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Isomorphisms of Direct Products of Cyclic Groups of Prime Power Order

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Summary. In this paper we formalized some theorems concerning the cyclic groups of prime power order. We formalize that every commutative cyclic group of prime power order is isomorphic to a direct product of family of cyclic groups [1], [18].

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The notation and terminology used in this paper have been introduced in the following articles: [2], [20], [6], [11], [7], [8], [24], [18], [25], [26], [27], [28], [13], [23], [16], [21], [3], [4], [15], [5], [9], [22], [17], [12], [30], [31], [14], [29], and [10].

1. Basic Properties of Cyclic Groups of Prime Power Order

Let G be a finite group. The functor $\mathrm{Ordset}(G)$ yielding a subset of $\mathbb{N}$ is defined by the term

(Def. 1) the set of all $\mathrm{ord}(a)$ where a is an element of G.

One can check that $\mathrm{Ordset}(G)$ is finite and non empty.

Now we state the propositions:

(1) Let us consider a finite group G. Then there exists an element g of G such that $\mathrm{ord}(g) = \sup \mathrm{Ordset}(G)$.

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(2) Let us consider a strict group G and a strict normal subgroup N of G. If G is commutative, then ${}^{G}/_{N}$ is commutative.

(3) Let us consider a finite group G and elements a, b of G. Then $b \in \mathrm{gr}(\{a\})$ if and only if there exists an element p of $\mathbb{N}$ such that $b = a^p$.

(4) Let us consider a finite group G, an element a of G, and elements n, p, s of $\mathbb{N}$. Suppose

(i) $\overline{\overline{\mathrm{gr}(\{a\})}} = n$, and

(ii) $n = p \cdot s$.

Then $\mathrm{ord}(a^p) = s$.

Let us consider an element k of $\mathbb{N}$, a finite group G, and an element a of G. Now we state the propositions:

(5) $\mathrm{gr}(\{a\}) = \mathrm{gr}(\{a^k\})$ if and only if $\gcd(k, \mathrm{ord}(a)) = 1$.

(6) If $\gcd(k, \mathrm{ord}(a)) = 1$, then $\mathrm{ord}(a) = \mathrm{ord}(a^k)$.

(7) $\mathrm{ord}(a) \mid k \cdot \mathrm{ord}(a^k)$.

Now we state the proposition:

(8) Let us consider a group G and elements a, b of G. Suppose $b \in \mathrm{gr}(\{a\})$. Then $\mathrm{gr}(\{b\})$ is a strict subgroup of $\mathrm{gr}(\{a\})$.

Let G be a strict commutative group and x be an element of $\mathrm{SubGr}\, G$. The functor $\mathrm{NormSp}_{\mathbb{R}}(x)$ yielding a normal strict subgroup of G is defined by the term

(Def. 2) x.

Now we state the propositions:

(9) Let us consider groups G, H, a subgroup K of H, and a homomorphism f from G to H. Then there exists a strict subgroup J of G such that the carrier of $J = f^{-1}$(the carrier of K). PROOF: Reconsider $I_3 = f^{-1}$(the carrier of K) as a non empty subset of the carrier of G. For every elements g_1, g_2 of G such that g_1, $g_2 \in I_3$ holds $g_1 \cdot g_2 \in I_3$ by [8, (38)], [25, (50)]. For every element g of G such that $g \in I_3$ holds $g^{-1} \in I_3$ by [8, (38)], [25, (51)], [28, (32)]. Consider J being a strict subgroup of G such that the carrier of $J = f^{-1}$(the carrier of K). □

(10) Let us consider a natural number p, a finite group G, and elements x, d of G. Suppose

(i) $\mathrm{ord}(d) = p$, and

(ii) p is prime, and

(iii) $x \in \mathrm{gr}(\{d\})$.

Then

(iv) $x = \mathbf{1}_G$, or

(v) $\mathrm{gr}(\{x\}) = \mathrm{gr}(\{d\})$.

The theorem is a consequence of (8). Proof: If $\mathrm{gr}(\{x\}) = \{\mathbf{1}\}_{\mathrm{gr}(\{d\})}$, then $x = \mathbf{1}_G$ by [19, (2)], [25, (44)]. □

(11) Let us consider a group G and normal subgroups H, K of G. Suppose (the carrier of H) $\cap$ (the carrier of K) $= \{\mathbf{1}_G\}$. Then (the canonical homomorphism onto cosets of H)$\restriction$(the carrier of K) is one-to-one. Proof: Set $f =$ the canonical homomorphism onto cosets of H. Set $g = f\restriction$the carrier of K. For every elements x_1, x_2 such that $x_1, x_2 \in \operatorname{dom} g$ and $g(x_1) = g(x_2)$ holds $x_1 = x_2$ by [30, (57)], [7, (49)], [25, (46), (103), (51)]. □

Let us consider finite commutative groups G, F, an element a of G, and a homomorphism f from G to F. Now we state the propositions:

(12) The carrier of $\mathrm{gr}(\{f(a)\}) = f^{\circ}$the carrier of $\mathrm{gr}(\{a\})$.

(13) $\mathrm{ord}(f(a)) \leqslant \mathrm{ord}(a)$.

(14) If f is one-to-one, then $\mathrm{ord}(f(a)) = \mathrm{ord}(a)$.

Now we state the propositions:

(15) Let us consider groups G, F, a subgroup H of G, and a homomorphism f from G to F. Then $f\restriction$the carrier of H is a homomorphism from H to F. Proof: Reconsider $g = f\restriction$the carrier of H as a function from the carrier of H into the carrier of F. For every elements a, b of H, $g(a \cdot b) = g(a) \cdot g(b)$ by [25, (40)], [7, (49)], [25, (43)]. □

(16) Let us consider finite commutative groups G, F, an element a of G, and a homomorphism f from G to F. Suppose $f\restriction$the carrier of $\mathrm{gr}(\{a\})$ is one-to-one. Then $\mathrm{ord}(f(a)) = \mathrm{ord}(a)$. The theorem is a consequence of (15) and (14).

(17) Let us consider a finite commutative group G, a prime number p, a natural number m, and an element a of G. Suppose

 (i) $\overline{\overline{G}} = p^m$, and

 (ii) $a \neq \mathbf{1}_G$.

 Then there exists a natural number n such that $\mathrm{ord}(a) = p^{n+1}$.

(18) Let us consider a prime number p and natural numbers j, m, k. If $m = p^k$ and $p \nmid j$, then $\gcd(j, m) = 1$.

2. Isomorphism of Cyclic Groups of Prime Power Order

Let us consider a strict finite commutative group G, a prime number p, and a natural number m. Now we state the propositions:

(19) Suppose $\overline{\overline{G}} = p^m$. Then there exists a normal strict subgroup K of G and there exist natural numbers n, k and there exists an element g of G such that $\mathrm{ord}(g) = \sup \mathrm{Ordset}(G)$ and K is finite and commutative and

(the carrier of K) $\cap$ (the carrier of $\operatorname{gr}(\{g\})) = \{\mathbf{1}_G\}$ and for every element x of G, there exist elements b_1, a_1 of G such that $b_1 \in K$ and $a_1 \in \operatorname{gr}(\{g\})$ and $x = b_1 \cdot a_1$ and $\operatorname{ord}(g) = p^n$ and $k = m - n$ and $n \leqslant m$ and $\overline{\overline{K}} = p^k$ and there exists a homomorphism F from $\prod\langle K, \operatorname{gr}(\{g\})\rangle$ to G such that F is bijective and for every elements a, b of G such that $a \in K$ and $b \in \operatorname{gr}(\{g\})$ holds $F(\langle a, b\rangle) = a \cdot b$.

(20) Suppose $\overline{\overline{G}} = p^m$. Then there exists a non zero natural number k and there exists a k-element finite sequence a of elements of G and there exists a k-element finite sequence I_2 of elements of $\mathbb{N}$ and there exists an associative group-like commutative multiplicative magma family F of $\operatorname{Seg} k$ and there exists a homomorphism H_1 from $\prod F$ to G such that for every natural number i such that $i \in \operatorname{Seg} k$ there exists an element a_2 of G such that $a_2 = a(i)$ and $F(i) = \operatorname{gr}(\{a_2\})$ and $\operatorname{ord}(a_2) = p^{I_2(i)}$ and for every natural number i such that $1 \leqslant i \leqslant k - 1$ holds $I_2(i) \leqslant I_2(i + 1)$ and for every elements p, q of $\operatorname{Seg} k$ such that $p \neq q$ holds (the carrier of $F(p)$) $\cap$ (the carrier of $F(q)) = \{\mathbf{1}_G\}$ and H_1 is bijective and for every (the carrier of G)-valued total $\operatorname{Seg} k$-defined function x such that for every element p of $\operatorname{Seg} k$, $x(p) \in F(p)$ holds $x \in \prod F$ and $H_1(x) = \prod x$.

(21) Suppose $\overline{\overline{G}} = p^m$. Then there exists a non zero natural number k and there exists a k-element finite sequence a of elements of G and there exists a k-element finite sequence I_2 of elements of $\mathbb{N}$ and there exists an associative group-like commutative multiplicative magma family F of $\operatorname{Seg} k$ such that for every natural number i such that $i \in \operatorname{Seg} k$ there exists an element a_2 of G such that $a_2 = a(i)$ and $F(i) = \operatorname{gr}(\{a_2\})$ and $\operatorname{ord}(a_2) = p^{I_2(i)}$ and for every natural number i such that $1 \leqslant i \leqslant k - 1$ holds $I_2(i) \leqslant I_2(i + 1)$ and for every elements p, q of $\operatorname{Seg} k$ such that $p \neq q$ holds (the carrier of $F(p)$) $\cap$ (the carrier of $F(q)) = \{\mathbf{1}_G\}$ and for every element y of G, there exists a (the carrier of G)-valued total $\operatorname{Seg} k$-defined function x such that for every element p of $\operatorname{Seg} k$, $x(p) \in F(p)$ and $y = \prod x$ and for every (the carrier of G)-valued total $\operatorname{Seg} k$-defined functions x_1, x_2 such that for every element p of $\operatorname{Seg} k$, $x_1(p) \in F(p)$ and for every element p of $\operatorname{Seg} k$, $x_2(p) \in F(p)$ and $\prod x_1 = \prod x_2$ holds $x_1 = x_2$.

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