

# Riemann Integral of Functions from $\mathbb{R}$ into $n$ -dimensional Real Normed Space

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**Summary.** In this article, we define the Riemann integral on functions  $\mathbb{R}$  into  $n$ -dimensional real normed space and prove the linearity of this operator. As a result, the Riemann integration can be applied to the wider range. Our method refers to the [21].

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The terminology and notation used in this paper have been introduced in the following papers: [23], [24], [6], [2], [25], [8], [7], [1], [4], [3], [5], [20], [10], [14], [12], [13], [18], [22], [19], [26], [9], [11], [15], [17], and [16].

## 1. ON THE FUNCTIONS FROM $\mathbb{R}$ INTO $n$ -DIMENSIONAL REAL SPACE

For simplicity, we adopt the following convention:  $X$  denotes a set,  $n$  denotes an element of  $\mathbb{N}$ ,  $a, b, c, d, e, r, x_0$  denote real numbers,  $A$  denotes a non empty closed-interval subset of  $\mathbb{R}$ ,  $f, g, h$  denote partial functions from  $\mathbb{R}$  to  $\mathcal{R}^n$ , and  $E$  denotes an element of  $\mathcal{R}^n$ . We now state a number of propositions:

- (1) If  $a \leq c \leq b$ , then  $c \in [a, b]$  and  $[a, c] \subseteq [a, b]$  and  $[c, b] \subseteq [a, b]$ .

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- (2) If  $a \leq c \leq d \leq b$  and  $[a, b] \subseteq X$ , then  $[c, d] \subseteq X$ .
- (3) If  $a \leq b$  and  $c, d \in [a, b]$  and  $[a, b] \subseteq X$ , then  $[\min(c, d), \max(c, d)] \subseteq X$ .
- (4) If  $a \leq c \leq d \leq b$  and  $[a, b] \subseteq \text{dom } f$  and  $[a, b] \subseteq \text{dom } g$ , then  $[c, d] \subseteq \text{dom}(f + g)$ .
- (5) If  $a \leq c \leq d \leq b$  and  $[a, b] \subseteq \text{dom } f$  and  $[a, b] \subseteq \text{dom } g$ , then  $[c, d] \subseteq \text{dom}(f - g)$ .
- (6) Let  $f$  be a partial function from  $\mathbb{R}$  to  $\mathbb{R}$ . Suppose  $a \leq c \leq d \leq b$  and  $f$  is integrable on  $[a, b]$  and  $f|_{[a, b]}$  is bounded and  $[a, b] \subseteq \text{dom } f$ . Then  $r \cdot f$  is integrable on  $[c, d]$  and  $(r \cdot f)|_{[c, d]}$  is bounded.
- (7) Let  $f, g$  be partial functions from  $\mathbb{R}$  to  $\mathbb{R}$ . Suppose that  $a \leq c \leq d \leq b$  and  $f$  is integrable on  $[a, b]$  and  $g$  is integrable on  $[a, b]$  and  $f|_{[a, b]}$  is bounded and  $g|_{[a, b]}$  is bounded and  $[a, b] \subseteq \text{dom } f$  and  $[a, b] \subseteq \text{dom } g$ . Then  $f - g$  is integrable on  $[c, d]$  and  $(f - g)|_{[c, d]}$  is bounded.
- (8) Suppose  $a \leq b$  and  $f$  is integrable on  $[a, b]$  and  $f|_{[a, b]}$  is bounded and  $[a, b] \subseteq \text{dom } f$  and  $c \in [a, b]$ . Then  $f$  is integrable on  $[a, c]$  and  $f$  is integrable on  $[c, b]$  and  $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ .
- (9) Suppose  $a \leq c \leq d \leq b$  and  $f$  is integrable on  $[a, b]$  and  $f|_{[a, b]}$  is bounded and  $[a, b] \subseteq \text{dom } f$ . Then  $f$  is integrable on  $[c, d]$  and  $f|_{[c, d]}$  is bounded.
- (10) Suppose that  $a \leq c \leq d \leq b$  and  $f$  is integrable on  $[a, b]$  and  $g$  is integrable on  $[a, b]$  and  $f|_{[a, b]}$  is bounded and  $g|_{[a, b]}$  is bounded and  $[a, b] \subseteq \text{dom } f$  and  $[a, b] \subseteq \text{dom } g$ . Then  $f + g$  is integrable on  $[c, d]$  and  $(f + g)|_{[c, d]}$  is bounded.
- (11) Suppose  $a \leq c \leq d \leq b$  and  $f$  is integrable on  $[a, b]$  and  $f|_{[a, b]}$  is bounded and  $[a, b] \subseteq \text{dom } f$ . Then  $r \cdot f$  is integrable on  $[c, d]$  and  $(r \cdot f)|_{[c, d]}$  is bounded.
- (12) Suppose  $a \leq c \leq d \leq b$  and  $f$  is integrable on  $[a, b]$  and  $f|_{[a, b]}$  is bounded and  $[a, b] \subseteq \text{dom } f$ . Then  $-f$  is integrable on  $[c, d]$  and  $(-f)|_{[c, d]}$  is bounded.
- (13) Suppose that  $a \leq c \leq d \leq b$  and  $f$  is integrable on  $[a, b]$  and  $g$  is integrable on  $[a, b]$  and  $f|_{[a, b]}$  is bounded and  $g|_{[a, b]}$  is bounded and  $[a, b] \subseteq \text{dom } f$  and  $[a, b] \subseteq \text{dom } g$ . Then  $f - g$  is integrable on  $[c, d]$  and  $(f - g)|_{[c, d]}$  is bounded.
- (14) Let  $n$  be a non empty element of  $\mathbb{N}$  and  $f$  be a function from  $A$  into  $\mathcal{R}^n$ . Then  $f$  is bounded if and only if  $|f|$  is bounded.
- (15) If  $f$  is bounded and  $A \subseteq \text{dom } f$ , then  $f|_A$  is bounded.
- (16) Let  $f$  be a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$  and  $g$  be a function from  $A$  into  $\mathcal{R}^n$ . If  $f$  is bounded and  $f = g$ , then  $g$  is bounded.
- (17) For every partial function  $f$  from  $\mathbb{R}$  to  $\mathcal{R}^n$  and for every function  $g$  from

$A$  into  $\mathcal{R}^n$  such that  $f = g$  holds  $|f| = |g|$ .

(18) If  $A \subseteq \text{dom } h$ , then  $|h \upharpoonright A| = |h| \upharpoonright A$ .

(19) Let  $n$  be a non empty element of  $\mathbb{N}$  and  $h$  be a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$ . If  $A \subseteq \text{dom } h$  and  $h \upharpoonright A$  is bounded, then  $|h| \upharpoonright A$  is bounded.

(20) Let  $n$  be a non empty element of  $\mathbb{N}$  and  $h$  be a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$ . Suppose  $A \subseteq \text{dom } h$  and  $h \upharpoonright A$  is bounded and  $h$  is integrable on  $A$  and  $|h|$  is integrable on  $A$ . Then  $|\int_A h(x)dx| \leq \int_A |h|(x)dx$ .

(21) Let  $n$  be a non empty element of  $\mathbb{N}$  and  $h$  be a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$ . Suppose  $a \leq b$  and  $[a, b] \subseteq \text{dom } h$  and  $h$  is integrable on  $[a, b]$  and  $|h|$  is integrable on  $[a, b]$  and  $h \upharpoonright [a, b]$  is bounded. Then  $|\int_a^b h(x)dx| \leq \int_a^b |h|(x)dx$ .

(22) Let  $n$  be a non empty element of  $\mathbb{N}$  and  $f$  be a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$ . Suppose that  $a \leq b$  and  $f$  is integrable on  $[a, b]$  and  $|f|$  is integrable on  $[a, b]$  and  $f \upharpoonright [a, b]$  is bounded and  $[a, b] \subseteq \text{dom } f$  and  $c, d \in [a, b]$ . Then  $|f|$  is integrable on  $[\min(c, d), \max(c, d)]$  and  $|f| \upharpoonright [\min(c, d), \max(c, d)]$  is bounded and  $|\int_c^d f(x)dx| \leq \int_{\min(c, d)}^{\max(c, d)} |f|(x)dx$ .

(23) Let  $n$  be a non empty element of  $\mathbb{N}$  and  $f$  be a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$ . Suppose that  $a \leq b$  and  $c \leq d$  and  $f$  is integrable on  $[a, b]$  and  $|f|$  is integrable on  $[a, b]$  and  $f \upharpoonright [a, b]$  is bounded and  $[a, b] \subseteq \text{dom } f$  and  $c, d \in [a, b]$ . Then  $|f|$  is integrable on  $[c, d]$  and  $|f| \upharpoonright [c, d]$  is bounded and  $|\int_c^d f(x)dx| \leq \int_c^d |f|(x)dx$  and  $|\int_d^c f(x)dx| \leq \int_c^d |f|(x)dx$ .

(24) Let  $n$  be a non empty element of  $\mathbb{N}$  and  $f$  be a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$ . Suppose that  $a \leq b$  and  $c \leq d$  and  $f$  is integrable on  $[a, b]$  and  $|f|$  is integrable on  $[a, b]$  and  $f \upharpoonright [a, b]$  is bounded and  $[a, b] \subseteq \text{dom } f$  and  $c, d \in [a, b]$  and for every real number  $x$  such that  $x \in [c, d]$  holds  $|f_x| \leq e$ . Then  $|\int_c^d f(x)dx| \leq e \cdot (d - c)$  and  $|\int_d^c f(x)dx| \leq e \cdot (d - c)$ .

(25) If  $a \leq b$  and  $f$  is integrable on  $[a, b]$  and  $f \upharpoonright [a, b]$  is bounded and  $[a, b] \subseteq \text{dom } f$  and  $c, d \in [a, b]$ , then  $\int_c^d (r \cdot f)(x)dx = r \cdot \int_c^d f(x)dx$ .

(26) If  $a \leq b$  and  $f$  is integrable on  $[a, b]$  and  $f \upharpoonright [a, b]$  is bounded and  $[a, b] \subseteq \text{dom } f$  and  $c, d \in [a, b]$ , then  $\int_c^d (-f)(x)dx = -\int_c^d f(x)dx$ .

(27) Suppose that  $a \leq b$  and  $f$  is integrable on  $[a, b]$  and  $g$  is integrable on  $[a, b]$  and  $f \upharpoonright [a, b]$  is bounded and  $g \upharpoonright [a, b]$  is bounded and  $[a, b] \subseteq \text{dom } f$  and  $[a, b] \subseteq \text{dom } g$  and  $c, d \in [a, b]$ . Then  $\int_c^d (f + g)(x)dx = \int_c^d f(x)dx + \int_c^d g(x)dx$ .

(28) Suppose that  $a \leq b$  and  $f$  is integrable on  $[a, b]$  and  $g$  is integrable on  $[a, b]$  and  $f \upharpoonright [a, b]$  is bounded and  $g \upharpoonright [a, b]$  is bounded and  $[a, b] \subseteq \text{dom } f$  and  $[a, b] \subseteq \text{dom } g$  and  $c, d \in [a, b]$ . Then  $\int_c^d (f - g)(x)dx = \int_c^d f(x)dx - \int_c^d g(x)dx$ .

(29) Suppose  $a \leq b$  and  $[a, b] \subseteq \text{dom } f$  and for every real number  $x$  such that  $x \in [a, b]$  holds  $f(x) = E$ . Then  $f$  is integrable on  $[a, b]$  and  $f \upharpoonright [a, b]$  is bounded and  $\int_a^b f(x)dx = (b - a) \cdot E$ .

(30) Suppose  $a \leq b$  and for every real number  $x$  such that  $x \in [a, b]$  holds  $f(x) = E$  and  $[a, b] \subseteq \text{dom } f$  and  $c, d \in [a, b]$ . Then  $\int_c^d f(x)dx = (d - c) \cdot E$ .

(31) If  $a \leq b$  and  $f$  is integrable on  $[a, b]$  and  $f \upharpoonright [a, b]$  is bounded and  $[a, b] \subseteq \text{dom } f$  and  $c, d \in [a, b]$ , then  $\int_a^d f(x)dx = \int_a^c f(x)dx + \int_c^d f(x)dx$ .

(32) Suppose that  $a \leq b$  and  $f$  is integrable on  $[a, b]$  and  $f \upharpoonright [a, b]$  is bounded and  $[a, b] \subseteq \text{dom } f$  and  $c, d \in [a, b]$  and for every real number  $x$  such that  $x \in [\min(c, d), \max(c, d)]$  holds  $|f_x| \leq e$ . Then  $|\int_c^d f(x)dx| \leq n \cdot e \cdot |d - c|$ .

$$(33) \quad \int_b^a f(x)dx = - \int_a^b f(x)dx.$$

## 2. ON THE FUNCTIONS FROM $\mathbb{R}$ INTO $n$ -DIMENSIONAL REAL NORMED SPACE

Let  $R$  be a real normed space, let  $X$  be a non empty set, and let  $g$  be a partial function from  $X$  to  $R$ . We say that  $g$  is bounded if and only if:

(Def. 1) There exists a real number  $r$  such that for every set  $y$  such that  $y \in \text{dom } g$  holds  $\|g_y\| < r$ .

Next we state a number of propositions:

- (34) Let  $f$  be a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$  and  $g$  be a partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . If  $f = g$ , then  $f$  is bounded iff  $g$  is bounded.
- (35) Let  $X, Y$  be sets and  $f_1, f_2$  be partial functions from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $f_1|_X$  is bounded and  $f_2|_Y$  is bounded. Then  $(f_1 + f_2)|(X \cap Y)$  is bounded and  $(f_1 - f_2)|(X \cap Y)$  is bounded.
- (36) Let  $f$  be a function from  $A$  into  $\mathcal{R}^n$ ,  $g$  be a function from  $A$  into  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ ,  $D$  be a Division of  $A$ ,  $p$  be a finite sequence of elements of  $\mathcal{R}^n$ , and  $q$  be a finite sequence of elements of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $f = g$  and  $p = q$ . Then  $p$  is a middle volume of  $f$  and  $D$  if and only if  $q$  is a middle volume of  $g$  and  $D$ .
- (37) Let  $f$  be a function from  $A$  into  $\mathcal{R}^n$ ,  $g$  be a function from  $A$  into  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ ,  $D$  be a Division of  $A$ ,  $p$  be a middle volume of  $f$  and  $D$ , and  $q$  be a middle volume of  $g$  and  $D$ . If  $f = g$  and  $p = q$ , then  $\text{middle sum}(f, p) = \text{middle sum}(g, q)$ .
- (38) Let  $f$  be a function from  $A$  into  $\mathcal{R}^n$ ,  $g$  be a function from  $A$  into  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ ,  $T$  be a division sequence of  $A$ ,  $p$  be a function from  $\mathbb{N}$  into  $(\mathcal{R}^n)^*$ , and  $q$  be a function from  $\mathbb{N}$  into  $(\text{the carrier of } \langle \mathcal{E}^n, \|\cdot\| \rangle)^*$ . Suppose  $f = g$  and  $p = q$ . Then  $p$  is a middle volume sequence of  $f$  and  $T$  if and only if  $q$  is a middle volume sequence of  $g$  and  $T$ .
- (39) Let  $f$  be a function from  $A$  into  $\mathcal{R}^n$ ,  $g$  be a function from  $A$  into  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ ,  $T$  be a division sequence of  $A$ ,  $S$  be a middle volume sequence of  $f$  and  $T$ , and  $U$  be a middle volume sequence of  $g$  and  $T$ . If  $f = g$  and  $S = U$ , then  $\text{middle sum}(f, S) = \text{middle sum}(g, U)$ .
- (40) Let  $f$  be a function from  $A$  into  $\mathcal{R}^n$ ,  $g$  be a function from  $A$  into  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ ,  $I$  be an element of  $\mathcal{R}^n$ , and  $J$  be a point of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $f = g$  and  $I = J$ . Then the following statements are equivalent
- (i) for every division sequence  $T$  of  $A$  and for every middle volume sequence  $S$  of  $f$  and  $T$  such that  $\delta_T$  is convergent and  $\lim(\delta_T) = 0$  holds  $\text{middle sum}(f, S)$  is convergent and  $\lim \text{middle sum}(f, S) = I$ ,
  - (ii) for every division sequence  $T$  of  $A$  and for every middle volume sequence  $S$  of  $g$  and  $T$  such that  $\delta_T$  is convergent and  $\lim(\delta_T) = 0$  holds  $\text{middle sum}(g, S)$  is convergent and  $\lim \text{middle sum}(g, S) = J$ .
- (41) Let  $f$  be a function from  $A$  into  $\mathcal{R}^n$  and  $g$  be a function from  $A$  into  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $f = g$  and  $f$  is bounded. Then  $f$  is integrable if and only if  $g$  is integrable.
- (42) Let  $f$  be a function from  $A$  into  $\mathcal{R}^n$  and  $g$  be a function from  $A$  into  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $f = g$  and  $f$  is bounded and integrable. Then  $g$  is integrable and  $\text{integral } f = \text{integral } g$ .
- (43) Let  $f$  be a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$  and  $g$  be a partial function

from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $f = g$  and  $f \upharpoonright A$  is bounded and  $A \subseteq \text{dom } f$ . Then  $f$  is integrable on  $A$  if and only if  $g$  is integrable on  $A$ .

(44) Let  $f$  be a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$  and  $g$  be a partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $f = g$  and  $f \upharpoonright A$  is bounded and  $A \subseteq \text{dom } f$  and  $f$  is integrable on  $A$ . Then  $g$  is integrable on  $A$  and  $\int_A f(x)dx = \int_A g(x)dx$ .

(45) Let  $f$  be a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$  and  $g$  be a partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $f = g$  and  $a \leq b$  and  $f \upharpoonright [a, b]$  is bounded and  $[a, b] \subseteq \text{dom } f$  and  $f$  is integrable on  $[a, b]$ . Then  $\int_a^b f(x)dx = \int_a^b g(x)dx$ .

(46) Let  $f, g$  be partial functions from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $a \leq b$  and  $f$  is integrable on  $[a, b]$  and  $g$  is integrable on  $[a, b]$  and  $[a, b] \subseteq \text{dom } f$  and  $[a, b] \subseteq \text{dom } g$ . Then  $\int_a^b (f + g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$  and  $\int_a^b (f - g)(x)dx = \int_a^b f(x)dx - \int_a^b g(x)dx$ .

(47) For every partial function  $f$  from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  such that  $a \leq b$  and  $[a, b] \subseteq \text{dom } f$  holds  $\int_b^a f(x)dx = -\int_a^b f(x)dx$ .

(48) Let  $f$  be a partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  and  $g$  be a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$ . Suppose  $f = g$  and  $a \leq b$  and  $[a, b] \subseteq \text{dom } f$  and  $f \upharpoonright [a, b]$  is bounded and  $f$  is integrable on  $[a, b]$  and  $c, d \in [a, b]$ . Then  $\int_c^d f(x)dx = \int_c^d g(x)dx$ .

(49) Let  $f, g$  be partial functions from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose that  $a \leq b$  and  $f$  is integrable on  $[a, b]$  and  $g$  is integrable on  $[a, b]$  and  $f \upharpoonright [a, b]$  is bounded and  $g \upharpoonright [a, b]$  is bounded and  $[a, b] \subseteq \text{dom } f$  and  $[a, b] \subseteq \text{dom } g$  and  $c, d \in [a, b]$ . Then  $\int_c^d (f + g)(x)dx = \int_c^d f(x)dx + \int_c^d g(x)dx$ .

(50) Let  $f, g$  be partial functions from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose that  $a \leq b$  and  $f$  is integrable on  $[a, b]$  and  $g$  is integrable on  $[a, b]$  and  $f \upharpoonright [a, b]$  is bounded and  $g \upharpoonright [a, b]$  is bounded and  $[a, b] \subseteq \text{dom } f$  and  $[a, b] \subseteq \text{dom } g$  and  $c, d \in [a, b]$ . Then  $\int_c^d (f - g)(x)dx = \int_c^d f(x)dx - \int_c^d g(x)dx$ .

(51) Let  $E$  be a point of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  and  $f$  be a partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $a \leq b$  and  $[a, b] \subseteq \text{dom } f$  and for every real number

$x$  such that  $x \in [a, b]$  holds  $f(x) = E$ . Then  $f$  is integrable on  $[a, b]$  and  $f|_{[a, b]}$  is bounded and  $\int_a^b f(x)dx = (b - a) \cdot E$ .

(52) Let  $E$  be a point of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  and  $f$  be a partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $a \leq b$  and  $[a, b] \subseteq \text{dom } f$  and for every real number  $x$  such that  $x \in [a, b]$  holds  $f(x) = E$  and  $c, d \in [a, b]$ . Then  $\int_c^d f(x)dx = (d - c) \cdot E$ .

(53) Let  $f$  be a partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $a \leq b$  and  $f$  is integrable on  $[a, b]$  and  $f|_{[a, b]}$  is bounded and  $[a, b] \subseteq \text{dom } f$  and  $c, d \in [a, b]$ . Then  $\int_a^d f(x)dx = \int_a^c f(x)dx + \int_c^d f(x)dx$ .

(54) Let  $f$  be a partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose that  $a \leq b$  and  $f$  is integrable on  $[a, b]$  and  $f|_{[a, b]}$  is bounded and  $[a, b] \subseteq \text{dom } f$  and  $c, d \in [a, b]$  and for every real number  $x$  such that  $x \in [\min(c, d), \max(c, d)]$  holds  $\|f_x\| \leq e$ . Then  $\|\int_c^d f(x)dx\| \leq n \cdot e \cdot |d - c|$ .

### 3. FUNDAMENTAL THEOREM OF CALCULUS

The following two propositions are true:

(55)<sup>2</sup> Let  $n$  be a non empty element of  $\mathbb{N}$  and  $F, f$  be partial functions from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose that  $a \leq b$  and  $f$  is integrable on  $[a, b]$  and  $f|_{[a, b]}$  is bounded and  $[a, b] \subseteq \text{dom } f$  and  $]a, b[ \subseteq \text{dom } F$  and for every real number  $x$  such that  $x \in ]a, b[$  holds  $F(x) = \int_a^x f(x)dx$  and  $x_0 \in ]a, b[$  and  $f$  is continuous in  $x_0$ . Then  $F$  is differentiable in  $x_0$  and  $F'(x_0) = f_{x_0}$ .

(56) Let  $n$  be a non empty element of  $\mathbb{N}$  and  $f$  be a partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $a \leq b$  and  $f$  is integrable on  $[a, b]$  and  $f|_{[a, b]}$  is bounded and  $[a, b] \subseteq \text{dom } f$  and  $x_0 \in ]a, b[$  and  $f$  is continuous in  $x_0$ . Then there exists a partial function  $F$  from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  such that  $]a, b[ \subseteq \text{dom } F$  and for every real number  $x$  such that  $x \in ]a, b[$  holds  $F(x) = \int_a^x f(x)dx$  and  $F$  is differentiable in  $x_0$  and  $F'(x_0) = f_{x_0}$ .

<sup>2</sup>Fundamental Theorem of Calculus (for  $\mathcal{R}^n$ )

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