

Riemann Integral of Functions from \mathbb{R} into Real Banach Space¹

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Summary. In this article we deal with the Riemann integral of functions from \mathbb{R} into a real Banach space. The last theorem establishes the integrability of continuous functions on the closed interval of reals. To prove the integrability we defined uniform continuity for functions from \mathbb{R} into a real normed space, and proved related theorems. We also stated some properties of finite sequences of elements of a real normed space and finite sequences of real numbers.

In addition we proved some theorems about the convergence of sequences. We applied definitions introduced in the previous article [21] to the proof of integrability.

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The notation and terminology used in this paper have been introduced in the following articles: [6], [1], [7], [22], [4], [8], [14], [9], [10], [21], [15], [16], [17], [18], [28], [26], [5], [27], [2], [23], [24], [3], [11], [19], [25], [32], [33], [30], [12], [20], [31], and [13].

1. SOME PROPERTIES OF CONTINUOUS FUNCTIONS

In this paper s_1, s_2, q_1 denote sequences of real numbers.

Let X be a real normed space and f be a partial function from \mathbb{R} to the carrier of X . We say that f is uniformly continuous if and only if

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(Def. 1) Let us consider a real number r . Suppose $0 < r$. Then there exists a real number s such that

- (i) $0 < s$, and
- (ii) for every real numbers x_1, x_2 such that $x_1, x_2 \in \text{dom } f$ and $|x_1 - x_2| < s$ holds $\|f_{x_1} - f_{x_2}\| < r$.

Now we state the propositions:

- (1) Let us consider a set X , a real normed space Y , and a partial function f from \mathbb{R} to the carrier of Y . Then $f \upharpoonright X$ is uniformly continuous if and only if for every real number r such that $0 < r$ there exists a real number s such that $0 < s$ and for every real numbers x_1, x_2 such that $x_1, x_2 \in \text{dom}(f \upharpoonright X)$ and $|x_1 - x_2| < s$ holds $\|f_{x_1} - f_{x_2}\| < r$. PROOF: If $f \upharpoonright X$ is uniformly continuous, then for every real number r such that $0 < r$ there exists a real number s such that $0 < s$ and for every real numbers x_1, x_2 such that $x_1, x_2 \in \text{dom}(f \upharpoonright X)$ and $|x_1 - x_2| < s$ holds $\|f_{x_1} - f_{x_2}\| < r$ by [11, (80)]. Consider s being a real number such that $0 < s$ and for every real numbers x_1, x_2 such that $x_1, x_2 \in \text{dom}(f \upharpoonright X)$ and $|x_1 - x_2| < s$ holds $\|f_{x_1} - f_{x_2}\| < r$. \square
- (2) Let us consider sets X, X_1 , a real normed space Y , and a partial function f from \mathbb{R} to the carrier of Y . Suppose
 - (i) $f \upharpoonright X$ is uniformly continuous, and
 - (ii) $X_1 \subseteq X$.

Then $f \upharpoonright X_1$ is uniformly continuous. The theorem is a consequence of (1).

- (3) Let us consider a real normed space X , a partial function f from \mathbb{R} to the carrier of X , and a subset Z of \mathbb{R} . Suppose
 - (i) $Z \subseteq \text{dom } f$, and
 - (ii) Z is compact, and
 - (iii) $f \upharpoonright Z$ is continuous.

Then $f \upharpoonright Z$ is uniformly continuous. The theorem is a consequence of (1).

2. SOME PROPERTIES OF SEQUENCES

Now we state the proposition:

- (4) Let us consider a real normed space X , natural numbers n, m , a function a from $\text{Seg } n \times \text{Seg } m$ into X , and finite sequences p, q of elements of X . Suppose
 - (i) $\text{dom } p = \text{Seg } n$, and

- (ii) for every natural number i such that $i \in \text{dom } p$ there exists a finite sequence r of elements of X such that $\text{dom } r = \text{Seg } m$ and $p(i) = \sum r$ and for every natural number j such that $j \in \text{dom } r$ holds $r(j) = a(i, j)$, and
- (iii) $\text{dom } q = \text{Seg } m$, and
- (iv) for every natural number j such that $j \in \text{dom } q$ there exists a finite sequence s of elements of X such that $\text{dom } s = \text{Seg } n$ and $q(j) = \sum s$ and for every natural number i such that $i \in \text{dom } s$ holds $s(i) = a(i, j)$.

Then $\sum p = \sum q$. PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every natural number m for every function a from $\text{Seg } \mathbb{N}_1 \times \text{Seg } m$ into X for every finite sequences p, q of elements of X such that $\text{dom } p = \text{Seg } \mathbb{N}_1$ and for every natural number i such that $i \in \text{dom } p$ there exists a finite sequence r of elements of X such that $\text{dom } r = \text{Seg } m$ and $p(i) = \sum r$ and for every natural number j such that $j \in \text{dom } r$ holds $r(j) = a(i, j)$ and $\text{dom } q = \text{Seg } m$ and for every natural number j such that $j \in \text{dom } q$ there exists a finite sequence s of elements of X such that $\text{dom } s = \text{Seg } \mathbb{N}_1$ and $q(j) = \sum s$ and for every natural number i such that $i \in \text{dom } s$ holds $s(i) = a(i, j)$ holds $\sum p = \sum q$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [4, (5)], [2, (11)], [13, (95)]. For every natural number n , $\mathcal{P}[n]$ from [2, Sch. 2]. \square

Let A be a subset of \mathbb{R} . The extension of $\text{vol}(A)$ yielding a real number is defined by the term

$$\text{(Def. 2)} \quad \begin{cases} 0, & \text{if } A \text{ is empty,} \\ \text{vol}(A), & \text{otherwise.} \end{cases}$$

In the sequel n denotes an element of \mathbb{N} and a, b denote real numbers.

Now we state the propositions:

- (5) Let us consider a real bounded subset A of \mathbb{R} . Then $0 \leq$ the extension of $\text{vol}(A)$.
- (6) Let us consider a non empty closed interval subset A of \mathbb{R} , a Division D of A , and a finite sequence q of elements of \mathbb{R} . Suppose
 - (i) $\text{dom } q = \text{Seg } \text{len } D$, and
 - (ii) for every natural number i such that $i \in \text{Seg } \text{len } D$ holds $q(i) = \text{vol}(\text{divset}(D, i))$.

Then $\sum q = \text{vol}(A)$. PROOF: Set $p = \text{lower_volume}(\chi_{A,A}, D)$. For every natural number k such that $k \in \text{dom } q$ holds $q(k) = p(k)$ by [15, (19)]. \square

- (7) Let us consider a real normed space Y , a point E of Y , a finite sequence q of elements of \mathbb{R} , and a finite sequence S of elements of Y . Suppose
 - (i) $\text{len } S = \text{len } q$, and

- (ii) for every natural number i such that $i \in \text{dom } S$ there exists a real number r such that $r = q(i)$ and $S(i) = r \cdot E$.

Then $\sum S = \sum q \cdot E$. PROOF: Define \mathcal{P} [natural number] \equiv for every finite sequence q of elements of \mathbb{R} for every finite sequence S of elements of Y such that $\$1 = \text{len } S$ and $\text{len } S = \text{len } q$ and for every natural number i such that $i \in \text{dom } S$ there exists a real number r such that $r = q(i)$ and $S(i) = r \cdot E$ holds $\sum S = \sum q \cdot E$. $\mathcal{P}[0]$ by [30, (10)], [12, (72)], [30, (43)]. For every natural number i , $\mathcal{P}[i]$ from [2, Sch. 2]. \square

- (8) Let us consider a non empty closed interval subset A of \mathbb{R} , a Division D of A , a non empty closed interval subset B of \mathbb{R} , and a finite sequence v of elements of \mathbb{R} . Suppose

- (i) $B \subseteq A$, and
(ii) $\text{len } D = \text{len } v$, and
(iii) for every natural number i such that
 $i \in \text{dom } v$ holds $v(i) = \text{the extension of } \text{vol}(B \cap \text{divset}(D, i))$.

Then $\sum v = \text{vol}(B)$. The theorem is a consequence of (5). PROOF: Define \mathcal{P} [natural number] \equiv for every non empty closed interval subset A of \mathbb{R} for every Division D of A for every non empty closed interval subset B of \mathbb{R} for every finite sequence v of elements of \mathbb{R} such that $\$1 = \text{len } D$ and $B \subseteq A$ and $\text{len } D = \text{len } v$ and for every natural number k such that $k \in \text{dom } v$ holds $v(k) = \text{the extension of } \text{vol}(B \cap \text{divset}(D, k))$ holds $\sum v = \text{vol}(B)$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [29, (29)], [4, (4)], [2, (11)]. For every natural number i , $\mathcal{P}[i]$ from [2, Sch. 2]. \square

- (9) Let us consider a real normed space Y , a finite sequence x_3 of elements of Y , and a finite sequence y of elements of \mathbb{R} . Suppose

- (i) $\text{len } x_3 = \text{len } y$, and
(ii) for every element i of \mathbb{N} such that $i \in \text{dom } x_3$ there exists a point v of Y such that $v = x_{3i}$ and $y(i) = \|v\|$.

Then $\|\sum x_3\| \leq \sum y$. PROOF: Define \mathcal{P} [natural number] \equiv for every finite sequence x_3 of elements of Y for every finite sequence y of elements of \mathbb{R} such that $\$1 = \text{len } x_3$ and $\text{len } x_3 = \text{len } y$ and for every element i of \mathbb{N} such that $i \in \text{dom } x_3$ there exists a point v of Y such that $v = x_{3i}$ and $y(i) = \|v\|$ holds $\|\sum x_3\| \leq \sum y$. $\mathcal{P}[0]$ by [30, (43)], [12, (72)]. For every natural number i , $\mathcal{P}[i]$ from [2, Sch. 2]. \square

- (10) Let us consider a real normed space Y , a finite sequence p of elements of Y , and a finite sequence q of elements of \mathbb{R} . Suppose

- (i) $\text{len } p = \text{len } q$, and
(ii) for every natural number j such that $j \in \text{dom } p$ holds $\|p_j\| \leq q(j)$.

Then $\|\sum p\| \leq \sum q$. The theorem is a consequence of (9). PROOF: Define $\mathcal{Q}[\text{natural number, set}] \equiv$ there exists a point v of Y such that $v = p_{\mathbb{S}_1}$ and $\mathbb{S}_2 = \|v\|$. For every natural number i such that $i \in \text{Seg len } p$ there exists an element x of \mathbb{R} such that $\mathcal{Q}[i, x]$. Consider u being a finite sequence of elements of \mathbb{R} such that $\text{dom } u = \text{Seg len } p$ and for every natural number i such that $i \in \text{Seg len } p$ holds $\mathcal{Q}[i, u(i)]$ from [4, Sch. 5]. For every element i of \mathbb{N} such that $i \in \text{dom } p$ there exists a point v of Y such that $v = p_i$ and $u(i) = \|v\|$. \square

- (11) Let us consider an element j of \mathbb{N} , a non empty closed interval subset A of \mathbb{R} , and a Division D_1 of A . Suppose $j \in \text{dom } D_1$. Then $\text{vol}(\text{divset}(D_1, j)) \leq \delta_{D_1}$.
- (12) Let us consider a non empty closed interval subset A of \mathbb{R} , a Division D of A , and a real number r . Suppose $\delta_D < r$. Let us consider a natural number i and real numbers s, t . If $i \in \text{dom } D$ and $s, t \in \text{divset}(D, i)$, then $|s - t| < r$. The theorem is a consequence of (11).
- (13) Let us consider a real Banach space X , a non empty closed interval subset A of \mathbb{R} , and a function h from A into the carrier of X . Suppose a real number r . Suppose $0 < r$. Then there exists a real number s such that

- (i) $0 < s$, and
- (ii) for every real numbers x_1, x_2 such that $x_1, x_2 \in \text{dom } h$ and $|x_1 - x_2| < s$ holds $\|h_{x_1} - h_{x_2}\| < r$.

Let us consider a division sequence T of A and a middle volume sequence S of h and T . Suppose

- (iii) δ_T is convergent, and
- (iv) $\lim \delta_T = 0$.

Then middle sum(h, S) is convergent. The theorem is a consequence of (8), (7), (4), (12), (5), (10), and (6). PROOF: For every division sequence T of A and for every middle volume sequence S of h and T such that δ_T is convergent and $\lim \delta_T = 0$ holds middle sum(h, S) is convergent by [32, (57)], [15, (9)], [17, (9)]. \square

The scheme *ExRealSeq2X* deals with a non empty set \mathcal{D} and a unary functor \mathcal{F}, \mathcal{G} yielding an element of \mathcal{D} and states that

- (Sch. 1) There exists a sequence s of \mathcal{D} such that for every natural number n , $s(2 \cdot n) = \mathcal{F}(n)$ and $s(2 \cdot n + 1) = \mathcal{G}(n)$.

Now we state the propositions:

- (14) Let us consider a natural number n . Then there exists a natural number k such that $n = 2 \cdot k$ or $n = 2 \cdot k + 1$.

- (15) Let us consider a non empty closed interval subset A of \mathbb{R} and division sequences T_2, T of A . Then there exists a division sequence T_1 of A such that for every natural number i , $T_1(2 \cdot i) = T_2(i)$ and $T_1(2 \cdot i + 1) = T(i)$. The theorem is a consequence of (14).
- (16) Let us consider a non empty closed interval subset A of \mathbb{R} and division sequences T_2, T, T_1 of A . Suppose
- (i) δ_{T_2} is convergent, and
 - (ii) $\lim \delta_{T_2} = 0$, and
 - (iii) δ_T is convergent, and
 - (iv) $\lim \delta_T = 0$, and
 - (v) for every natural number i , $T_1(2 \cdot i) = T_2(i)$ and $T_1(2 \cdot i + 1) = T(i)$.

Then

- (vi) δ_{T_1} is convergent, and
- (vii) $\lim \delta_{T_1} = 0$.

The theorem is a consequence of (14).

- (17) Let us consider a real normed space X , a non empty closed interval subset A of \mathbb{R} , a function h from A into the carrier of X , division sequences T_2, T, T_1 of A , a middle volume sequence S_7 of h and T_2 , and a middle volume sequence S of h and T . Suppose a natural number i . Then
- (i) $T_1(2 \cdot i) = T_2(i)$, and
 - (ii) $T_1(2 \cdot i + 1) = T(i)$.

Then there exists a middle volume sequence S_1 of h and T_1 such that for every natural number i , $S_1(2 \cdot i) = S_7(i)$ and $S_1(2 \cdot i + 1) = S(i)$. The theorem is a consequence of (14). PROOF: Reconsider $S_2 = S_7$, $S_3 = S$ as a sequence of (the carrier of X)^{*}. Define \mathcal{F} (natural number) = $S_{2\mathbb{S}_1}$. Define \mathcal{G} (natural number) = $S_{3\mathbb{S}_1}$. Consider S_1 being a sequence of (the carrier of X)^{*} such that for every natural number n , $S_1(2 \cdot n) = \mathcal{F}(n)$ and $S_1(2 \cdot n + 1) = \mathcal{G}(n)$ from *ExRealSeq2X*. For every element i of \mathbb{N} , $S_1(i)$ is a middle volume of h and $T_1(i)$. \square

- (18) Let us consider a real normed space X and sequences S_4, S_6, S_5 of X . Suppose
- (i) S_5 is convergent, and
 - (ii) for every natural number i , $S_5(2 \cdot i) = S_4(i)$ and $S_5(2 \cdot i + 1) = S_6(i)$.
- Then
- (iii) S_4 is convergent, and
 - (iv) $\lim S_4 = \lim S_5$, and
 - (v) S_6 is convergent, and

(vi) $\lim S_6 = \lim S_5$.

The theorem is a consequence of (14). PROOF: For every real number r such that $0 < r$ there exists a natural number m_1 such that for every natural number i such that $m_1 \leq i$ holds $\|S_4(i) - \lim S_5\| < r$ by [2, (11)]. For every real number r such that $0 < r$ there exists a natural number m_1 such that for every natural number i such that $m_1 \leq i$ holds $\|S_6(i) - \lim S_5\| < r$ by [2, (11)]. \square

- (19) Let us consider a real Banach space X and a continuous partial function f from \mathbb{R} to the carrier of X . If $a \leq b$ and $[a, b] \subseteq \text{dom } f$, then f is integrable on $[a, b]$. The theorem is a consequence of (3), (13), (15), (17), (16), and (18). PROOF: Set $A = [a, b]$. Reconsider $h = f \upharpoonright A$ as a function from A into the carrier of X . Consider T_2 being a division sequence of A such that δ_{T_2} is convergent and $\lim \delta_{T_2} = 0$. Set $S_7 =$ the middle volume sequence of h and T_2 . Set $I = \lim \text{middle sum}(h, S_7)$. For every division sequence T of A and for every middle volume sequence S of h and T such that δ_T is convergent and $\lim \delta_T = 0$ holds $\text{middle sum}(h, S)$ is convergent and $\lim \text{middle sum}(h, S) = I$. \square

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