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Contracting Mapping on Normed Linear Space[1]

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Summary. In this article, we described the contracting mapping on normed linear space. Furthermore, we applied that mapping to ordinary differential equations on real normed space. Our method is based on the one presented by Schwarz [29].

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The papers [28], [3], [20], [8], [26], [32], [4], [5], [18], [16], [17], [12], [34], [30], [2], [33], [23], [15], [22], [21], [24], [19], [25], [1], [6], [10], [13], [27], [9], [38], [39], [35], [36], [11], [31], [37], [14], and [7] provide the notation and terminology for this paper.

1. The Principle of Contracting Mapping on Normed Linear Space

We use the following convention: n denotes a non empty element of $\mathbb{N}$ and a, b, r, t denote real numbers.

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Let f be a function. We say that f has unique fixpoint if and only if:

(Def. 1) There exists a set x such that x is a fixpoint of f and for every set y such that y is a fixpoint of f holds $x = y$.

Next we state two propositions:

(1) Every set x is a fixpoint of $\{\langle x, x\rangle\}$.

(2) For all sets x, y, z such that x is a fixpoint of $\{\langle y, z\rangle\}$ holds $x = y$.

Let x be a set. Observe that $\{\langle x, x\rangle\}$ has unique fixpoint.

Next we state three propositions:

(3) Let X be a real normed space and x be a point of X. If for every real number e such that $e > 0$ holds $\|x\| < e$, then $x = 0_X$.

(4) Let X be a real normed space and x, y be points of X. If for every real number e such that $e > 0$ holds $\|x - y\| < e$, then $x = y$.

(5) For all real numbers K, L, e such that $0 < K < 1$ and $0 < e$ there exists a natural number n such that $|L \cdot K^n| < e$.

Let X be a real normed space. Note that every function from X into X which is constant is also contraction.

Let X be a real Banach space. One can verify that every function from X into X which is contraction also has unique fixpoint.

One can prove the following three propositions:

(6) Let X be a real Banach space and f be a function from X into X. Suppose f is contraction. Then there exists a point x_1 of X such that $f(x_1) = x_1$ and for every point x of X such that $f(x) = x$ holds $x_1 = x$.

(7) Let X be a real Banach space and f be a function from X into X such that there exists a natural number n_0 such that f^{n_0} is contraction. Then f has unique fixpoint.

(8) Let X be a real Banach space and f be a function from X into X. Given an element n_0 of $\mathbb{N}$ such that f^{n_0} is contraction. Then there exists a point x_1 of X such that $f(x_1) = x_1$ and for every point x of X such that $f(x) = x$ holds $x_1 = x$.

2. The Real Banach Space C([a,b],X)

We now state the proposition

(9) Let X be a non empty closed interval subset of $\mathbb{R}$, Y be a real normed space, and f be a continuous partial function from $\mathbb{R}$ to Y. If $\operatorname{dom} f = X$, then f is a bounded function from X into Y.

Let X be a non empty closed interval subset of $\mathbb{R}$ and let Y be a real normed space. The continuous functions of X and Y yields a subset of the set of bounded real sequences from X into Y and is defined by the condition (Def. 2).

(Def. 2) Let x be a set. Then $x \in$ the continuous functions of X and Y if and only if there exists a continuous partial function f from $\mathbb{R}$ to Y such that $x = f$ and $\operatorname{dom} f = X$.

Let X be a non empty closed interval subset of $\mathbb{R}$ and let Y be a real normed space. Note that the continuous functions of X and Y is non empty.

Let X be a non empty closed interval subset of $\mathbb{R}$ and let Y be a real normed space. Observe that the continuous functions of X and Y is linearly closed.

Let X be a non empty closed interval subset of $\mathbb{R}$ and let Y be a real normed space. The $\mathbb{R}$-vector space of continuous functions of X and Y yielding a strict real linear space is defined by the condition (Def. 3).

(Def. 3) The $\mathbb{R}$-vector space of continuous functions of X and $Y = \langle$the continuous functions of X and Y, Zero(the continuous functions of X and Y, the set of bounded real sequences from X into Y), Add(the continuous functions of X and Y, the set of bounded real sequences from X into Y), Mult(the continuous functions of X and Y, the set of bounded real sequences from X into Y)$\rangle$.

Let X be a non empty closed interval subset of $\mathbb{R}$ and let Y be a real normed space. Observe that the $\mathbb{R}$-vector space of continuous functions of X and Y is Abelian, add-associative, right zeroed, right complementable, vector distributive, scalar distributive, scalar associative, and scalar unital.

One can prove the following three propositions:

(10) Let X be a non empty closed interval subset of $\mathbb{R}$, Y be a real normed space, f, g, h be vectors of the $\mathbb{R}$-vector space of continuous functions of X and Y, and f_9, g_9, h_9 be continuous partial functions from $\mathbb{R}$ to Y. Suppose $f_9 = f$ and $g_9 = g$ and $h_9 = h$ and $\operatorname{dom} f_9 = X$ and $\operatorname{dom} g_9 = X$ and $\operatorname{dom} h_9 = X$. Then $h = f + g$ if and only if for every element x of X holds $(h_9)_x = (f_9)_x + (g_9)_x$.

(11) Let X be a non empty closed interval subset of $\mathbb{R}$, Y be a real normed space, f, h be vectors of the $\mathbb{R}$-vector space of continuous functions of X and Y, and f_9, h_9 be continuous partial functions from $\mathbb{R}$ to Y. Suppose $f_9 = f$ and $h_9 = h$ and $\operatorname{dom} f_9 = X$ and $\operatorname{dom} h_9 = X$. Then $h = a \cdot f$ if and only if for every element x of X holds $(h_9)_x = a \cdot (f_9)_x$.

(12) Let X be a non empty closed interval subset of $\mathbb{R}$ and Y be a real normed space. Then $0_{\text{the } \mathbb{R}\text{-vector space of continuous functions of } X \text{ and } Y} = X \longmapsto 0_Y$.

Let X be a non empty closed interval subset of $\mathbb{R}$ and let Y be a real normed space. The continuous functions norm of X and Y yields a function from the continuous functions of X and Y into $\mathbb{R}$ and is defined as follows:

(Def. 4) The continuous functions norm of X and $Y = \operatorname{BdFuncsNorm}(X, Y) \restriction$the continuous functions of X and Y.

Let X be a non empty closed interval subset of $\mathbb{R}$, let Y be a real normed space, and let f be a set. Let us assume that $f \in$ the continuous functions of X and Y. The functor modetrans(f, X, Y) yielding a continuous partial function from $\mathbb{R}$ to Y is defined by:

(Def. 5) modetrans$(f, X, Y) = f$ and dom modetrans$(f, X, Y) = X$.

Let X be a non empty closed interval subset of $\mathbb{R}$ and let Y be a real normed space. The $\mathbb{R}$-norm space of continuous functions of X and Y yields a strict non empty normed structure and is defined by the condition (Def. 6).

(Def. 6) The $\mathbb{R}$-norm space of continuous functions of X and $Y = \langle$the continuous functions of X and Y, Zero(the continuous functions of X and Y, the set of bounded real sequences from X into Y), Add(the continuous functions of X and Y, the set of bounded real sequences from X into Y), Mult(the continuous functions of X and Y, the set of bounded real sequences from X into Y), the continuous functions norm of X and $Y\rangle$.

We now state several propositions:

(13) Let X be a non empty closed interval subset of $\mathbb{R}$, Y be a real normed space, and f be a continuous partial function from $\mathbb{R}$ to Y. If dom $f = X$, then modetrans$(f, X, Y) = f$.

(14) Let X be a non empty closed interval subset of $\mathbb{R}$ and Y be a real normed space. Then $X \longmapsto 0_Y = 0_{\text{the } \mathbb{R}\text{-norm space of continuous functions of } X \text{ and } Y}$.

(15) Let X be a non empty closed interval subset of $\mathbb{R}$, Y be a real normed space, f, g, h be points of the $\mathbb{R}$-norm space of continuous functions of X and Y, and f_9, g_9, h_9 be continuous partial functions from $\mathbb{R}$ to Y. Suppose $f_9 = f$ and $g_9 = g$ and $h_9 = h$ and dom $f_9 = X$ and dom $g_9 = X$ and dom $h_9 = X$. Then $h = f + g$ if and only if for every element x of X holds $(h_9)_x = (f_9)_x + (g_9)_x$.

(16) Let X be a non empty closed interval subset of $\mathbb{R}$, Y be a real normed space, f, h be points of the $\mathbb{R}$-norm space of continuous functions of X and Y, and f_9, h_9 be continuous partial functions from $\mathbb{R}$ to Y. Suppose $f_9 = f$ and $h_9 = h$ and dom $f_9 = X$ and dom $h_9 = X$. Then $h = a \cdot f$ if and only if for every element x of X holds $(h_9)_x = a \cdot (f_9)_x$.

(17) Let X be a non empty closed interval subset of $\mathbb{R}$, Y be a real normed space, f be a point of the $\mathbb{R}$-norm space of continuous functions of X and Y, and g be a point of the real normed space of bounded functions from X into Y. If $f = g$, then $\|f\| = \|g\|$.

(18) Let X be a non empty closed interval subset of $\mathbb{R}$, Y be a real normed space, f, g be points of the $\mathbb{R}$-norm space of continuous functions of X and Y, and f_1, g_1 be points of the real normed space of bounded functions from X into Y. If $f_1 = f$ and $g_1 = g$, then $f + g = f_1 + g_1$.

(19) Let X be a non empty closed interval subset of $\mathbb{R}$, Y be a real normed space, f be a point of the $\mathbb{R}$-norm space of continuous functions of X and

Y, and f_1 be a point of the real normed space of bounded functions from X into Y. If $f_1 = f$, then $a \cdot f = a \cdot f_1$.

Let X be a non empty closed interval subset of $\mathbb{R}$ and let Y be a real normed space. Observe that the $\mathbb{R}$-norm space of continuous functions of X and Y is reflexive, discernible, real normed space-like, vector distributive, scalar distributive, scalar associative, scalar unital, Abelian, add-associative, right zeroed, and right complementable.

One can prove the following propositions:

(20) Let X be a non empty closed interval subset of $\mathbb{R}$, Y be a real normed space, f, g, h be points of the $\mathbb{R}$-norm space of continuous functions of X and Y, and f_9, g_9, h_9 be continuous partial functions from $\mathbb{R}$ to Y. Suppose $f_9 = f$ and $g_9 = g$ and $h_9 = h$ and $\operatorname{dom} f_9 = X$ and $\operatorname{dom} g_9 = X$ and $\operatorname{dom} h_9 = X$. Then $h = f - g$ if and only if for every element x of X holds $(h_9)_x = (f_9)_x - (g_9)_x$.

(21) Let X be a non empty closed interval subset of $\mathbb{R}$, Y be a real normed space, f, g be points of the $\mathbb{R}$-norm space of continuous functions of X and Y, and f_1, g_1 be points of the real normed space of bounded functions from X into Y. If $f_1 = f$ and $g_1 = g$, then $f - g = f_1 - g_1$.

Let X be a non empty closed interval subset of $\mathbb{R}$ and let Y be a real normed space. Note that there exists a subset of the real normed space of bounded functions from X into Y which is closed.

The following two propositions are true:

(22) Let X be a non empty closed interval subset of $\mathbb{R}$ and Y be a real normed space. Then the continuous functions of X and Y is a closed subset of the real normed space of bounded functions from X into Y.

(23) Let X be a non empty closed interval subset of $\mathbb{R}$, Y be a real normed space, and s_1 be a sequence of the $\mathbb{R}$-norm space of continuous functions of X and Y. Suppose Y is complete and s_1 is Cauchy sequence by norm. Then s_1 is convergent.

Let X be a non empty closed interval subset of $\mathbb{R}$ and let Y be a real Banach space. One can check that the $\mathbb{R}$-norm space of continuous functions of X and Y is complete.

We now state four propositions:

(24) Let X be a non empty closed interval subset of $\mathbb{R}$, Y be a real normed space, v be a point of the $\mathbb{R}$-norm space of continuous functions of X and Y, and g be a partial function from $\mathbb{R}$ to Y. If $g = v$, then for every real number t such that $t \in X$ holds $\|g_t\| \leq \|v\|$.

(25) Let X be a non empty closed interval subset of $\mathbb{R}$, Y be a real normed space, K be a real number, v be a point of the $\mathbb{R}$-norm space of continuous functions of X and Y, and g be a partial function from $\mathbb{R}$ to Y. Suppose

$g = v$ and for every real number t such that $t \in X$ holds $\|g_t\| \leq K$. Then $\|v\| \leq K$.

(26) Let X be a non empty closed interval subset of $\mathbb{R}$, Y be a real normed space, v_1, v_2 be points of the $\mathbb{R}$-norm space of continuous functions of X and Y, and g_1, g_2 be partial functions from $\mathbb{R}$ to Y. Suppose $g_1 = v_1$ and $g_2 = v_2$. Let t be a real number. If $t \in X$, then $\|(g_1)_t - (g_2)_t\| \leq \|v_1 - v_2\|$.

(27) Let X be a non empty closed interval subset of $\mathbb{R}$, Y be a real normed space, K be a real number, v_1, v_2 be points of the $\mathbb{R}$-norm space of continuous functions of X and Y, and g_1, g_2 be partial functions from $\mathbb{R}$ to Y. Suppose $g_1 = v_1$ and $g_2 = v_2$ and for every real number t such that $t \in X$ holds $\|(g_1)_t - (g_2)_t\| \leq K$. Then $\|v_1 - v_2\| \leq K$.

3. Differential Equations

The following propositions are true:

(28) Let n, i be natural numbers, f be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$, and A be a subset of $\mathbb{R}$. Then $\mathrm{proj}(i, n) \cdot (f \restriction A) = (\mathrm{proj}(i, n) \cdot f) \restriction A$.

(29) For every continuous partial function g from $\mathbb{R}$ to $\mathcal{R}^n$ such that $\mathrm{dom}\, g = [a, b]$ holds $g \restriction [a, b]$ is bounded.

(30) For every continuous partial function g from $\mathbb{R}$ to $\mathcal{R}^n$ such that $\mathrm{dom}\, g = [a, b]$ holds g is integrable on $[a, b]$.

(31) Let f, F be partial functions from $\mathbb{R}$ to $\mathcal{R}^n$. Suppose $a \leq b$ and $\mathrm{dom}\, f = [a, b]$ and $\mathrm{dom}\, F = [a, b]$ and f is continuous and for every real number t such that $t \in [a, b]$ holds $F(t) = \int\limits_a^t f(x)dx$. Let x be a real number. If $x \in [a, b]$, then F is continuous in x.

(32) For every continuous partial function f from $\mathbb{R}$ to $\langle \mathcal{E}^n, \| \cdot \| \rangle$ such that $\mathrm{dom}\, f = [a, b]$ holds $f \restriction [a, b]$ is bounded.

(33) For every continuous partial function f from $\mathbb{R}$ to $\langle \mathcal{E}^n, \| \cdot \| \rangle$ such that $\mathrm{dom}\, f = [a, b]$ holds f is integrable on $[a, b]$.

(34) Let f be a continuous partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \| \cdot \| \rangle$ and F be a partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \| \cdot \| \rangle$. Suppose $a \leq b$ and $\mathrm{dom}\, f = [a, b]$ and $\mathrm{dom}\, F = [a, b]$ and for every real number t such that $t \in [a, b]$ holds $F(t) = \int\limits_a^t f(x)dx$. Let x be a real number. If $x \in [a, b]$, then F is continuous in x.

(35) Let R be a partial function from $\mathbb{R}$ to $\mathbb{R}$. Suppose R is total. Then R is rest-like if and only if for every real number r such that $r > 0$ there exists

a real number d such that $d > 0$ and for every real number z such that $z \neq 0$ and $|z| < d$ holds $|z|^{-1} \cdot |R_z| < r$.

In the sequel Z denotes an open subset of $\mathbb{R}$, y_0 denotes a vector of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and G denotes a function from $\langle \mathcal{E}^n, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$.

One can prove the following propositions:

(36) Let f be a continuous partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and g be a partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $a \leq b$ and $\operatorname{dom} f = [a, b]$ and $\operatorname{dom} g = [a, b]$ and $Z =]a, b[$ and for every real number t such that $t \in [a, b]$ holds $g(t) = y_0 + \int\limits_a^t f(x)dx$. Then g is continuous and $g_a = y_0$ and g is differentiable on Z and for every real number t such that $t \in Z$ holds $g'(t) = f_t$.

(37) For every natural number i and for all points y_1, y_2 of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ holds $(\operatorname{proj}(i, n))(y_1 + y_2) = (\operatorname{proj}(i, n))(y_1) + (\operatorname{proj}(i, n))(y_2)$.

(38) For every natural number i and for every point y_1 of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and for every real number r holds $(\operatorname{proj}(i, n))(r \cdot y_1) = r \cdot (\operatorname{proj}(i, n))(y_1)$.

(39) Let g be a partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, x_0 be a real number, and i be a natural number. Suppose $1 \leq i \leq n$ and g is differentiable in x_0. Then $\operatorname{proj}(i, n) \cdot g$ is differentiable in x_0 and $(\operatorname{proj}(i, n))(g'(x_0)) = (\operatorname{proj}(i, n) \cdot g)'(x_0)$.

(40) Let f be a partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and X be a set. Suppose that for every natural number i such that $1 \leq i \leq n$ holds $(\operatorname{proj}(i, n) \cdot f) \restriction X$ is constant. Then $f \restriction X$ is constant.

(41) Let f be a partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $]a, b[\subseteq \operatorname{dom} f$ and f is differentiable on $]a, b[$ and for every real number x such that $x \in]a, b[$ holds $f'(x) = 0_{\langle \mathcal{E}^n, \|\cdot\| \rangle}$. Then $f \restriction]a, b[$ is constant.

(42) Let f be a continuous partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $a < b$ and $[a, b] = \operatorname{dom} f$ and $f \restriction]a, b[$ is constant. Let x be a real number. If $x \in [a, b]$, then $f(x) = f(a)$.

(43) Let y, G_1 be continuous partial functions from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and g be a partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose that $a < b$ and $Z =]a, b[$ and $\operatorname{dom} y = [a, b]$ and $\operatorname{dom} g = [a, b]$ and $\operatorname{dom} G_1 = [a, b]$ and y is differentiable on Z and $y_a = y_0$ and for every real number t such that $t \in Z$ holds $y'(t) = (G_1)_t$ and for every real number t such that $t \in [a, b]$ holds $g(t) = y_0 + \int\limits_a^t G_1(x)dx$. Then $y = g$.

(44) Let a, b, c, d be real numbers and f be a partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose that $a \leq b$ and f is integrable on $[a, b]$ and $\|f\|$ is integrable on $[a, b]$ and $f \restriction [a, b]$ is bounded and $[a, b] \subseteq \operatorname{dom} f$ and c, $d \in [a, b]$. Then

$\|f\|$ is integrable on $[\min(c,d),\max(c,d)]$ and $\|f\|\restriction[\min(c,d),\max(c,d)]$ is bounded and $\|\int\limits_{c}^{d} f(x)dx\| \leq \int\limits_{\min(c,d)}^{\max(c,d)} \|f\|(x)dx$.

(45) Let a, b, c, d, e be real numbers and f be a partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\|\rangle$. Suppose that $a \leq b$ and $c \leq d$ and f is integrable on $[a,b]$ and $\|f\|$ is integrable on $[a,b]$ and $f\restriction[a,b]$ is bounded and $[a,b] \subseteq \operatorname{dom} f$ and c, $d \in [a,b]$ and for every real number x such that $x \in [c,d]$ holds $\|f_x\| \leq e$. Then $\|\int\limits_{c}^{d} f(x)dx\| \leq e\cdot(d-c)$ and $\|\int\limits_{d}^{c} f(x)dx\| \leq e\cdot(d-c)$.

(46) Let n be a natural number and g be a function from $\mathbb{R}$ into $\mathbb{R}$. Suppose that for every real number x holds $g(x) = (x-a)^{n+1}$. Let x be a real number. Then g is differentiable in x and $g'(x) = (n+1)\cdot(x-a)^n$.

(47) Let n be a natural number and g be a function from $\mathbb{R}$ into $\mathbb{R}$. Suppose that for every real number x holds $g(x) = \frac{(x-a)^{n+1}}{(n+1)!}$. Let x be a real number. Then g is differentiable in x and $g'(x) = \frac{(x-a)^n}{n!}$.

(48) Let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$. Suppose that $a \leq t$ and $[a,t] \subseteq \operatorname{dom} f$ and f is integrable on $[a,t]$ and $f\restriction[a,t]$ is bounded and $[a,t] \subseteq \operatorname{dom} g$ and g is integrable on $[a,t]$ and $g\restriction[a,t]$ is bounded and for every real number x such that $x \in [a,t]$ holds $f(x) \leq g(x)$. Then $\int\limits_{a}^{t} f(x)dx \leq \int\limits_{a}^{t} g(x)dx$.

Let n be a non empty element of $\mathbb{N}$, let y_0 be a vector of $\langle \mathcal{E}^n, \|\cdot\|\rangle$, let G be a function from $\langle \mathcal{E}^n, \|\cdot\|\rangle$ into $\langle \mathcal{E}^n, \|\cdot\|\rangle$, and let a, b be real numbers. Let us assume that $a \leq b$ and G is continuous on $\operatorname{dom} G$. The functor $\operatorname{Fredholm}(G,a,b,y_0)$ yielding a function from the $\mathbb{R}$-norm space of continuous functions of $[a,b]$ and $\langle \mathcal{E}^n, \|\cdot\|\rangle$ into the $\mathbb{R}$-norm space of continuous functions of $[a,b]$ and $\langle \mathcal{E}^n, \|\cdot\|\rangle$ is defined by the condition (Def. 7).

(Def. 7) Let x be a vector of the $\mathbb{R}$-norm space of continuous functions of $[a,b]$ and $\langle \mathcal{E}^n, \|\cdot\|\rangle$. Then there exist continuous partial functions f, g, G_1 from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\|\rangle$ such that $x = f$ and $(\operatorname{Fredholm}(G,a,b,y_0))(x) = g$ and $\operatorname{dom} f = [a,b]$ and $\operatorname{dom} g = [a,b]$ and $G_1 = G\cdot f$ and for every real number t such that $t \in [a,b]$ holds $g(t) = y_0 + \int\limits_{a}^{t} G_1(x)dx$.

We now state several propositions:

(49) Suppose $a \leq b$ and $0 < r$ and for all vectors y_1, y_2 of $\langle \mathcal{E}^n, \|\cdot\|\rangle$ holds $\|G_{y_1} - G_{y_2}\| \leq r\cdot\|y_1 - y_2\|$. Let u, v be vectors of the $\mathbb{R}$-norm space of continuous functions of $[a,b]$ and $\langle \mathcal{E}^n, \|\cdot\|\rangle$ and g, h be continuous partial

functions from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $g = (\mathrm{Fredholm}(G, a, b, y_0))(u)$ and $h = (\mathrm{Fredholm}(G, a, b, y_0))(v)$. Let t be a real number. If $t \in [a, b]$, then $\|g_t - h_t\| \leq r \cdot (t - a) \cdot \|u - v\|$.

(50) Suppose $a \leq b$ and $0 < r$ and for all vectors y_1, y_2 of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ holds $\|G_{y_1} - G_{y_2}\| \leq r \cdot \|y_1 - y_2\|$. Let u, v be vectors of the $\mathbb{R}$-norm space of continuous functions of $[a, b]$ and $\langle \mathcal{E}^n, \|\cdot\| \rangle$, m be an element of $\mathbb{N}$, and g, h be continuous partial functions from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $g = (\mathrm{Fredholm}(G, a, b, y_0))^{m+1}(u)$ and $h = (\mathrm{Fredholm}(G, a, b, y_0))^{m+1}(v)$. Let t be a real number. If $t \in [a, b]$, then $\|g_t - h_t\| \leq \frac{(r \cdot (t-a))^{m+1}}{(m+1)!} \cdot \|u - v\|$.

(51) Let m be a natural number. Suppose $a \leq b$ and $0 < r$ and for all vectors y_1, y_2 of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ holds $\|G_{y_1} - G_{y_2}\| \leq r \cdot \|y_1 - y_2\|$. Let u, v be vectors of the $\mathbb{R}$-norm space of continuous functions of $[a, b]$ and $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Then $\|(\mathrm{Fredholm}(G, a, b, y_0))^{m+1}(u) - (\mathrm{Fredholm}(G, a, b, y_0))^{m+1}(v)\| \leq \frac{(r \cdot (b-a))^{m+1}}{(m+1)!} \cdot \|u - v\|$.

(52) Suppose $a < b$ and G is Lipschitzian on the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Then there exists a natural number m such that $(\mathrm{Fredholm}(G, a, b, y_0))^{m+1}$ is contraction.

(53) If $a < b$ and G is Lipschitzian on the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, then $\mathrm{Fredholm}(G, a, b, y_0)$ has unique fixpoint.

(54) Let f, g be continuous partial functions from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $\mathrm{dom}\, f = [a, b]$ and $\mathrm{dom}\, g = [a, b]$ and $Z =]a, b[$ and $a < b$ and G is Lipschitzian on the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and $g = (\mathrm{Fredholm}(G, a, b, y_0))(f)$. Then $g_a = y_0$ and g is differentiable on Z and for every real number t such that $t \in Z$ holds $g'(t) = (G \cdot f)_t$.

(55) Let y be a continuous partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose that $a < b$ and $Z =]a, b[$ and G is Lipschitzian on the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and $\mathrm{dom}\, y = [a, b]$ and y is differentiable on Z and $y_a = y_0$ and for every real number t such that $t \in Z$ holds $y'(t) = G(y_t)$. Then y is a fixpoint of $\mathrm{Fredholm}(G, a, b, y_0)$.

(56) Let y_1, y_2 be continuous partial functions from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose that $a < b$ and $Z =]a, b[$ and G is Lipschitzian on the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and $\mathrm{dom}\, y_1 = [a, b]$ and y_1 is differentiable on Z and $(y_1)_a = y_0$ and for every real number t such that $t \in Z$ holds ${y_1}'(t) = G((y_1)_t)$ and $\mathrm{dom}\, y_2 = [a, b]$ and y_2 is differentiable on Z and $(y_2)_a = y_0$ and for every real number t such that $t \in Z$ holds ${y_2}'(t) = G((y_2)_t)$. Then $y_1 = y_2$.

(57) Suppose $a < b$ and $Z =]a, b[$ and G is Lipschitzian on the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Then there exists a continuous partial function y from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$ such that $\mathrm{dom}\, y = [a, b]$ and y is differentiable on Z and $y_a = y_0$ and for every real number t such that $t \in Z$ holds $y'(t) = G(y_t)$.

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