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$\mathbb{Z}$-modules

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Summary. In this article, we formalize $\mathbb{Z}$-module, that is a module over integer ring. $\mathbb{Z}$-module is necassary for lattice problems, LLL (Lenstra-Lenstra-Lovász) base reduction algorithm and cryptographic systems with lattices [11].

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The papers [10], [17], [18], [7], [2], [9], [14], [8], [6], [13], [5], [1], [15], [4], [3], [19], [16], and [12] provide the terminology and notation for this paper.

1. Definition of $\mathbb{Z}$-module

We introduce $\mathbb{Z}$-module structures which are extensions of additive loop structure and are systems

⟨ a carrier, a zero, an addition, an external multiplication ⟩,

where the carrier is a set, the zero is an element of the carrier, the addition is a binary operation on the carrier, and the external multiplication is a function from $\mathbb{Z} \times$ the carrier into the carrier.

Let us mention that there exists a $\mathbb{Z}$-module structure which is non empty.

Let V be a $\mathbb{Z}$-module structure. A vector of V is an element of V.

In the sequel V denotes a non empty $\mathbb{Z}$-module structure and v denotes a vector of V.

Let us consider V, v and let a be an integer number. The functor $a \cdot v$ yields an element of V and is defined by:

(Def. 1) $a \cdot v =$ (the external multiplication of V)(a, v).

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Let Z_1 be a non empty set, let O be an element of Z_1, let F be a binary operation on Z_1, and let G be a function from $\mathbb{Z} \times Z_1$ into Z_1. One can verify that $\langle Z_1, O, F, G \rangle$ is non empty.

Let I_1 be a non empty $\mathbb{Z}$-module structure. We say that I_1 is vector distributive if and only if:

(Def. 2) For every a and for all vectors v, w of I_1 holds $a \cdot (v + w) = a \cdot v + a \cdot w$.

We say that I_1 is scalar distributive if and only if:

(Def. 3) For all a, b and for every vector v of I_1 holds $(a + b) \cdot v = a \cdot v + b \cdot v$.

We say that I_1 is scalar associative if and only if:

(Def. 4) For all a, b and for every vector v of I_1 holds $(a \cdot b) \cdot v = a \cdot (b \cdot v)$.

We say that I_1 is scalar unital if and only if:

(Def. 5) For every vector v of I_1 holds $1 \cdot v = v$.

The strict $\mathbb{Z}$-module structure the trivial structure of $\mathbb{Z}$-module is defined as follows:

(Def. 6) The trivial structure of $\mathbb{Z}$-module $= \langle 1, \mathrm{op}_0, \mathrm{op}_2, \pi_2(\mathbb{Z} \times 1) \rangle$.

Let us observe that the trivial structure of $\mathbb{Z}$-module is trivial and non empty.

Let us observe that there exists a non empty $\mathbb{Z}$-module structure which is strict, Abelian, add-associative, right zeroed, right complementable, scalar distributive, vector distributive, scalar associative, and scalar unital.

A $\mathbb{Z}$-module is an Abelian add-associative right zeroed right complementable scalar distributive vector distributive scalar associative scalar unital non empty $\mathbb{Z}$-module structure.

In the sequel v, w denote vectors of V.

Let I_1 be a non empty $\mathbb{Z}$-module structure. We say that I_1 inherits cancelable on multiplication if and only if:

(Def. 7) For every a and for every vector v of I_1 such that $a \cdot v = 0_{(I_1)}$ holds $a = 0$ or $v = 0_{(I_1)}$.

The following propositions are true:

(1) If $a = 0$ or $v = 0_V$, then $a \cdot v = 0_V$.

(2) $-v = (-1) \cdot v$.

(3) If V inherits cancelable on multiplication and $v = -v$, then $v = 0_V$.

(4) If V inherits cancelable on multiplication and $v + v = 0_V$, then $v = 0_V$.

(5) $a \cdot -v = (-a) \cdot v$.

(6) $a \cdot -v = -a \cdot v$.

(7) $(-a) \cdot -v = a \cdot v$.

(8) $a \cdot (v - w) = a \cdot v - a \cdot w$.

(9) $(a - b) \cdot v = a \cdot v - b \cdot v$.

(10) If V inherits cancelable on multiplication and $a \neq 0$ and $a \cdot v = a \cdot w$, then $v = w$.

(11) If V inherits cancelable on multiplication and $v \neq 0_V$ and $a \cdot v = b \cdot v$, then $a = b$.

For simplicity, we follow the rules: V is a $\mathbb{Z}$-module, u, v, w are vectors of V, F, G, H, I are finite sequences of elements of V, j, k, n are elements of $\mathbb{N}$, and f_9 is a function from $\mathbb{N}$ into the carrier of V.

Next we state several propositions:

(12) If $\operatorname{len} F = \operatorname{len} G$ and for all k, v such that $k \in \operatorname{dom} F$ and $v = G(k)$ holds $F(k) = a \cdot v$, then $\sum F = a \cdot \sum G$.

(13) For every $\mathbb{Z}$-module V and for every integer a holds $a \cdot \sum(\varepsilon_{(\text{the carrier of } V)}) = 0_V$.

(14) For every $\mathbb{Z}$-module V and for every integer a and for all vectors v, u of V holds $a \cdot \sum\langle v, u\rangle = a \cdot v + a \cdot u$.

(15) For every $\mathbb{Z}$-module V and for every integer a and for all vectors v, u, w of V holds $a \cdot \sum\langle v, u, w\rangle = a \cdot v + a \cdot u + a \cdot w$.

(16) $(-a) \cdot v = -a \cdot v$.

(17) If $\operatorname{len} F = \operatorname{len} G$ and for every k such that $k \in \operatorname{dom} F$ holds $G(k) = a \cdot F_k$, then $\sum G = a \cdot \sum F$.

2. Submodules and Cosets of Submodules in $\mathbb{Z}$-module

We use the following convention: V, X are $\mathbb{Z}$-modules, V_1, V_2, V_3 are subsets of V, and x is a set.

Let us consider V, V_1. We say that V_1 is linearly closed if and only if:

(Def. 8) For all v, u such that $v, u \in V_1$ holds $v + u \in V_1$ and for all a, v such that $v \in V_1$ holds $a \cdot v \in V_1$.

One can prove the following propositions:

(18) If $V_1 \neq \emptyset$ and V_1 is linearly closed, then $0_V \in V_1$.

(19) If V_1 is linearly closed, then for every v such that $v \in V_1$ holds $-v \in V_1$.

(20) If V_1 is linearly closed, then for all v, u such that $v, u \in V_1$ holds $v - u \in V_1$.

(21) If the carrier of $V = V_1$, then V_1 is linearly closed.

(22) If V_1 is linearly closed and V_2 is linearly closed and $V_3 = \{v + u : v \in V_1 \ \wedge\ u \in V_2\}$, then V_3 is linearly closed.

Let us consider V. Observe that $\{0_V\}$ is linearly closed.

Let us consider V. Note that there exists a subset of V which is linearly closed.

Let us consider V and let V_1, V_2 be linearly closed subsets of V. Note that $V_1 \cap V_2$ is linearly closed.

Let us consider V. A $\mathbb{Z}$-module is called a submodule of V if it satisfies the conditions (Def. 9).

(Def. 9)(i) The carrier of it $\subseteq$ the carrier of V,

(ii) $0_{\rm it} = 0_V$,

(iii) the addition of it $=$ (the addition of V) $\restriction$ (the carrier of it), and

(iv) the external multiplication of it $=$ (the external multiplication of V)$\restriction$($\mathbb{Z} \times$ the carrier of it).

In the sequel W_2 denotes a submodule of V and w, w_1, w_2 denote vectors of W.

We now state a number of propositions:

(23) If $x \in W_1$ and W_1 is a submodule of W_2, then $x \in W_2$.

(24) If $x \in W$, then $x \in V$.

(25) w is a vector of V.

(26) $0_W = 0_V$.

(27) $0_{(W_1)} = 0_{(W_2)}$.

(28) If $w_1 = v$ and $w_2 = u$, then $w_1 + w_2 = v + u$.

(29) If $w = v$, then $a \cdot w = a \cdot v$.

(30) If $w = v$, then $-v = -w$.

(31) If $w_1 = v$ and $w_2 = u$, then $w_1 - w_2 = v - u$.

(32) V is a submodule of V.

(33) $0_V \in W$.

(34) $0_{(W_1)} \in W_2$.

(35) $0_W \in V$.

(36) If u, $v \in W$, then $u + v \in W$.

(37) If $v \in W$, then $a \cdot v \in W$.

(38) If $v \in W$, then $-v \in W$.

(39) If u, $v \in W$, then $u - v \in W$.

In the sequel d_1 is an element of D, A is a binary operation on D, and M is a function from $\mathbb{Z} \times D$ into D.

We now state several propositions:

(40) Suppose $V_1 = D$ and $d_1 = 0_V$ and $A =$ (the addition of V) $\restriction$ (V_1) and $M =$ (the external multiplication of V)$\restriction$($\mathbb{Z} \times V_1$). Then $\langle D, d_1, A, M \rangle$ is a submodule of V.

(41) For all strict $\mathbb{Z}$-modules V, X such that V is a submodule of X and X is a submodule of V holds $V = X$.

(42) If V is a submodule of X and X is a submodule of Y, then V is a submodule of Y.

(43) If the carrier of $W_1 \subseteq$ the carrier of W_2, then W_1 is a submodule of W_2.

(44) If for every v such that $v \in W_1$ holds $v \in W_2$, then W_1 is a submodule of W_2.

Let us consider V. Note that there exists a submodule of V which is strict.

Next we state several propositions:

(45) For all strict submodules W_1, W_2 of V such that the carrier of W_1 = the carrier of W_2 holds $W_1 = W_2$.

(46) For all strict submodules W_1, W_2 of V such that for every v holds $v \in W_1$ iff $v \in W_2$ holds $W_1 = W_2$.

(47) Let V be a strict $\mathbb{Z}$-module and W be a strict submodule of V. If the carrier of W = the carrier of V, then $W = V$.

(48) Let V be a strict $\mathbb{Z}$-module and W be a strict submodule of V. If for every vector v of V holds $v \in W$ iff $v \in V$, then $W = V$.

(49) If the carrier of $W = V_1$, then V_1 is linearly closed.

(50) If $V_1 \neq \emptyset$ and V_1 is linearly closed, then there exists a strict submodule W of V such that V_1 = the carrier of W.

Let us consider V. The functor $\mathbf{0}_V$ yielding a strict submodule of V is defined by:

(Def. 10) The carrier of $\mathbf{0}_V = \{0_V\}$.

Let us consider V. The functor Ω_V yields a strict submodule of V and is defined by:

(Def. 11) Ω_V = the $\mathbb{Z}$-module structure of V.

We now state several propositions:

(51) $\mathbf{0}_W = \mathbf{0}_V$.

(52) $\mathbf{0}_{(W_1)} = \mathbf{0}_{(W_2)}$.

(53) $\mathbf{0}_W$ is a submodule of V.

(54) $\mathbf{0}_V$ is a submodule of W.

(55) $\mathbf{0}_{(W_1)}$ is a submodule of W_2.

(56) Every strict $\mathbb{Z}$-module V is a submodule of Ω_V.

Let us consider V, v, W. The functor $v + W$ yields a subset of V and is defined as follows:

(Def. 12) $v + W = \{v + u : u \in W\}$.

Let us consider V, W. A subset of V is called a coset of W if:

(Def. 13) There exists v such that it $= v + W$.

In the sequel B, C are cosets of W.

The following propositions are true:

(57) $0_V \in v + W$ iff $v \in W$.

(58) $v \in v + W$.

(59) $0_V + W$ = the carrier of W.

(60) $v + \mathbf{0}_V = \{v\}$.

(61) $v + \Omega_V$ = the carrier of V.

(62) $0_V \in v + W$ iff $v + W =$ the carrier of W.

(63) $v \in W$ iff $v + W =$ the carrier of W.

(64) If $v \in W$, then $a \cdot v + W =$ the carrier of W.

(65) $u \in W$ iff $v + W = v + u + W$.

(66) $u \in W$ iff $v + W = (v - u) + W$.

(67) $v \in u + W$ iff $u + W = v + W$.

(68) If $u \in v_1 + W$ and $u \in v_2 + W$, then $v_1 + W = v_2 + W$.

(69) If $v \in W$, then $a \cdot v \in v + W$.

(70) $u + v \in v + W$ iff $u \in W$.

(71) $v - u \in v + W$ iff $u \in W$.

(72) $u \in v + W$ iff there exists v_1 such that $v_1 \in W$ and $u = v + v_1$.

(73) $u \in v + W$ iff there exists v_1 such that $v_1 \in W$ and $u = v - v_1$.

(74) There exists v such that $v_1, v_2 \in v + W$ iff $v_1 - v_2 \in W$.

(75) If $v + W = u + W$, then there exists v_1 such that $v_1 \in W$ and $v + v_1 = u$.

(76) If $v + W = u + W$, then there exists v_1 such that $v_1 \in W$ and $v - v_1 = u$.

(77) For all strict submodules W_1, W_2 of V such that $v + W_1 = v + W_2$ holds $W_1 = W_2$.

(78) For all strict submodules W_1, W_2 of V such that $v + W_1 = u + W_2$ holds $W_1 = W_2$.

(79) C is linearly closed iff $C =$ the carrier of W.

(80) For all strict submodules W_1, W_2 of V and for every coset C_1 of W_1 and for every coset C_2 of W_2 such that $C_1 = C_2$ holds $W_1 = W_2$.

(81) $\{v\}$ is a coset of $\mathbf{0}_V$.

(82) If V_1 is a coset of $\mathbf{0}_V$, then there exists v such that $V_1 = \{v\}$.

(83) The carrier of W is a coset of W.

(84) The carrier of V is a coset of Ω_V.

(85) If V_1 is a coset of Ω_V, then $V_1 =$ the carrier of V.

(86) $0_V \in C$ iff $C =$ the carrier of W.

(87) $u \in C$ iff $C = u + W$.

(88) If $u, v \in C$, then there exists v_1 such that $v_1 \in W$ and $u + v_1 = v$.

(89) If $u, v \in C$, then there exists v_1 such that $v_1 \in W$ and $u - v_1 = v$.

(90) There exists C such that $v_1, v_2 \in C$ iff $v_1 - v_2 \in W$.

(91) If $u \in B$ and $u \in C$, then $B = C$.

3. Operations on Submodules in $\mathbb{Z}$-module

For simplicity, we use the following convention: V is a $\mathbb{Z}$-module, W, W_1, W_2, W_3 are submodules of V, u, u_1, u_2, v, v_1, v_2 are vectors of V, a, a_1, a_2 are integer numbers, and X, Y, y, y_1, y_2 are sets.

Let us consider V, W_1, W_2. The functor $W_1 + W_2$ yielding a strict submodule of V is defined by:

(Def. 14) The carrier of $W_1 + W_2 = \{v + u : v \in W_1 \land u \in W_2\}$.

Let us notice that the functor $W_1 + W_2$ is commutative.

Let us consider V, W_1, W_2. The functor $W_1 \cap W_2$ yields a strict submodule of V and is defined as follows:

(Def. 15) The carrier of $W_1 \cap W_2 =$ (the carrier of W_1) $\cap$ (the carrier of W_2).

Let us observe that the functor $W_1 \cap W_2$ is commutative.

We now state a number of propositions:

(92) $x \in W_1 + W_2$ iff there exist v_1, v_2 such that $v_1 \in W_1$ and $v_2 \in W_2$ and $x = v_1 + v_2$.

(93) If $v \in W_1$ or $v \in W_2$, then $v \in W_1 + W_2$.

(94) $x \in W_1 \cap W_2$ iff $x \in W_1$ and $x \in W_2$.

(95) For every strict submodule W of V holds $W + W = W$.

(96) $W_1 + (W_2 + W_3) = (W_1 + W_2) + W_3$.

(97) W_1 is a submodule of $W_1 + W_2$.

(98) For every strict submodule W_2 of V holds W_1 is a submodule of W_2 iff $W_1 + W_2 = W_2$.

(99) For every strict submodule W of V holds $\mathbf{0}_V + W = W$.

(100) $\mathbf{0}_V + \Omega_V =$ the $\mathbb{Z}$-module structure of V.

(101) $\Omega_V + W =$ the $\mathbb{Z}$-module structure of V.

(102) For every strict $\mathbb{Z}$-module V holds $\Omega_V + \Omega_V = V$.

(103) For every strict submodule W of V holds $W \cap W = W$.

(104) $W_1 \cap (W_2 \cap W_3) = (W_1 \cap W_2) \cap W_3$.

(105) $W_1 \cap W_2$ is a submodule of W_1.

(106) For every strict submodule W_1 of V holds W_1 is a submodule of W_2 iff $W_1 \cap W_2 = W_1$.

(107) $\mathbf{0}_V \cap W = \mathbf{0}_V$.

(108) $\mathbf{0}_V \cap \Omega_V = \mathbf{0}_V$.

(109) For every strict submodule W of V holds $\Omega_V \cap W = W$.

(110) For every strict $\mathbb{Z}$-module V holds $\Omega_V \cap \Omega_V = V$.

(111) $W_1 \cap W_2$ is a submodule of $W_1 + W_2$.

(112) For every strict submodule W_2 of V holds $W_1 \cap W_2 + W_2 = W_2$.

(113) For every strict submodule W_1 of V holds $W_1 \cap (W_1 + W_2) = W_1$.

(114) $W_1 \cap W_2 + W_2 \cap W_3$ is a submodule of $W_2 \cap (W_1 + W_3)$.

(115) If W_1 is a submodule of W_2, then $W_2 \cap (W_1 + W_3) = W_1 \cap W_2 + W_2 \cap W_3$.

(116) $W_2 + W_1 \cap W_3$ is a submodule of $(W_1 + W_2) \cap (W_2 + W_3)$.

(117) If W_1 is a submodule of W_2, then $W_2 + W_1 \cap W_3 = (W_1 + W_2) \cap (W_2 + W_3)$.

(118) If W_1 is a strict submodule of W_3, then $W_1 + W_2 \cap W_3 = (W_1 + W_2) \cap W_3$.

(119) For all strict submodules W_1, W_2 of V holds $W_1 + W_2 = W_2$ iff $W_1 \cap W_2 = W_1$.

(120) For all strict submodules W_2, W_3 of V such that W_1 is a submodule of W_2 holds $W_1 + W_3$ is a submodule of $W_2 + W_3$.

(121) There exists W such that the carrier of W = (the carrier of W_1) $\cup$ (the carrier of W_2) if and only if W_1 is a submodule of W_2 or W_2 is a submodule of W_1.

Let us consider V. The functor $\mathrm{Sub}(V)$ yields a set and is defined by:

(Def. 16) For every x holds $x \in \mathrm{Sub}(V)$ iff x is a strict submodule of V.

Let us consider V. One can verify that $\mathrm{Sub}(V)$ is non empty.

We now state the proposition

(122) For every strict $\mathbb{Z}$-module V holds $V \in \mathrm{Sub}(V)$.

Let us consider V, W_1, W_2. We say that V is the direct sum of W_1 and W_2 if and only if:

(Def. 17) The $\mathbb{Z}$-module structure of $V = W_1 + W_2$ and $W_1 \cap W_2 = \mathbf{0}_V$.

Let V be a $\mathbb{Z}$-module and let W be a submodule of V. We say that W has linear complement if and only if:

(Def. 18) There exists a submodule C of V such that V is the direct sum of C and W.

Let V be a $\mathbb{Z}$-module. Observe that there exists a submodule of V which has linear complement.

Let V be a $\mathbb{Z}$-module and let W be a submodule of V. Let us assume that W has linear complement. A submodule of V is called a linear complement of W if:

(Def. 19) V is the direct sum of it and W.

One can prove the following propositions:

(123) Let V be a $\mathbb{Z}$-module and W_1, W_2 be submodules of V. Suppose V is the direct sum of W_1 and W_2. Then W_2 is a linear complement of W_1.

(124) Let V be a $\mathbb{Z}$-module, W be a submodule of V with linear complement, and L be a linear complement of W. Then V is the direct sum of L and W and the direct sum of W and L.

(125) Let V be a $\mathbb{Z}$-module, W be a submodule of V with linear complement, and L be a linear complement of W. Then $W+L =$ the $\mathbb{Z}$-module structure of V.

(126) Let V be a $\mathbb{Z}$-module, W be a submodule of V with linear complement, and L be a linear complement of W. Then $W \cap L = \mathbf{0}_V$.

(127) If V is the direct sum of W_1 and W_2, then V is the direct sum of W_2 and W_1.

(128) Let V be a $\mathbb{Z}$-module, W be a submodule of V with linear complement, and L be a linear complement of W. Then W is a linear complement of L.

(129) Every $\mathbb{Z}$-module V is the direct sum of $\mathbf{0}_V$ and Ω_V and the direct sum of Ω_V and $\mathbf{0}_V$.

(130) For every $\mathbb{Z}$-module V holds $\mathbf{0}_V$ is a linear complement of Ω_V and Ω_V is a linear complement of $\mathbf{0}_V$.

In the sequel C is a coset of W, C_1 is a coset of W_1, and C_2 is a coset of W_2.

Next we state several propositions:

(131) If C_1 meets C_2, then $C_1 \cap C_2$ is a coset of $W_1 \cap W_2$.

(132) Let V be a $\mathbb{Z}$-module and W_1, W_2 be submodules of V. Then V is the direct sum of W_1 and W_2 if and only if for every coset C_1 of W_1 and for every coset C_2 of W_2 there exists a vector v of V such that $C_1 \cap C_2 = \{v\}$.

(133) Let V be a $\mathbb{Z}$-module and W_1, W_2 be submodules of V. Then $W_1 + W_2 =$ the $\mathbb{Z}$-module structure of V if and only if for every vector v of V there exist vectors v_1, v_2 of V such that $v_1 \in W_1$ and $v_2 \in W_2$ and $v = v_1 + v_2$.

(134) If V is the direct sum of W_1 and W_2 and $v_1 + v_2 = u_1 + u_2$ and v_1, $u_1 \in W_1$ and v_2, $u_2 \in W_2$, then $v_1 = u_1$ and $v_2 = u_2$.

(135) Suppose $V = W_1 + W_2$ and there exists v such that for all v_1, v_2, u_1, u_2 such that $v_1 + v_2 = u_1 + u_2$ and v_1, $u_1 \in W_1$ and v_2, $u_2 \in W_2$ holds $v_1 = u_1$ and $v_2 = u_2$. Then V is the direct sum of W_1 and W_2.

Let us consider V, v, W_1, W_2. Let us assume that V is the direct sum of W_1 and W_2. The functor $v_{\langle W_1, W_2 \rangle}$ yields an element of (the carrier of V) $\times$ (the carrier of V) and is defined as follows:

(Def. 20) $v = (v_{\langle W_1,W_2\rangle})_{\mathbf{1}} + (v_{\langle W_1,W_2\rangle})_{\mathbf{2}}$ and $(v_{\langle W_1,W_2\rangle})_{\mathbf{1}} \in W_1$ and $(v_{\langle W_1,W_2\rangle})_{\mathbf{2}} \in W_2$.

Next we state several propositions:

(136) If V is the direct sum of W_1 and W_2, then $(v_{\langle W_1,W_2\rangle})_{\mathbf{1}} = (v_{\langle W_2,W_1\rangle})_{\mathbf{2}}$.

(137) If V is the direct sum of W_1 and W_2, then $(v_{\langle W_1,W_2\rangle})_{\mathbf{2}} = (v_{\langle W_2,W_1\rangle})_{\mathbf{1}}$.

(138) Let V be a $\mathbb{Z}$-module, W be a submodule of V with linear complement, L be a linear complement of W, v be a vector of V, and t be an element

of (the carrier of V) $\times$ (the carrier of V). If $t_{\mathbf{1}} + t_{\mathbf{2}} = v$ and $t_{\mathbf{1}} \in W$ and $t_{\mathbf{2}} \in L$, then $t = v_{\langle W,L \rangle}$.

(139) Let V be a $\mathbb{Z}$-module, W be a submodule of V with linear complement, L be a linear complement of W, and v be a vector of V. Then $(v_{\langle W,L \rangle})_{\mathbf{1}} + (v_{\langle W,L \rangle})_{\mathbf{2}} = v$.

(140) Let V be a $\mathbb{Z}$-module, W be a submodule of V with linear complement, L be a linear complement of W, and v be a vector of V. Then $(v_{\langle W,L \rangle})_{\mathbf{1}} \in W$ and $(v_{\langle W,L \rangle})_{\mathbf{2}} \in L$.

(141) Let V be a $\mathbb{Z}$-module, W be a submodule of V with linear complement, L be a linear complement of W, and v be a vector of V. Then $(v_{\langle W,L \rangle})_{\mathbf{1}} = (v_{\langle L,W \rangle})_{\mathbf{2}}$.

(142) Let V be a $\mathbb{Z}$-module, W be a submodule of V with linear complement, L be a linear complement of W, and v be a vector of V. Then $(v_{\langle W,L \rangle})_{\mathbf{2}} = (v_{\langle L,W \rangle})_{\mathbf{1}}$.

In the sequel A_1, A_2, B are elements of $\mathrm{Sub}(V)$.

Let us consider V. The functor $\mathrm{SubJoin}\,V$ yielding a binary operation on $\mathrm{Sub}(V)$ is defined by:

(Def. 21) For all A_1, A_2, W_1, W_2 such that $A_1 = W_1$ and $A_2 = W_2$ holds $(\mathrm{SubJoin}\,V)(A_1, A_2) = W_1 + W_2$.

Let us consider V. The functor $\mathrm{SubMeet}\,V$ yields a binary operation on $\mathrm{Sub}(V)$ and is defined by:

(Def. 22) For all A_1, A_2, W_1, W_2 such that $A_1 = W_1$ and $A_2 = W_2$ holds $(\mathrm{SubMeet}\,V)(A_1, A_2) = W_1 \cap W_2$.

One can prove the following proposition

(143) $\langle \mathrm{Sub}(V), \mathrm{SubJoin}\,V, \mathrm{SubMeet}\,V \rangle$ is a lattice.

Let us consider V. Note that $\langle \mathrm{Sub}(V), \mathrm{SubJoin}\,V, \mathrm{SubMeet}\,V \rangle$ is lattice-like.

We now state several propositions:

(144) For every $\mathbb{Z}$-module V holds $\langle \mathrm{Sub}(V), \mathrm{SubJoin}\,V, \mathrm{SubMeet}\,V \rangle$ is lower-bounded.

(145) For every $\mathbb{Z}$-module V holds $\langle \mathrm{Sub}(V), \mathrm{SubJoin}\,V, \mathrm{SubMeet}\,V \rangle$ is upper-bounded.

(146) For every $\mathbb{Z}$-module V holds $\langle \mathrm{Sub}(V), \mathrm{SubJoin}\,V, \mathrm{SubMeet}\,V \rangle$ is a bound lattice.

(147) For every $\mathbb{Z}$-module V holds $\langle \mathrm{Sub}(V), \mathrm{SubJoin}\,V, \mathrm{SubMeet}\,V \rangle$ is modular.

(148) Let V be a $\mathbb{Z}$-module and W_1, W_2, W_3 be strict submodules of V. If W_1 is a submodule of W_2, then $W_1 \cap W_3$ is a submodule of $W_2 \cap W_3$.

(149) Let V be a $\mathbb{Z}$-module and W be a strict submodule of V. Suppose that for every vector v of V holds $v \in W$. Then $W =$ the $\mathbb{Z}$-module structure

of V.

(150) There exists C such that $v \in C$.

4. Transformation of Abelian Group to $\mathbb{Z}$-module

Let A_3 be a non empty additive loop structure. The left integer multiplication of A_3 yielding a function from $\mathbb{Z} \times$ the carrier of A_3 into the carrier of A_3 is defined by the condition (Def. 23).

(Def. 23) Let i be an element of $\mathbb{Z}$ and a be an element of A_3. Then

(i) if $i \geq 0$, then (the left integer multiplication of A_3)(i,a) = (Nat-mult-left A_3)(i,a), and

(ii) if $i < 0$, then (the left integer multiplication of A_3)(i,a) = (Nat-mult-left A_3)$(-i,-a)$.

The following propositions are true:

(151) Let R be a non empty additive loop structure, a be an element of R, i be an element of $\mathbb{Z}$, and i_1 be an element of $\mathbb{N}$. If $i = i_1$, then (the left integer multiplication of R)$(i,a) = i_1 \cdot a$.

(152) Let R be a non empty additive loop structure, a be an element of R, and i be an element of $\mathbb{Z}$. If $i = 0$, then (the left integer multiplication of R)$(i,a) = 0_R$.

(153) Let R be an add-associative right zeroed right complementable non empty additive loop structure and i be an element of $\mathbb{N}$. Then (Nat-mult-left R)$(i,0_R) = 0_R$.

(154) Let R be an add-associative right zeroed right complementable non empty additive loop structure and i be an element of $\mathbb{Z}$. Then (the left integer multiplication of R)$(i,0_R) = 0_R$.

(155) Let R be a right zeroed non empty additive loop structure, a be an element of R, and i be an element of $\mathbb{Z}$. If $i = 1$, then (the left integer multiplication of R)$(i,a) = a$.

(156) Let R be an Abelian right zeroed add-associative right complementable non empty additive loop structure, a be an element of R, and i, j, k be elements of $\mathbb{N}$. If $i \leq j$ and $k = j - i$, then (Nat-mult-left R)(k,a) = (Nat-mult-left R)(j,a) − (Nat-mult-left R)(i,a).

(157) Let R be an Abelian right zeroed add-associative right complementable non empty additive loop structure, a be an element of R, and i be an element of $\mathbb{N}$. Then −(Nat-mult-left R)(i,a) = (Nat-mult-left R)$(i,-a)$.

(158) Let R be an Abelian right zeroed add-associative right complementable non empty additive loop structure, a be an element of R, and i, j be elements of $\mathbb{Z}$. Suppose $i \in \mathbb{N}$ and $j \notin \mathbb{N}$. Then (the left integer multipli-

cation of $R)(i+j,a) =$ (the left integer multiplication of $R)(i,a) +$ (the left integer multiplication of $R)(j,a)$.

(159) Let R be an Abelian right zeroed add-associative right complementable non empty additive loop structure, a be an element of R, and i, j be elements of $\mathbb{Z}$. Then (the left integer multiplication of $R)(i+j,a) =$ (the left integer multiplication of $R)(i,a) +$ (the left integer multiplication of $R)(j,a)$.

(160) Let R be an Abelian right zeroed add-associative right complementable non empty additive loop structure, a, b be elements of R, and i be an element of $\mathbb{N}$. Then (Nat-mult-left $R)(i,a+b) =$ (Nat-mult-left $R)(i,a) +$ (Nat-mult-left $R)(i,b)$.

(161) Let R be an Abelian right zeroed add-associative right complementable non empty additive loop structure, a, b be elements of R, and i be an element of $\mathbb{Z}$. Then (the left integer multiplication of $R)(i,a+b) =$ (the left integer multiplication of $R)(i,a) +$ (the left integer multiplication of $R)(i,b)$.

(162) Let R be an Abelian right zeroed add-associative right complementable non empty additive loop structure, a be an element of R, and i, j be elements of $\mathbb{N}$. Then (Nat-mult-left $R)(i \cdot j,a) =$ (Nat-mult-left $R)(i,$ (Nat-mult-left $R)(j,a))$.

(163) Let R be an Abelian right zeroed add-associative right complementable non empty additive loop structure, a be an element of R, and i, j be elements of $\mathbb{Z}$. Then (the left integer multiplication of $R)(i \cdot j,a) =$ (the left integer multiplication of $R)(i,$ (the left integer multiplication of $R)(j,a))$.

(164) Let A_3 be a non empty Abelian add-associative right zeroed right complementable additive loop structure. Then $\langle$the carrier of A_3, the zero of A_3, the addition of A_3, the left integer multiplication of $A_3\rangle$ is a $\mathbb{Z}$-module.

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