) - z| < e$ . For every  $e$  such that  $0 < e$  there exists  $N$  such that for every  $n$  such that  $n \geq N$  holds  $|(\text{the lim in the first coordinate of } R)(n) - z| < e$  by [4, (63), (60)]. For every  $e$  such that  $0 < e$

there exists  $N$  such that for every  $n$  such that  $n \geq N$  holds |(the lim in the first coordinate of  $R$ )( $n$ ) - P-lim  $R$ |  $< e$  by [4, (60), (63)].  $\square$

One can verify that every function from  $\mathbb{N} \times \mathbb{N}$  into  $\mathbb{R}$  which is non-decreasing and upper bounded is also p-convergent convergent in the first coordinate and convergent in the second coordinate and every function from  $\mathbb{N} \times \mathbb{N}$  into  $\mathbb{R}$  which is non-increasing and lower bounded is also p-convergent convergent in the first coordinate and convergent in the second coordinate.

Now we state the propositions:

- (5) Suppose  $R$  is uniformly convergent in the first coordinate and the lim in the first coordinate of  $R$  is convergent. Then
  - (i)  $R$  is p-convergent, and
  - (ii) P-lim  $R$  = the first coordinate major iterated lim of  $R$ .
- (6) Suppose  $R$  is uniformly convergent in the second coordinate and the lim in the second coordinate of  $R$  is convergent. Then
  - (i)  $R$  is p-convergent, and
  - (ii) P-lim  $R$  = the second coordinate major iterated lim of  $R$ .

Let us consider  $R$ . We say that  $R$  is Cauchy if and only if

- (Def. 13) Let us consider a real number  $e$ . Suppose  $e > 0$ . Then there exists a natural number  $N$  such that for every natural numbers  $n_1, n_2, m_1, m_2$  such that  $N \leq n_1 \leq n_2$  and  $N \leq m_1 \leq m_2$  holds  $|R(n_2, m_2) - R(n_1, m_1)| < e$ .

Now we state the propositions:

- (7)  $R$  is p-convergent if and only if  $R$  is Cauchy. PROOF: Define  $\mathcal{R}$ (element of  $\mathbb{N}$ ) =  $R(\$_1, \$_1)$ . Consider  $s_1$  being a function from  $\mathbb{N}$  into  $\mathbb{R}$  such that for every element  $n$  of  $\mathbb{N}$ ,  $s_1(n) = \mathcal{R}(n)$  from [7, Sch. 4]. Reconsider  $z = \lim s_1$  as a complex number. For every  $e$  such that  $0 < e$  there exists  $N$  such that for every  $n$  and  $m$  such that  $n \geq N$  and  $m \geq N$  holds  $|R(n, m) - z| < e$  by [4, (63)].  $\square$
- (8) Let us consider a function  $R$  from  $\mathbb{N} \times \mathbb{N}$  into  $\mathbb{R}$ . Suppose
  - (i)  $R$  is non-decreasing, or
  - (ii)  $R$  is non-increasing.

Then  $R$  is p-convergent if and only if  $R$  is lower bounded and upper bounded.

Let  $X, Y$  be non empty sets,  $H$  be a binary operation on  $Y$ , and  $f, g$  be functions from  $X$  into  $Y$ . Observe that the functor  $H_{f,g}$  yields a function from  $X \times X$  into  $Y$ . Now we state the propositions:

- (9) (i)  $\cdot_{\mathbb{R}_{r_1, r_2}}$  is convergent in the first coordinate and convergent in the second coordinate, and
  - (ii) the lim in the first coordinate of  $\cdot_{\mathbb{R}_{r_1, r_2}}$  is convergent, and

- (iii) the first coordinate major iterated lim of  $\cdot_{\mathbb{R} r_1, r_2} = \lim r_1 \cdot \lim r_2$ , and
- (iv) the lim in the second coordinate of  $\cdot_{\mathbb{R} r_1, r_2}$  is convergent, and
- (v) the second coordinate major iterated lim of  $\cdot_{\mathbb{R} r_1, r_2} = \lim r_1 \cdot \lim r_2$ , and
- (vi)  $\cdot_{\mathbb{R} r_1, r_2}$  is p-convergent, and
- (vii) P-lim  $\cdot_{\mathbb{R} r_1, r_2} = \lim r_1 \cdot \lim r_2$ .

PROOF: Set  $R = \cdot_{\mathbb{R} r_1, r_2}$ . For every  $n$  and  $m$ ,  $R(n, m) = r_1(n) \cdot r_2(m)$  by [5, (77)]. For every element  $m$  of  $\mathbb{N}$  and for every real number  $e$  such that  $0 < e$  there exists  $N$  such that for every  $n$  such that  $n \geq N$  holds  $|(\text{curry}'(R, m))(n) - \lim r_1 \cdot r_2(m)| < e$  by [4, (47), (65), (44)]. For every element  $m$  of  $\mathbb{N}$ ,  $\text{curry}'(R, m)$  is convergent. For every element  $m$  of  $\mathbb{N}$  and for every real number  $e$  such that  $0 < e$  there exists  $N$  such that for every  $n$  such that  $n \geq N$  holds  $|(\text{curry}(R, m))(n) - r_1(m) \cdot \lim r_2| < e$  by [4, (47), (65), (44)]. For every element  $m$  of  $\mathbb{N}$ ,  $\text{curry}(R, m)$  is convergent. For every  $e$  such that  $0 < e$  there exists  $N$  such that for every  $n$  such that  $n \geq N$  holds  $|(\text{the lim in the first coordinate of } R)(n) - \lim r_1 \cdot \lim r_2| < e$  by [4, (46), (65)]. For every  $e$  such that  $0 < e$  there exists  $N$  such that for every  $n$  such that  $n \geq N$  holds  $|(\text{the lim in the second coordinate of } R)(n) - \lim r_1 \cdot \lim r_2| < e$  by [4, (46), (65)]. For every  $e$  such that  $0 < e$  there exists  $N$  such that for every  $n$  and  $m$  such that  $n \geq N$  and  $m \geq N$  holds  $|R(n, m) - \lim r_1 \cdot \lim r_2| < e$  by [12, (3)], [4, (63), (46), (65)].  $\square$

- (10) (i)  $+_{\mathbb{R} r_1, r_2}$  is convergent in the first coordinate and convergent in the second coordinate, and
- (ii) the lim in the first coordinate of  $+_{\mathbb{R} r_1, r_2}$  is convergent, and
- (iii) the first coordinate major iterated lim of  $+_{\mathbb{R} r_1, r_2} = \lim r_1 + \lim r_2$ , and
- (iv) the lim in the second coordinate of  $+_{\mathbb{R} r_1, r_2}$  is convergent, and
- (v) the second coordinate major iterated lim of  $+_{\mathbb{R} r_1, r_2} = \lim r_1 + \lim r_2$ , and
- (vi)  $+_{\mathbb{R} r_1, r_2}$  is p-convergent, and
- (vii) P-lim  $+_{\mathbb{R} r_1, r_2} = \lim r_1 + \lim r_2$ .

PROOF: Set  $R = +_{\mathbb{R} r_1, r_2}$ . For every  $n$  and  $m$ ,  $R(n, m) = r_1(n) + r_2(m)$  by [5, (77)]. For every element  $m$  of  $\mathbb{N}$  and for every real number  $e$  such that  $0 < e$  there exists a natural number  $N$  such that for every natural number  $n$  such that  $n \geq N$  holds  $|(\text{curry}'(R, m))(n) - (\lim r_1 + r_2(m))| < e$ . For every element  $m$  of  $\mathbb{N}$ ,  $\text{curry}'(R, m)$  is convergent. For every element  $m$  of  $\mathbb{N}$  and for every real number  $e$  such that  $0 < e$  there exists  $N$  such that for every  $n$  such that  $n \geq N$  holds  $|(\text{curry}(R, m))(n) - (r_1(m) + \lim r_2)| < e$ . For every element  $m$  of  $\mathbb{N}$ ,  $\text{curry}(R, m)$  is convergent. For every  $e$  such

that  $0 < e$  there exists  $N$  such that for every  $n$  such that  $n \geq N$  holds  $|(the\ lim\ in\ the\ first\ coordinate\ of\ R)(n) - (\lim r_1 + \lim r_2)| < e$ . For every  $e$  such that  $0 < e$  there exists  $N$  such that for every  $n$  such that  $n \geq N$  holds  $|(the\ lim\ in\ the\ second\ coordinate\ of\ R)(n) - (\lim r_1 + \lim r_2)| < e$ . For every  $e$  such that  $0 < e$  there exists  $N$  such that for every  $n$  and  $m$  such that  $n \geq N$  and  $m \geq N$  holds  $|R(n, m) - (\lim r_1 + \lim r_2)| < e$  by [4, (56)].  $\square$

- (11) Suppose  $R_1$  is p-convergent and  $R_2$  is p-convergent. Then
- (i)  $R_1 + R_2$  is p-convergent, and
  - (ii)  $P\text{-lim}(R_1 + R_2) = P\text{-lim } R_1 + P\text{-lim } R_2$ .
- (12) Suppose  $R_1$  is p-convergent and  $R_2$  is p-convergent. Then
- (i)  $R_1 - R_2$  is p-convergent, and
  - (ii)  $P\text{-lim}(R_1 - R_2) = P\text{-lim } R_1 - P\text{-lim } R_2$ .
- (13) Let us consider a function  $R$  from  $\mathbb{N} \times \mathbb{N}$  into  $\mathbb{R}$  and a real number  $r$ . Suppose  $R$  is p-convergent. Then
- (i)  $r \cdot R$  is p-convergent, and
  - (ii)  $P\text{-lim}(r \cdot R) = r \cdot P\text{-lim } R$ .
- (14) If  $R$  is p-convergent and for every natural numbers  $n, m$ ,  $R(n, m) \geq r$ , then  $P\text{-lim } R \geq r$ .
- (15) Suppose  $R_1$  is p-convergent and  $R_2$  is p-convergent and for every natural numbers  $n, m$ ,  $R_1(n, m) \leq R_2(n, m)$ . Then  $P\text{-lim } R_1 \leq P\text{-lim } R_2$ . The theorem is a consequence of (12) and (14).
- (16) Suppose  $R_1$  is p-convergent and  $R_2$  is p-convergent and  $P\text{-lim } R_1 = P\text{-lim } R_2$  and for every natural numbers  $n, m$ ,  $R_1(n, m) \leq R(n, m) \leq R_2(n, m)$ . Then
- (i)  $R$  is p-convergent, and
  - (ii)  $P\text{-lim } R = P\text{-lim } R_1$ .

PROOF: For every  $e$  such that  $0 < e$  there exists  $N$  such that for every  $n$  and  $m$  such that  $n \geq N$  and  $m \geq N$  holds  $|R(n, m) - P\text{-lim } R_1| < e$  by [14, (4), (5), (1)].  $\square$

Let  $X$  be a non empty set and  $s_1$  be a function from  $\mathbb{N} \times \mathbb{N}$  into  $X$ . A subsequence of  $s_1$  is a function from  $\mathbb{N} \times \mathbb{N}$  into  $X$  and is defined by

- (Def. 14) There exist increasing sequences  $N, M$  of  $\mathbb{N}$  such that for every natural numbers  $n, m$ ,  $it(n, m) = s_1(N(n), M(m))$ .

Let us consider  $P_1$ . Observe that every subsequence of  $P_1$  is p-convergent.

Now we state the proposition:

- (17) Let us consider a subsequence  $P_2$  of  $P_1$ . Then  $P\text{-lim } P_2 = P\text{-lim } P_1$ .

Let  $R$  be a convergent in the first coordinate function from  $\mathbb{N} \times \mathbb{N}$  into  $\mathbb{R}$ . Note that every subsequence of  $R$  is convergent in the first coordinate.

Now we state the proposition:

- (18) Let us consider a subsequence  $R_1$  of  $R$ . Suppose
- (i)  $R$  is convergent in the first coordinate, and
  - (ii) the lim in the first coordinate of  $R$  is convergent.

Then

- (iii) the lim in the first coordinate of  $R_1$  is convergent, and
- (iv) the first coordinate major iterated lim of  $R_1 =$  the first coordinate major iterated lim of  $R$ .

PROOF: Consider  $I_1, I_2$  being increasing sequences of  $\mathbb{N}$  such that for every natural numbers  $n, m$ ,  $R_1(n, m) = R(I_1(n), I_2(m))$ . For every  $e$  such that  $0 < e$  there exists  $N$  such that for every  $m$  such that  $m \geq N$  holds  $|( \text{the lim in the first coordinate of } R_1 )(m) - \text{the first coordinate major iterated lim of } R| < e$ .  $\square$

Let  $R$  be a convergent in the second coordinate function from  $\mathbb{N} \times \mathbb{N}$  into  $\mathbb{R}$ . One can check that every subsequence of  $R$  is convergent in the second coordinate.

Now we state the proposition:

- (19) Let us consider a subsequence  $R_1$  of  $R$ . Suppose
- (i)  $R$  is convergent in the second coordinate, and
  - (ii) the lim in the second coordinate of  $R$  is convergent.

Then

- (iii) the lim in the second coordinate of  $R_1$  is convergent, and
- (iv) the second coordinate major iterated lim of  $R_1 =$  the second coordinate major iterated lim of  $R$ .

PROOF: Consider  $I_1, I_2$  being increasing sequences of  $\mathbb{N}$  such that for every  $n$  and  $m$ ,  $R_1(n, m) = R(I_1(n), I_2(m))$ . For every  $e$  such that  $0 < e$  there exists  $N$  such that for every  $m$  such that  $m \geq N$  holds  $|( \text{the lim in the second coordinate of } R_1 )(m) - \text{the second coordinate major iterated lim of } R| < e$ .  $\square$

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